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Research Article

Biwave Maps into Manifolds

Yuan-Jen Chiang

Department of Mathematics, University of Mary Washington, Fredericksburg, VA 22401, USA

Correspondence should be addressed to Yuan-Jen Chiang, ychiang@umw.edu

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We generalize wave maps to biwave maps. We prove that the composition of a biwave map and a totally geodesic map is a biwave map. We give examples of biwave nonwave maps. We show that if f is a biwave map into a Riemannian manifold under certain circumstance, then f is a wave map. We verify that if f is a stable biwave map into a Riemannian manifold with positive constant sectional curvature satisfying the conservation law, then f is a wave map. We finally obtain a theorem involving an unstable biwave map.

1. Introduction

Harmonic maps between Riemannian manifolds were first introduced and established by Eells and Sampson [1] in 1964. Afterwards, there were two reports on harmonic maps by Eells and Lemaire [2, 3] in 1978 and 1988. Biharmonic maps, which generalized harmonic maps, were first studied by Jiang [4, 5] in 1986. In this decade, there has been progress in biharmonic maps made by Caddeo et al. [6, 7], Loubeau and Oniciuc [8], Montaldo and Oniciuc [9], Chiang and Wolak [10], Chiang and Sun [11, 12], Chang et al. [13], Wang [14, 15], and so forth.

Wave maps are harmonic maps on Minkowski spaces, and their equations are the second-order hyperbolic systems of partial differential equations, which are related to Einstein's equations and Yang-Mills fields. In recent years, there have been many new developments involving local well-posedness and global-well posedness of wave maps into Riemannian manifolds achieved by Klainerman and Machedon [16, 17], Shatah and Struwe [18, 19], Tao [20, 21], Tataru [22, 23], and so forth. Furthermore, Nahmod et al. [24] also studied wave maps from $R \times R^m$ into (compact) Lie groups or Riemannian symmetric spaces, that is, gauged wave maps when $m \geq 4$, and established global existence and uniqueness, provided that the initial data are small. Moreover, Chiang and Yang [25] , Chiang and Wolak [26] have investigated exponential wave maps and transversal wave maps.

Bi-Yang-Mills fields, which generalize Yang-Mills fields, have been introduced by Ichiyama et al. [27] recently. The following connection between bi-Yang Mills fields and biwave equations motivates one to study biwave maps.

Let P be a principal fiber bundle over a manifold M with structure group G and canonical projection π, and let $\mathcal{G}$ be the Lie algebra of G. A connection A can be considered as a $\mathcal{G}$-valued 1-form $A = A_\mu(x)dx^\mu$ locally. The curvature of the connection A is given by the 2-form $F = F_{\mu\nu}dx^\mu dx^\nu$ with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \left[A_\mu, A_\nu\right]. \tag{1.1}$$

The bi-Yang-Mills Lagrangian is defined

$$L_2(A) = \frac{1}{2}\int_M \|\delta F\|^2 dv_M, \tag{1.2}$$

where δ is the adjoint operator of the exterior differentiation d on the space of E-valued smooth forms on M ($E = \mathrm{End}(P)$, the endormorphisms of P). Then the Euler-Lagrange equation describing the critical point of (1.2) has the form

$$(\delta d + F)\delta F = 0, \tag{1.3}$$

which is the bi-Yang-Mills system. In particular, letting $M = R \times R^2$ and $G = \mathrm{SO}(2)$, the group of orthogonal transformations on R^2, we have that $A_\mu(x)$ is a 2×2 skew symmetric matrix A_μ^{ij}. The appropriate equivariant ansatz has the form

$$A_\mu^{ij}(x) = \left(\delta_\mu^i x^j - \delta_\mu^j x^i\right)h(t, |x|), \tag{1.4}$$

where $h : M \to R$ is a spatially radial function. Setting $u = r^2 h$ and $r = |x|$, the bi-Yang-Mills system (1.3) becomes the following equation for $u(t, r)$:

$$u_{tttt} - u_{rrrr} - \frac{3}{r}u_{rrr} + \frac{2}{r^2}u_{rr} - \frac{2}{r^3}u_r = k(t, r), \tag{1.5}$$

which is a linear nonhomogeneous biwave equation, where $k(t, r)$ is a function of t and r.

Biwave maps are biharmonic maps on Minkowski spaces. It is interesting to study biwave maps since their equations are the fourth-order hyperbolic systems of partial differential equations, which generalize wave maps. This is the first attempt to study biwave maps and their relationship with wave maps. There are interesting and difficult problems involving local well posedness and global well posedness of biwave maps into Riemannian manifolds or Lie groups (or Riemannian symmetric spaces), that is, gauged biwave maps for future exploration.

In Section 2, we compute the first variation of the bi-energy functional of a biharmonic map using tensor technique, which is different but much easier than Jiang's [4] original computation. In Section 3, we prove in Theorem 3.3 that *if* $f : R^{m,1} \to N_1$ *is a biwave map and* $f_1 : N_1 \to N_2$ *is a totally geodesic map, then* $f_1 \circ f : R^{m,1} \to N_2$ *is a biwave map.* Then we can apply this theorem to provide many biwave maps (see Example 3.4). We also can construct biwave nonwave maps as follow: *Let* $h : \Omega \subset R^{m,1} \rightarrow S^n(1/\sqrt{2})$ *be a wave map on a compact domain and let* $i : S^n(1/\sqrt{2}) \rightarrow S^{n+1}(1)$ *be an inclusion map. The map* $f = i \circ h : \Omega \rightarrow S^{n+1}(1)$ *is a biwave nonwave map if and only if* h *has constant energy density,* compare with Theorem 3.5. Afterwards, we show that *if* $f : \Omega \rightarrow N$ *is a biwave map on a compact domain into a Riemannian manifold satisfying*

