Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2008, Article ID 805305, 12 pages
doi:10.1155/2008/805305

Research Article

Adjoint and Trivial Cohomologies of Nilpotent Complex Lie Algebras of Dimension ⩽7

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Received 8 May 2008; Accepted 19 August 2008

Recommended by Aloys Krieg

We present tables for adjoint and trivial cohomologies of complex nilpotent Lie algebras of dimension ⩽7. Attention is paid to quadratic Lie algebras, Poincaré duality, and harmonic cocycles.

1. Introduction

There are several classifications of the complex nilpotent Lie algebras (NLA) of dimension ⩽7 (see [1–7], and the comparisons in [3, 5, 8]). In dimension ⩽6, there are also historical notations which go back to Dixmier [9] and Vergne [10]. In dimension 7, out of the several classifications, Carles' classification ([2] with a couple corrections as in [8]) is of particular interest, as it is based on weight systems, with notations taking into account the rank and the labeling of the weight systems, in the spirit of the historical notations. We label the NLAs of dimension ⩽7 according to the historical notations in dimensions ⩽6 and according to Carles' notations in dimension 7. We refer to [8] for commutation relations (they are given in a basis that diagonalizes a maximal torus, i.e., a maximal Abelian subalgebra of the derivation algebra consisting of semisimple elements) and discussion of the classification, as well as for comparison with other classifications. Let us simply recall here that there are 98 (nonequivalent) weight systems for complex 7-dimensional indecomposable NLAs: 1 in rank 0, 24 in rank 1, 45 in rank 2, 24 in rank 3, and 4 in rank 4. The indecomposable NLAs of dimension 7 are almost classified by their weight system with a few exceptions, and one gets 123 nonisomorphic indecomposable NLAs of dimension 7, of which 6 continuous 1-parameter series (each continuous series counts as one algebra). We also recall the isomorphisms for the continuous series: $\mathfrak{g}_{7,0.4(\lambda')} \cong \mathfrak{g}_{7,0.4(\lambda)} \Leftrightarrow \lambda' = \pm\lambda$; $\mathfrak{g}_{7,1.1(i_{\lambda'})} \cong \mathfrak{g}_{7,1.1(i_\lambda)} \Leftrightarrow \lambda' = \lambda$; $\mathfrak{g}_{7,1.2(i_{\lambda'})} \cong \mathfrak{g}_{7,1.2(i_\lambda)} \Leftrightarrow (\lambda' = \lambda \text{ or } \lambda\lambda' = 1)$; $\mathfrak{g}_{7,1.3(i_{\lambda'})} \cong \mathfrak{g}_{7,1.3(i_\lambda)} \Leftrightarrow \lambda' = \lambda$; $\mathfrak{g}_{7,2.1(i_{\lambda'})} \cong \mathfrak{g}_{7,2.1(i_\lambda)} \Leftrightarrow \lambda' = \lambda$. As to $\mathfrak{g}_{7,3.1(i_\lambda)}$, $\mathfrak{g}_{7,3.1(i_{\lambda'})} \cong \mathfrak{g}_{7,3.1(i_\lambda)}$ if and only if $\lambda' = s(\lambda)$ with s any element of

Table 1: Singular values for continuous series of 7-dimensional NLAs.

Algebra	Singular values
$\mathfrak{g}_{7,0.4(\lambda)}$:	No singular value
$\mathfrak{g}_{7,1.1(i_\lambda)}$:	$\lambda = 1$, $\lambda = 3$, $\lambda \in \{-\omega, -\omega^2\}$
$\mathfrak{g}_{7,1.2(i_\lambda)}$ $(\lambda \neq -1)$	$\lambda = 1$, $\lambda \in \{-\omega, -\omega^2\}$
$\mathfrak{g}_{7,1.3(i_\lambda)}$	$\lambda = 1$;
$\mathfrak{g}_{7,2.1(i_\lambda)}$	$\lambda = 0$, $\lambda = \dfrac{1}{2}$, $\lambda = 2$, $\lambda \in \{-\omega, -\omega^2\}$
$\mathfrak{g}_{7,3.1(i_\lambda)}$	$\lambda = 0, 1$, $\lambda = -1, 2, \dfrac{1}{2}$, $\lambda \in \{-\omega, -\omega^2\}$

the group of transformations of $\lambda : G = \{\lambda, 1/\lambda, 1-\lambda, 1/(1-\lambda), 1-1/\lambda, \lambda/(\lambda-1)\}$, which is isomorphic to the symmetric group $\mathcal{S}_3$.