$$-|\tau_\square f|_t^2 + \sum_{i=1}^m |\tau_\square f|_{x^i}^2 - R'^{\alpha}_{\beta\gamma\mu}\left(-f_t^\beta f_t^\gamma + \sum_{i=1}^m f_i^\beta f_i^\gamma\right)\tau_\square(f)^\mu \ \geq 0, \tag{1.6}$$

then f is a wave map (cf. Theorem 3.6). This theorem is different than the theorem obtained by Jiang [4]: *if* f *is a biharmonic map from a compact manifold into a Riemannian manifold with nonpositive curvature, then* f *is a harmonic map.* In Section 4, we verify that *if* f *is a stable biwave map into a Riemannian manifold with positive constant sectional curvature satisfying the conservation law, then* f *is a wave map* (cf. Theorem 4.5). We also prove that *if* $h : \Omega \rightarrow S^n(1/\sqrt{2})$ *is a wave map on a compact domain with constant energy density, then* $f = i \circ h : \Omega \rightarrow S^{n+1}(1)$ *is an unstable biwave map* (cf. Theorem 4.7).

2. Biharmonic Maps

A biharmonic map $f : (M^m, g_{ij}) \rightarrow (N^n, h_{\alpha\beta})$ from an m-dimensional Riemannian manifold M into an n-dimensional Riemannian manifold N is the critical point of the bi-energy functional

$$E_2(f) = \frac{1}{2}\int_M \left\|(d + d^*)^2 f\right\|^2 dv = \frac{1}{2}\int_M \|(d^* d) f\|^2 dv = \frac{1}{2}\int_M \|\tau(f)\|^2 dv, \tag{2.1}$$

where dv is the volume form on M.

Notations

d^* is the adjoint of d and $\tau(f) = \operatorname{trace}(Ddf) = (Ddf)(e_i, e_i) = (D_{e_i} df)(e_i)$ is the tension field. Here D is the Riemannian connection on $T^*M \otimes f^{-1}TN$ induced by the Levi-Civita connections on M and N, and $\{e_i\}$ is the local frame at a point of M. The tension field has components

$$\tau(f)^\alpha = g^{ij} f^\alpha_{i|j} = g^{ij}\left(f^\alpha_{ij} - \Gamma^k_{ij} f^\alpha_k + \Gamma'^\alpha_{\beta\gamma} f^\beta_i f^\gamma_j\right), \tag{2.2a}$$

where Γ^k_{ij} and $\Gamma'^\gamma_{\alpha\beta}$ are the Christoffel symbols on M and N, respectively.

In order to compute the Euler-Lagrange equation of the bi-energy functional, we consider a one-parameter family of maps $\{f_t\} \in C^\infty(M \times I, N)$ from a compact manifold M (without boundary) into a Riemannian manifold N. Here $f_t(x)$ is the endpoint of a segment starting at $f(x)(=f_0(x))$, determined in length and direction by the vector field $\dot{f}(x)$ along $f(x)$. For a nonclosed manifold M, we assume that the compact support of $\dot{f}(x)$ is contained in the interior of M (we need this assumption when we compute $\tau(f)$ by applying the divergence theorem). Then we have

$$\frac{d}{dt}E_2(f_t)|_{t=0} = \dot{E}_2(f) = \int_M (D_t\tau f, \tau f)_{t=0} dv. \tag{2.3}$$

Let $\xi = \partial f_t/\partial t$. The components of $D_t\tau f$ are $f^\alpha_{i|j|t} = (\partial f^\alpha_{i|j}/\partial t) + \Gamma'^\alpha_{\mu\gamma} f^\mu_{i|j}\xi^\gamma$. We can use the curvature formula on $M \times I \to N$ and get

$$f^\alpha_{i|j|t} = f^\alpha_{i|t|j} + R'^\alpha_{\beta\gamma\mu} f^\beta_i f^\gamma_j \xi^\mu, \tag{2.4}$$

where R' is the Riemannian curvature of N. But $f^\alpha_{i|t} = f^\alpha_{t|i} = \xi^\alpha_{|i}$, therefore, $D_t\tau f$ has components $\xi^\alpha_{|i|j} + R'^\alpha_{\beta\gamma\mu} f^\beta_i f^\gamma_j \xi^\mu$. We can rewrite (2.3) as

$$\frac{d}{dt}E_2(f_t)|_{t=0} = \int_M (J_f(\tau f), \tau f) dv, \tag{2.5}$$

where

$$J^\alpha_f(\xi) = g^{ij}\xi^\alpha_{|i|j} + g^{ij} R'^\alpha_{\beta\gamma\mu} f^\beta_i f^\gamma_j \xi^\mu = \Delta\xi^\alpha + R'^\alpha(df, df)\xi \tag{2.6}$$

is a linear equation for $\xi(= \tau(f))$, and $\Delta(\xi) = D^*D(\xi)$ is an operator from $f^{-1}TN$ to $f^{-1}TN$. Solutions of $J_f(\xi) = 0$ are called Jacobi fields. Hence, we obtain the following definition from (2.3), (2.5), and (2.6).

Definition 2.1. $f : M \to N$ is a biharmonic map if and only if the bitension field

$$\begin{aligned} \tau_2(f)^\alpha &= J_f(\tau f)^\alpha = \Delta\tau(f)^\alpha + R'\alpha(df, df)\tau(f) \\ &= g^{ij}\left(f^\alpha_{ij} - \Gamma^k_{ij} f^\alpha_k + \Gamma'^\alpha_{\beta\gamma} f^\beta_i f^\gamma_j\right) + g^{ij} R'^\alpha_{\beta\gamma\mu} f^\beta_i f^\gamma_j \tau(f)^\mu = 0, \end{aligned} \tag{2.7}$$

that is, the tension field $\tau(f)$, is a Jacobi field.

If $\tau(f) = 0$, then $\tau_2(f) = 0$. Thus, harmonic maps are obviously biharmonic. Biharmonic maps satisfy the fourth-order elliptic systems of PDEs, which generalize harmonic maps. Our computation for the first variation of the bi-energy functional presented here using tensor technique is different but much easier than Jiang's [4] original computation (it took him four pages).

Caddeo et al. [7] showed that a biharmonic curve on a surface of nonpositive Gaussian curvature is a geodesic (i.e., is harmonic) and gave examples of biharmonic nonharmonic curves on spheres, ellipses, unduloids, and nodoids.

Theorem 2.2 (see [4]). *Let $f : M^m \to S^{m+1}(1)$ be an isometric embedding of an m-dimensional compact Riemannian manifold M into an $(m+1)$-dimensional unit sphere $S^{m+1}(1)$ with nonzero constant mean curvature. The map f is biharmonic if and only if $\|B(f)\|^2 = m$, where $B(f)$ is the second fundamental form of f.*

Example 2.3. In $S^{m+1}(1)$, the compact hypersurfaces, whose Gauss maps are isometric embeddings, are the Clifford surfaces [28]:

$$M_k^m(1) = S^k\left(\frac{1}{\sqrt{2}}\right) \times S^{m-k}\left(\frac{1}{\sqrt{2}}\right), \quad 0 \le k \le m. \tag{2.8}$$

Let $f : M_k^m(1) \to S^{m+1}(1)$ be a standard embedding such that $k \neq m/2$. Because $\|B(f)\|^2 = k + m - k = m$ and $\tau(f) = k - (m-k) = 2k - m \neq 0$, f is a biharmonic nonharmonic map by Theorem 2.2.

3. Biwave Maps

Let $R^{m,1}$ be an $m+1$ dimensional Minkowski space $R \times R^m$ with the metric $(g_{ij}) = (-1, 1, \ldots, 1)$ and the coordinates $x^0 = t, x^1, x^2, \ldots, x^m$ and let $(N, h_{\alpha\beta})$ be an n-dimensional Riemannian manifold. A wave map is a harmonic map on the Minkowski space $R^{m,1}$ with the energy functional

$$E(f) = \frac{1}{2}\int_{R^{m,1}} \left(-|f_t|^2 + |\nabla_x f|^2\right) dt\, dx = \frac{1}{2}\int_{R^{m,1}} h_{\alpha\beta}\left(-f_t^\alpha f_t^\beta + \sum_{i=1}^m f_i^\alpha f_i^\beta\right) dt\, dx_i. \tag{3.1}$$

The Euler-Lagrange equation describing the critical point of (3.1) is

$$\tau_\square^\alpha(f) = \square f^\alpha + \Gamma'^\alpha_{\beta\gamma}\left(-f_t^\beta f_t^\gamma + \sum_{i=1}^m f_i^\beta f_i^\gamma\right) = 0, \tag{3.2}$$

where $\square = -(\partial^2/\partial t^2) + \Delta_x$ is the wave operator on $R^{m,1}$ and $\Gamma'^\alpha_{\beta\gamma}$ are the Christoffel symbols of N. f is a wave map iff the wave field $\tau_\square^\alpha(f)$ (i.e., the tension field on a Minkowski space) vanishes. The wave map equation is invariant with respect to the dimensionless scaling $f(t,x) \to f(ct, cx)$, $c \in R$. But, the energy is scale invariant in dimension $m = 2$.

If $f : R^{m,1} \to N$ is a smooth map from a Minkowski space $R^{m,1}$ into a Riemannian manifold N, then the bi-energy functional is, from (2.1),

$$\begin{aligned} E_2(f) &= \frac{1}{2}\int_{R^{m,1}} \left\|(d+d^*)^2 f\right\|^2 dt\, dx \\ &= \frac{1}{2}\int_{R^{m,1}} \|d^* df\|^2 dt\, dx = \frac{1}{2}\int_{R^{m,1}} \|\tau_\square(f)\|^2 dt\, dx. \end{aligned} \tag{3.3}$$

The Euler-Lagrange equation describing the critical point of (3.3), from (2.5), is

$$(\tau_2)_\square(f) = J_f(\tau_\square f) = \Delta\tau_\square(f) + R'(df, df)\tau_\square(f) = 0. \tag{3.4}$$

Definition 3.1. $f : R^{m,1} \to N$ from a Minkowski space into a Riemannian manifold is a biwave map if and only if the biwave field (i.e., the bitension field on a Minkowski space),

$$\begin{aligned}(\tau_2)_\square(f)^\alpha &= J_f(\tau_\square f)^\alpha = \Delta\tau_\square(f)^\alpha + R'^\alpha(df, df)\tau_\square(f)\\ &= \square\tau_\square(f)^\alpha + \Gamma'^\alpha_{\mu\gamma}\left(-\tau_\square(f)^\mu_t\tau_\square(f)^\gamma_t + \sum_{i=1}^m \tau_\square(f)^\mu_i\tau_\square(f)^\gamma_i\right)\\ &\quad + R'^\alpha_{\beta\gamma\mu}\left(-f^\beta_t f^\gamma_t + \sum_{i=1}^m f^\beta_i f^\gamma_i\right)\tau_\square(f)^\mu = 0,\end{aligned} \tag{3.5}$$

that is, the wave field $\tau_\square(f)$, is a Jacobi field on the Minkowski space.