Now, given some 7-dimensional NLA, it is not always easy to match it (up to isomorphism) to an algebra of the list. For that purpose, however, the adjoint cohomology is very effective. Adjoint and trivial cohomologies for all complex 7-dimensional indecomposable NLAs, along with their weight systems under the action of the maximal torus have been computed in [11] (see also [12], and for trivial cohomology [13]). (Beside the Abelian case, there is a couple special instances in which there are formulae valid in any dimension: for standard filiform [11] and Heisenberg Lie algebras [14, 15].)

However, on one hand that work is unpublished, and on the other hand, when identifying a particular NLA, one has to look up quickly some particular cohomology sequence. Hence there is a point in publishing a handy list of cohomology for all NLAs of dimension $\leqslant 7$. In the present paper, we write down such a list. For each NLA $\mathfrak{g}$, we give the sequences $(\dim Z^j(\mathfrak{g},\mathfrak{g}))_{0\leqslant j\leqslant \dim\mathfrak{g}}$ and $(\dim H^j(\mathfrak{g},\mathfrak{g}))_{0\leqslant j\leqslant \dim\mathfrak{g}}$ for, respectively, the spaces of adjoint cocycles and cohomology groups, along with the sequence of Betti numbers, that is, the trivial cohomology $(\dim H^j(\mathfrak{g},\mathbb{C}))_{0\leqslant j\leqslant \dim\mathfrak{g}}$. We also pay attention to quadratic Lie algebras, Poincaré duality and harmonic cocycles.

Nonisomorphic NLAs of dimension $\leqslant 6$ have different adjoint cohomologies, even though their trivial cohomologies may be equal. In dimension 7, up to 14 nonisomorphic NLAs (of which 2 continuous series) may share the same trivial cohomology (e.g., $(1,2,3,4,4,3,2,1)$), hence the trivial cohomology is ineffective in separating nonisomorphic Lie algebras; it does not refine the classification by weight systems of the algebras either. However, the adjoint cohomology does separate all but 13 pairs of NLAs, and refines the classification by weight systems of the algebras, with only 4 exceptions. For any continuous series, the adjoint cohomology is the same for all but some *singular values* of the parameter at which gaps occur. The singular values for the 6 continuous series are listed in Table 1. Throughout the paper, we denote $\omega = \exp(2i\pi/3)$. For continuous series, the term *generic* will refer to the values of the parameter which are not singular.

The nonisomorphic 7-dimensional NLAs having the same adjoint cohomology come in 13 pairs, as shown in Table 2. Adjoint cohomology refines the classification by weight systems of the NLAs, except for the 4 pairs # 2, 3, 4, 8 in Table 2. In each of those 4 pairs, the weight system on the cohomology is identical for the 2 components [11].

2. Cohomology tables

The results appear as in Tables 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13. For continuous series, the places where gaps occur for the singular values are underlined.

Table 2: Pairs of 7-dimensional NLAs having same adjoint cohomology.

#	Pair	Adjoint cohomology
1	$(\mathfrak{g}_{7,0.4(\lambda)}, \mathfrak{g}_{7,0.5})$	$(1,4,9,15,16,12,7,2)$
2	$(\mathfrak{g}_{7,1.1(i_\lambda)}\ (\lambda=-\omega),\ \mathfrak{g}_{7,1.1(i_\lambda)}\ (\lambda=-\omega^2))$	$(1,4,9,14,16,12,6,2)$
3	$(\mathfrak{g}_{7,1.2(i_\lambda)}\ (\text{generic}),\ \mathfrak{g}_{7,1.2(ii)})$	$(1,6,16,27,30,23,12,3)$
4	$(\mathfrak{g}_{7,1.3(i_\lambda)}\ (\text{generic}),\ \mathfrak{g}_{7,1.3(iii)})$	$(1,7,19,30,32,23,11,3)$
5	$(\mathfrak{g}_{7,1.20},\ \mathfrak{g}_{7,1.21})$	$(1,5,12,17,17,15,10,3)$
6	$(\mathfrak{g}_{7,2.1(i_\lambda)}\ (\text{generic}),\ \mathfrak{g}_{7,1.3(ii)})$	$(1,8,20,31,33,24,12,3)$
7	$(\mathfrak{g}_{7,2.1(i_\lambda)}\ (\lambda=0),\ \mathfrak{g}_{7,2.11})$	$(2,9,21,33,34,24,12,3)$
8	$(\mathfrak{g}_{7,2.1(i_\lambda)}\ (\lambda=-\omega),\ \mathfrak{g}_{7,2.1(i_\lambda)}\ (\lambda=-\omega^2))$	$(1,8,20,31,34,25,12,3)$
9	$(\mathfrak{g}_{7,2.26},\ \mathfrak{g}_{7,3.4})$	$(2,8,19,32,36,26,12,3)$
10	$(\mathfrak{g}_{7,2.31},\ \mathfrak{g}_{7,2.41})$	$(1,7,18,29,32,24,12,3)$
11	$(\mathfrak{g}_{7,2.34},\ \mathfrak{g}_{7,2.35})$	$(2,7,18,29,33,25,11,3)$
12	$(\mathfrak{g}_{7,2.36},\ \mathfrak{g}_{6,12}\times\mathbb{C})$	$(2,11,29,47,51,38,18,4)$
13	$(\mathfrak{g}_{6,11}\times\mathbb{C},\ \mathfrak{g}_4\times\mathfrak{g}_3)$	$(2,12,31,48,51,38,18,4)$