Biwave maps satisfy the fourth-order hyperbolic systems of PDEs, which generalize wave maps. If $\tau_\square(f) = 0$, then $(\tau_2)_\square(f) = 0$. Waves maps are obviously biwave maps, but biwave maps are not necessarily wave maps.

Example 3.2. Let $u : R^{m,1} \to R$ be a function defined on a Minkowski space satisfying the following conditions:

$$\begin{aligned}&\square^2 u(t,x) = \square(\square u) = u_{tttt} - 2u_{ttxx} + u_{xxxx} = 0, \quad (t,x) \in (0,\infty)\times R^m,\\ &u = u_0, \quad u_t = u_1, \quad u = u_0, \quad \frac{\partial}{\partial t}u = \frac{\partial u}{\partial t} = u_1, \quad (t,x) \in \{t=0\}\times R^m,\end{aligned} \tag{3.6}$$

where the initial data u_0 and u_1 are given. Since this is a fourth-order homogeneous linear biwave equation with constant coefficients, it is well known that $u(t,x)$ can be solved by [18, 29].

Let $f : R^{m,1} \to N_1$ be a smooth map from a Minkowski space $R^{m,1}$ into a Riemannian manifold N_1 and let $f_1 : N_1 \to N_2$ be a smooth map between two Riemannian manifolds N_1 and N_2. Then the composition $f_1 \circ f : R^{m,1} \to N_2$ is a smooth map. Since $R^{m,1}$ is a semi-Riemannian manifold (i.e., a pseudo-Riemannian manifold), we can define a Levi-Civita connection on $R^{m,1}$ by O'Neill [30]. Let D, D', $\overline{D}$, $\overline{D}'$, $\overline{D}''$, $\widehat{D}$, $\widehat{D}'$, $\widehat{D}''$ be the connections on $R^{m,1}$, TN_1, $f^{-1}N_1$, $f_1^{-1}TN_2$, $(f_1\circ f)^{-1}TN_2$, $T^*R^{m,1}\otimes f^{-1}TN_1$, $T^*N_1\otimes f_1^{-1}TN_2$, $T^*R^{m,1}\otimes (f_1\circ f)^{-1}TN_2$, respectively, and let $R^{N_2}(,)$, $R^{f_1^{-1}TN_2}(,)$ be the curvatures on TN_2, $f_1^{-1}TN_2$, respectively. We first have the following two formulas:

$$\widehat{D}''_X d(f_1\circ f)(Y) = \left(\widehat{D}'_{df(X)} df_1\right) df(Y) + df_1 \circ \overline{D}_X df(Y), \tag{3.7a}$$

for $X, Y \in R^{m,1}$, and

$$R^{N_2}(df_1(X'), df_1(Y'))df_1(Z') = R^{f_1^{-1}TN_2}(X', Y')df_1(Z'), \tag{3.7b}$$

for X', Y', $Z' \in \Gamma(TN_1)$.

Theorem 3.3. *If $f : R^{m,1} \to N_1$ is a biwave map and $f_1 : N_1 \to N_2$ is totally geodesic between two Riemannian manifolds N_1 and N_2, then the composition $f_1\circ f : R^{m,1} \to N_2$ is a biwave map.*

Proof. Let $x^0 = t, x^1, \ldots, x^m$ be the coordinates of a point p in $R^{m,1}$ and let $e_0 = \partial/\partial t$, $e_1 = (1,0,\ldots,0)$, $e_2 = (0,1,0,\ldots,0)$, $\ldots$, $e_m = (0,\ldots,0,1)$ be the frame at p. We know from [4] that $\overline{D}'' * \overline{D}'' = \overline{D}''_{e_k}\overline{D}''_{e_k} - \overline{D}''_{D_{e_k}e_k}$. Since f_1 is totally geodesic, we have $\tau_\square(f_1 \circ f) = df_1 \circ \tau_\square(f)$ by applying the chain rule of the wave field to $f_1 \circ f$ as [1]. Then we get

$$\begin{aligned} \overline{D}'' * \overline{D}''_{\tau_\square}(f_1 \circ f) &= \overline{D}'' * \overline{D}''(df_1 \circ \tau_\square(f)) \\ &= \overline{D}''_{e_k}\overline{D}''_{e_k}(df_1 \circ \tau_\square(f)) - \overline{D}''_{D_{e_k}e_k}(df_1 \circ \tau_\square(f)). \end{aligned} \tag{3.8}$$

Recalling that $\tau_\square(f) = \widehat{D}_{e_j} df(e_j)$, we derive from (3.7a) that

$$\begin{aligned} \overline{D}''_{e_k}(df_1 \circ \tau_\square(f)) &= \overline{D}''_{e_k}\left(df_1 \circ \widehat{D}_{ej} df(ej)\right) \\ &= \left(\widehat{D}'_{\widehat{D}_{e_j}df(e_k)} df_1\right)\left(\widehat{D}_{e_j} df(e_j)\right) + df_1 \circ \overline{D}_{e_k}\left(\widehat{D}_{e_j} df(e_j)\right) = df_1 \circ \overline{D}_{e_k}\tau_\square(f), \end{aligned} \tag{3.9}$$

since f_1 is totally geodesic. Therefore, we have

$$\begin{aligned} \overline{D}''_{e_k}\overline{D}''_{e_k}(df_1 \circ \tau_\square(f)) &= \overline{D}''_{e_k}\left(df_1 \circ \overline{D}_{e_k}\tau(f)\right) = df_1 \circ \overline{D}_{e_k}\overline{D}_{e_k}\tau_\square(f), \\ \overline{D}''_{D_{e_k}e_k}(df_1 \circ \tau(f)) &= df_1 \circ \overline{D}_{D_{e_k}e_k}\tau_\square(f). \end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.8), we arrive at

$$\overline{D}'' * \overline{D}''\tau_\square(f_1 \circ f) = df_1 \circ \overline{D}^*\overline{D}\tau_\square(f), \tag{3.11}$$

where $\overline{D}^*\overline{D} = \overline{D}_{e_k}\overline{D}_{e_k} - \overline{D}_{D_{e_k}e_k}$.

On the other hand, we have by (3.7b)

$$\begin{aligned} &R^{N_2}(d(f_1 \circ f)(e_i), \tau_\square(f_1 \circ f))d(f_1 \circ f)(e_i) \\ &\quad = R^{f_1^{-1}TN_2}(df(e_i), \tau_\square(f))df_1(df(e_i)) = df_1 \circ R^{N_1}(df(e_i), \tau_\square(f))df(e_i). \end{aligned} \tag{3.12}$$

We obtain from (3.11) and (3.12)

$$\begin{aligned} &\overline{D}'' * \overline{D}''(f_1 \circ f) + R^{N_2}(d(f_1 \circ f)(e_i), \tau_\square(f_1 \circ f))d(f_1 \circ f)(e_i) \\ &\quad = df_1 \circ \left[\overline{D}^*\overline{D}\tau_\square(f) + R^{N_1}(df(e_i), \tau_\square(f))df(e_i)\right], \end{aligned} \tag{3.13}$$

that is, $(\tau_2)_\square(f_1 \circ f) = df_1 \circ (\tau_2)_\square(f)$. Hence, if f is a biwave map and f_1 is totally geodesic, then $f_1 \circ f$ is a biwave map. Note that the total geodesicity of f_1 cannot be weakened into a harmonic or biharmonic map. □

Example 3.4. Let N_1 be a submanifold of N. Are the biwave maps into N_1 also biwave maps into N? The answer is affirmative iff N_1 is a totally geodesic submanifold of N, that is, N_1 geodesics are N geodesics. N_1 is a geodesic $\gamma(t) = (\gamma^1, \ldots, \gamma^n) : R \to N \subset R^n$ with $|\dot{\gamma}(t)| = 1$ iff $\dot{\gamma}$ is parallel, that is, $D_{\partial/\partial t}\dot{\gamma} = 0$ iff $\ddot{\gamma} \perp T_\gamma N$. For a map $u : R^{m,1} \to R$, letting $f = \gamma \circ u = (f^1, \ldots, f^n) : R^{m,1} \to N \subset R^n$, we have by (3.13) the following:

$$(\tau_2)_\Box(f) = d\gamma \circ (\tau_2)_\Box(u) = d\gamma \circ \Box^2 u, \tag{3.14}$$

since γ is a geodesic. Hence, $f = \gamma \circ u$ is a biwave map if and only if u solves the fourth-order homogeneous linear biwave equation $\Box^2 u = 0$ as in Example 3.2. It follows from Theorem 3.3 that there are many biwave maps $f : R^{m,1} \to N$ provided by geodesics of N.

We also can construct examples of biwave nonwave maps from some wave maps with constant energy using Theorem 3.5. Let

$$S^n\left(\frac{1}{\sqrt{2}}\right) = S^n\left(\frac{1}{\sqrt{2}}\right) \times \left\{\frac{1}{\sqrt{2}}\right\} = \left\{\left(x_1, x_2, \ldots, x_{n+1}, \frac{1}{\sqrt{2}}\right) \mid x_1^2 + \cdots + x_{n+1}^2 = \frac{1}{2}\right\} \tag{3.15}$$

be a hypersphere of $S^{n+1}(1)$. Then $S^n(1/\sqrt{2})$ is a biharmonic nonminimal submanifold of $S^{n+1}(1)$ by Theorem 2.2 and Example 2.3. Let $\zeta = (x_1, \ldots, x_{n+1}, -1/\sqrt{2})$ be a unit section of the normal bundle of $S^n(1/\sqrt{2})$ in $S^{n+1}(1)$. Then the second fundamental form of the inclusion $i : S^n(1/\sqrt{2}) \to S^{n+1}(1)$ is $B(X,Y) = Ddi(X,Y) = -(X,Y)\zeta$. By computation, the tension field of i is $\tau(i) = -n\zeta$, and the bitension field is $\tau_2(i) = 0$.