Recall that $Z^0(\mathfrak{g},\mathfrak{g}) = H^0(\mathfrak{g},\mathfrak{g}) = \mathfrak{c}$ the center of $\mathfrak{g}$, and $Z^1(\mathfrak{g},\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$, $H^1(\mathfrak{g},\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})/\mathrm{ad}(\mathfrak{g})$. Recall also the following facts about Poincaré duality (PD) ([16, Theorem 6.10]). For any complex N-dimensional Lie algebra $\mathfrak{g}$ and any $\mathfrak{g}$-module V, cohomology and homology are related by the formulae (with V^* the contragredient $\mathfrak{g}$-module, and $0 \leqslant k \leqslant N$)

$$H_k(\mathfrak{g},V) \cong H^k(\mathfrak{g},V^*)^*, \tag{2.1}$$

$$H^k(\mathfrak{g},V) \cong H_{N-k}\left(\mathfrak{g}, V\otimes_{\mathbb{C}}\left(\bigwedge^N \mathfrak{g}\right)^*\right), \tag{2.2}$$

whence

$$H^k(\mathfrak{g},V) \cong H^{N-k}\left(\mathfrak{g}, V^*\otimes_{\mathbb{C}}\bigwedge^N \mathfrak{g}\right)^*. \tag{2.3}$$

Now, as the algebra $\mathfrak{g}$ considered in this paper is nilpotent, hence unimodular (i.e., $\mathrm{Tr}(\mathrm{ad}\,X) = 0$ for all $X \in \mathfrak{g}$), one has $\bigwedge^N \mathfrak{g} \cong \mathbb{C}$ (the trivial module), and (2.2), (2.3) read, respectively,

$$H^k(\mathfrak{g},V) \cong H_{N-k}(\mathfrak{g},V), \tag{2.4}$$

$$H^k(\mathfrak{g},V) \cong H^{N-k}(\mathfrak{g},V^*)^*. \tag{2.5}$$

Hence, $H^N(\mathfrak{g},\mathfrak{g}) \cong H_0(\mathfrak{g},\mathfrak{g}) \cong \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \cong (H^1(\mathfrak{g},\mathbb{C}))^*$, and $H^{N-k}(\mathfrak{g},\mathfrak{g}) \cong H^k(\mathfrak{g},\mathfrak{g}^*)^*$, $\mathfrak{g}^*$ being equipped with the coadjoint representation. We say that PD holds true for the cohomology of the $\mathfrak{g}$-module V if $H^{N-k}(\mathfrak{g},V) \cong H^k(\mathfrak{g},V)$. PD holds true for the trivial cohomology. However, it does not hold true in general for the adjoint cohomology. The NLAs satisfying Poincaré duality for the adjoint cohomology are signalled with a ‡; among them, those quadratic NLAs have a ■. Recall that a Lie algebra $\mathfrak{g}$ is called quadratic if there exists a nondegenerate symmetric bilinear form B on $\mathfrak{g}$ which is invariant, that is,

Table 3: Cohomology table for NLAs of dimension 7: rank 0.