Theorem 3.5. *Let $h : \Omega \to S^n(1/\sqrt{2})$ be a nonconstant wave map on a compact space-time domain $\Omega \subset R^{m,1}$ and let $i : S^n(1/\sqrt{2}) \to S^{n+1}(1)$ be an inclusion. The map $f = i \circ h : R^{m,1} \to S^{n+1}(1)$ is a biwave nonwave map if and only if h has constant energy density $e(h) = (1/2)|dh|^2$.*

Proof. Let $x^0 = t, x^1, \ldots, x^m$ be the coordinate of a point p in $\Omega \subset R^{m,1}$ and let $e_0 = \partial/\partial t$, $e_1 = (1,0,\ldots,0)$, $e_2 = (0,1,0,\ldots,0), \ldots,$ $e_m = (0,\ldots,0,1)$ be the frame at p. Recall that ζ is the unit section of the normal bundle. By applying the chain rule of the wave field to $f = i \circ h$, we have

$$\tau_\Box(f) = di(\tau_\Box(h)) + \text{trace } Ddi(dh, dh) = -2e(h)\zeta, \tag{3.16}$$

since h is a wave map. We can derive the following at the point p by straightforward calculation:

$$\begin{aligned} D^* D\tau_\Box(f) &= -D^f_{e_i} D^f_{e_i} \tau_\Box(f) = -D^f_{e_i} D^f_{e_i}(-2e(h)\zeta) \\ &= 2(e_i e_i e(h))\zeta - 2e(h)(dh(e_i), dh(e_i))\zeta + 4df[(e_i e(h))e_i] \\ &\quad + 2e(h)Ddh(e_i, e_i), \end{aligned} \tag{3.17}$$
$$R^{S^{n+1}}(df(e_i), \tau_\Box(f))df(e_i) = -(dh(e_i), dh(e_i))\tau(f) = 2(dh(e_i), dh(e_i))e(h)\zeta.$$

Therefore, we obtain

$$\tau_{2\square}(f) = -2(\Delta e(h))\zeta + 4df(\operatorname{grad} e(h)). \tag{3.18}$$

Suppose that $f = i \circ h : \Omega \to S^n(1/\sqrt{2}) \times \{1/\sqrt{2}\} \to S^{n+1}(1)$ is a biwave nonwave map ($\tau_\square(f) \neq 0$). As the ζ-part of $\tau_{2\square}(f)$, $\Delta e(h)$ vanishes, which implies that $e(h)$ is constant since Ω is compact. The converse is obvious. $\square$

Let $x^0 = t, x^1, \dots, x^m$ be the coordinates of a point in a compact space-time domain $\Omega \subset R^{m,1}$ and $e_0 = \partial/\partial t$, $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0), \dots,$ $e_m = (0, \dots, 0, 1)$ be the frame at the point. Suppose that $f : \Omega \to N$ is a biwave map from a compact domain Ω into a Riemannian manifold N such that the compact supports of $\partial f/\partial x_i$ and $D_{e_i}\partial f/\partial x_i$ are contained in the interior of Ω.

Theorem 3.6. *If $f : \Omega \to N$ is a biwave map from a compact domain into a Riemannian manifold such that*

$$-\left|\tau_\square f\right|_t^2 + \sum_{i=1}^m \left|\tau_\square f\right|_{x^i}^2 - R'^{\alpha}_{\beta\gamma\mu}\left(-f_t^\beta f_t^\gamma + \sum_{i=1}^m f_i^\beta f_i^\gamma\right)\tau_\square(f)^\mu \ \geq 0, \tag{3.19}$$

then f is a wave map.

Proof. Since f is a biwave map, we have by (3.4)

$$(\tau_2)_\square(f) = \Delta\tau_\square(f) + R'(df, df)\tau_\square(f). \tag{3.20}$$

Recall that $x^0 = t, x^1, \dots, x^m$ are the coordinates of a point in $\Omega \subset R^{m,1}$ and $e_0 = \partial/\partial t$, $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0), \dots,$ $e_m = (0, \dots, 0, 1)$. We compute

$$\begin{aligned}
\frac{1}{2}\Delta\left\|\tau_\square(f)\right\|^2 &= \left(D_{e_i}\tau_\square(f), D_{e_i}\tau_\square(f)\right) + \left(D^*D\tau_\square(f), \tau_\square(f)\right) \\
&= \sum_{i=0}^m \left(D_{e_i}\tau_\square(f), D_{e_i}\tau_\square(f)\right) - \left(R'^{\alpha}_{\beta\gamma\mu}\left(-f_t^\beta f_t^\gamma + \sum_{i=1}^m f_i^\beta f_i^\gamma\right)\tau_\square(f)^\mu, \tau_\square(f)\right) \\
&= -\left|\tau_\square f\right|_t^2 + \sum_{i=1}^m \left|\tau_\square f\right|_{x^i}^2 - \left(R'^{\alpha}_{\beta\gamma\mu}\left(-f_t^\beta f_t^\gamma + \sum_{i=1}^m f_i^\beta f_i^\gamma\right)\tau_\square(f)^\mu, \tau_\square(f)\right).
\end{aligned} \tag{3.21}$$

By applying the Bochner's technique from (3.19) and the assumption that the compact supports of $\partial f/\partial x_i$ and $D_{e_i}\partial f/\partial x_i$ are contained in the interior of Ω, we know that $\|\tau_\square(f)\|^2$ is constant, that is, $d\tau_\square(f) = 0$. If we use the identity

$$\int_\Omega \operatorname{div}(df, \tau(f))dz = \int_\Omega \left(\left|\tau(f)\right|^2 + (df, d\tau(f))\right)dz, \quad z = (t, x) \tag{3.22}$$

and the fact $d\tau_{\square}(f) = 0$, then we can conclude that $\tau_{\square}(f) = 0$ by applying the divergence theorem. □

Corollary 3.7. *If $f : \Omega \to N$ is a biwave map on a compact domain such that $\sum_{i=1}^{m} |\tau_{\square} f|_{x^i}^2 \geq |\tau_{\square} f|_t^2$ and $R'^{\alpha}_{\beta\gamma\mu}(-f_t^{\beta} f_t^{\gamma} + \sum_{i=1}^{m} f_i^{\beta} f_i^{\gamma})\tau_{\square}(f)^{\mu} \leq 0$, then f is a wave map.*

Proof. The result follows from (3.19) immediately. □

4. Stability of Biwave Maps

Let $x^0 = t, x^1, \ldots, x^m$ be the coordinates of a point in a compact space-time domain $\Omega \subset R^{m,1}$ and let $e_0 = \partial/\partial t$, $e_1 = (1, 0, \ldots, 0), \ldots,$ $e_m = (0, 0, \ldots, 1)$ be the frame at the point. Suppose that $f : \Omega \to N$ is a biwave map from a compact space-time domain Ω into a Riemannian manifold N such that the compact supports of $\partial f/\partial x_i$ and $D_{e_i}\partial f/\partial x_i$ are contained in the interior of Ω. Let $V \in \Gamma(f^{-1}TN)$ be a vector field such that $\partial f/\partial t|_{t=0} = V$. If we apply the second variation of a biharmonic map in [4] to a biwave map, we can have the following.

Lemma 4.1. *If $f : \Omega \to N$ is a biwave map from a compact domain into a Riemannian manifold, then*

$$\begin{aligned}
\frac{1}{2}\frac{d^2}{dt^2}E_2(f)|_{t=0} = &\int_{\Omega}\left\|\Delta V + R^N(df(e_i), V)df(e_i)\right\|^2 dz \\
&+ \int_{\Omega} < V, \left(D'_{df(e_i)}R^N\right)(f(e_i), \tau_{\square}(f))V \\
&+ \left(D'_{\tau_{\square}(f)}R^N\right)(df(e_i), V)df(e_i) + R^N(\tau_{\square}(f), V)\tau_{\square}(f) \\
&+ 2R^N(df(e_i), V)\overline{D}_{e_i}\tau_{\square}(f) + 2R^N(df(e_i), \tau_{\square}(f))\overline{D}_{e_i}V > dz,
\end{aligned} \tag{4.1}$$

where $z = (t, x) \in R \times R^m$, D' is the Riemannian connection on TN, and V is the vector field along f.

Definition 4.2. Let $f : R^{m,1} \to N$ be a biwave map. If $(d^2/dt^2)E_2(f)|_{t=0} \geq 0$, then f is a stable biwave map.

If we consider a wave map, that is, $\tau_{\square}(f) = 0$ as a biwave map, then by (4.1) we have $(d^2/dt^2)E_2(f)|_{t=0} \geq 0$ and f is automatically stable.

Definition 4.3. Let $f : R^{m,1} \to (N, h)$ be a smooth map from a Minkowski space into a Riemannian manifold (N, h). The *stress energy* is defined by $S(f) = e(f)g - f^*h$, where $e(f) = (1/2)|df|^2$ is the energy function and $g = \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}$. The map f satisfies the *conservation law* if $\operatorname{div} S(f) = 0$.

Proposition 4.4. *Let $f : R^{m,1} \to (N, h)$ be a smooth map from a Minkowski space into a Riemannian manifold (N, h). Then*

$$\operatorname{div} S(f)(X) = -\langle \tau_{\square}(f), df(X) \rangle, \qquad X \in R^{m,1}. \tag{4.2}$$

Proof. Let $x^0 = t, x^1, \ldots, x^m$ be the coordinates of a point in $R^{m,1}$, $e_0 = \partial/\partial t$, $e_1 = (1, 0, \ldots, 0)$, $\ldots$, $e_m = (0, 0, \ldots, 1)$ and $g = \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}$, where I is an $m \times m$ matrix. We calculate

$$\begin{aligned}
\operatorname{div} S(f)(X) &= D_{e_i} S(f)(e_i, X) = D_{e_i}\left(\frac{1}{2}|df|^2 \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix} - f^* h\right)(e_i, X) \\
&= D_{e_i}\left(\frac{1}{2}|df|^2 \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}\right)(e_i, X) - (D_{e_i} f^* h)(e_i, X) \\
&= \left(-\left(D\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right)(-1)\right)(e_0, X) + \left(D\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_i}\right) I(e_i, X) - D_{e_i}(f_* e_i, f_* X) \\
&= \left(D\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right)(e_0, X) + \left(D\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_i}\right)(e_i, X) - (D_{e_i} f_* e_i, f_* X) - (f_* e_i, D_{e_i} f_* X) \\
&= ((D_X df) e_i, f_* e_i) - (\tau_\square(f), f_* X) - (f_* e_i, D_{e_i} f_* X),
\end{aligned} \tag{4.3}$$

where the first term and the third term are canceled out and $D_{e_i} f_* e_i = \tau_\square(f)$. □

Theorem 4.5. *Let $\Omega \subset R^{m,1}$ be a compact domain and let (N, h) be a Riemannian manifold with constant sectional curvature $K > 0$. If $f : \Omega \to N$ is a stable biwave map satisfying the conservation law, then f is a wave map.*

Proof. Because N has constant sectional curvature, the second term of (4.1) disappears and (4.1) becomes

$$\begin{aligned}
\frac{1}{2}\frac{d^2}{dt^2} E_2(f_t)\bigg|_{t=0} &= \int_\Omega \left\| \Delta V + R^N(df(e_i), V) df(e_i) \right\|^2 dz \\
&\quad + \int_\Omega \Big\langle V,\ R^N(\tau_\square(f), V)\tau_\square(f) + 2R^N(df(e_i), V) D_{e_i}\tau_\square(f) \\
&\qquad + 2R^N(df(e_i), \tau_\square(f)) D_{e_i} V \Big\rangle dz.
\end{aligned} \tag{4.4}$$

In particular, let $V = \tau_\square(f)$. Recalling that f is a biwave map and N has constant sectional curvature $K > 0$, (4.4) can be reduced to

$$\begin{aligned}
\frac{1}{2}\frac{d^2}{dt^2} E_2(f)\bigg|_{t=0} &= 4\int_\Omega \Big\langle R^N(df(e_i), \tau_\square(f)) D_{e_i}\tau_\square(f), \tau_\square(f) \Big\rangle dz \\
&= 4K \int_\Omega \Big[\langle df(e_i), D_{e_i}\tau_\square(f)\rangle \|\tau_\square(f)\|^2 \\
&\qquad - \langle df(e_i), \tau_\square(f)\rangle \langle \tau_\square(f), D_{e_i}\tau_\square(f)\rangle\Big] dz.
\end{aligned} \tag{4.5}$$

Since f satisfies the conservation law, by Definition 4.3, Proposition 4.4, and (4.2) we have

$$\begin{gathered}\langle df(e_i), \tau_{\square}(f)\rangle = 0,\\ \langle df(e_i), D_{e_i}\tau_{\square}(f)\rangle = -\langle D_{e_i}df(e_i), \tau_{\square}(f)\rangle = -\|\tau_{\square}(f)\|^2.\end{gathered} \tag{4.6}$$

Substituting (4.6) into (4.5) and applying the stability of f, we get

$$\frac{1}{2}\frac{d^2}{dt^2}E_2(f_t)\bigg|_{t=0} = -4K\int_{\Omega}\|\tau_{\square}f\|^4 dz \geq 0, \tag{4.7}$$

which implies that $\tau_{\square}(f) = 0$, that is, $f : \Omega \to N$ is a wave map. $\square$

If we apply the Hessian of the bi-energy of a biharmonic map [4] to a biwave map $f : \Omega \to S^{n+1}(1)$, then we have the following.

Lemma 4.6. *Let $f : \Omega \to S^{n+1}(1)$ be a biwave map. The Hessian of the bi-energy functional E_2 of f is*

$$H(E_2)_f(X, Y) = \int_{\Omega}(I_f(X), Y)dz, \tag{4.8}$$

where

$$\begin{aligned} I_f(X) = {} & \Delta^f\left(\Delta^f X\right) + \Delta^f\left(\mathrm{trace}(X, df\cdot)df\cdot - |df|^2 X\right) + 2(d\tau_{\square}(f), df)X \\ & + |\tau_{\square}(f)|^2 X - 2\ \mathrm{trace}(X, d\tau_{\square}(f)\cdot)df - 2\ \mathrm{trace}(\tau_{\square}(f), dX\cdot)df\cdot \\ & - (\tau_{\square}(f), X)\tau_{\square}(f) + \mathrm{trace}\ \left(df\cdot, \Delta^f X\right)df\cdot + \mathrm{trace}\ (df, \mathrm{trace}\ (X, df\cdot)df\cdot)df\cdot \\ & - 2|df|^2\ \mathrm{trace}\ (df\cdot, X)df\cdot + 2(dX, df)\tau_{\square}(f) - |df|^2\Delta^f X + |df|^4 X, \end{aligned} \tag{4.9}$$

for $X, Y \in \Gamma(f^{-1}TS^{n+1}(1))$.

Theorem 4.7. *Let $h : \Omega \to S^n(1/\sqrt{2})$ be a wave map on a compact domain with constant energy and let $i : S^n(1/\sqrt{2}) \to S^{n+1}(1)$ be an inclusion map. Then $f = i \circ h : \Omega \to S^{n+1}(1)$ is an unstable biwave map.*

Proof. We have the following identities from Theorem 3.5:

$$\begin{gathered} |df|^2 = 2e(h), \qquad \mathrm{trace}\ (\zeta, df\cdot)df\cdot = 0, \qquad (d\tau_{\square}(f), df)\zeta = -4(e(h))^2\zeta, \\ |\tau_{\square}(f)|^2 = 4(e(h))^2, \qquad \mathrm{trace}\ (\zeta, d\tau_{\square}(f))df\cdot = 0, \qquad \mathrm{trace}\ (\tau(f), d\zeta\cdot)df = 0, \\ (\tau_{\square}(f), \zeta)\tau_{\square}(f) = 4(e(h))^2\zeta, \qquad \mathrm{trace}\ \left(df, \Delta^f\zeta\right)df\cdot = \left(\Delta^f\zeta\right)^T, \\ (d\zeta, df)\tau_{\square}(f) = -4(e(h))^2\zeta. \end{gathered} \tag{4.10}$$

Then we obtain the following formula from Lemma 4.6 and the previous identities:

$$\left(I_f(\zeta),\zeta\right)=\int_{\Omega}\left(\left|\Delta^f\zeta\right|^2-12e(h)^2-4e(h)\left(\Delta^f\zeta,\zeta\right)\right)dz, \tag{4.11}$$

which is strictly negative, where $\Delta^f\zeta=2e(h)\zeta$. Hence, f is an unstable biwave map. □

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