Algebra	Adjoint cocycles	Adjoint cohomology	Betti numbers
$\mathfrak{g}_{7,0.1}$	(1,10,48,114,147,109,44,7)	(1,4,9,15,16,11,6,2)	(1,2,3,4,4,3,2,1)
$\mathfrak{g}_{7,0.2}$	(1,10,49,113,147,109,44,7)	(1,4,10,15,15,11,6,2)	(1,2,3,4,4,3,2,1)
$\mathfrak{g}_{7,0.3}$	(1,11,50,116,149,110,44,7)	(1,5,12,19,20,14,7,2)	(1,2,4,6,6,4,2,1)
$\mathfrak{g}_{7,0.4(\lambda)}$	(1,10,48,114,147,110,44,7)	(1,4,9,15,16,12,7,2)	(1,2,3,4,4,3,2,1)
$\mathfrak{g}_{7,0.5}$	(1,10,48,114,147,110,44,7)	(1,4,9,15,16,12,7,2)	(1,2,3,4,4,3,2,1)
$\mathfrak{g}_{7,0.6}$	(1,10,48,114,148,110,44,7)	(1,4,9,15,17,13,7,2)	(1,2,3,5,5,3,2,1)
$\mathfrak{g}_{7,0.7}$	(1,10,49,115,148,110,44,7)	(1,4,10,17,18,13,7,2)	(1,2,3,4,4,3,2,1)
$\mathfrak{g}_{7,0.8}$	(1,10,50,114,148,112,45,7)	(1,4,11,17,17,15,10,3)	(1,3,4,4,4,4,3,1)

$B([x,y],z) + B(y,[x,z]) = 0$ for all $x,y,z \in \mathfrak{g}$. This amounts to the adjoint and coadjoint representations being equivalent, and hence implies PD for the adjoint cohomology. It is known that quadratic structures B on $\mathfrak{g}$ are in one-to-one correspondence with those elements $I \in \bigwedge^3\mathfrak{g}^*$ whose super-Poisson bracket $\{I,I\}$ vanishes [17], the correspondence being $B \mapsto I_B$, with $I_B(x,y,z) = B([x,y],z)$ for all $x,y,z \in \mathfrak{g}$. Recall that the super-Poisson bracket is, for $\omega \in \bigwedge^k\mathfrak{g}^*$, $\pi \in \bigwedge\mathfrak{g}^*$, $\{\omega,\pi\} = 2(-1)^k\sum_{i,j}B(y_i,y_j)(x_i\rfloor\omega)\wedge(x_j\rfloor\pi)$, with $\rfloor$ the (left) interior product, $(x_j)_{1\leqslant j\leqslant \dim\mathfrak{g}}$ the basis of $\mathfrak{g}$ (in which the commutation relations are given in [8]) and y_j such that $B(y_j,\cdot) = \omega^j$, with $(\omega^j)_{1\leqslant j\leqslant\dim\mathfrak{g}}$ the dual basis to $(x_j)_{1\leqslant j\leqslant\dim\mathfrak{g}}$. There are only 6 quadratic non-Abelian NLAs of dimension $\leqslant 7$ (only one indecomposable in each dimension 5,6,7). Each of them has only one quadratic structure, up to equivalence under the natural action of $\operatorname{Aut}\mathfrak{g}$. Here are Bs and I_Bs in the basis (ω^j).

$$\begin{aligned}
\mathfrak{g}_{7,2.4}: B &= \omega^1\otimes\omega^7+\omega^7\otimes\omega^1+\omega^2\otimes\omega^6+\omega^6\otimes\omega^2-\left(\omega^3\otimes\omega^5+\omega^5\otimes\omega^3\right)+\omega^4\otimes\omega^4;\\
I_B &= \omega^1\wedge\omega^3\wedge\omega^4-\omega^1\wedge\omega^2\wedge\omega^5;\\
\mathfrak{g}_{6,3}\times\mathbb{C}: B &= \omega^1\otimes\omega^6+\omega^6\otimes\omega^1-\left(\omega^2\otimes\omega^5+\omega^5\otimes\omega^2\right)+\omega^3\otimes\omega^4+\omega^4\otimes\omega^3+\omega^7\otimes\omega^7;\\
I_B &= \omega^1\wedge\omega^2\wedge\omega^3;\\
\mathfrak{g}_{5,4}\times\mathbb{C}^2: B &= \omega^1\otimes\omega^5+\omega^5\otimes\omega^1-\left(\omega^2\otimes\omega^4+\omega^4\otimes\omega^2\right)+\omega^3\otimes\omega^3+\omega^6\otimes\omega^6+\omega^7\otimes\omega^7;\\
I_B &= \omega^1\wedge\omega^2\wedge\omega^3;\\
\mathfrak{g}_{6,3}: B &= \omega^1\otimes\omega^6+\omega^6\otimes\omega^1-\left(\omega^2\otimes\omega^5+\omega^5\otimes\omega^2\right)+\omega^3\otimes\omega^4+\omega^4\otimes\omega^3;\\
I_B &= \omega^1\wedge\omega^2\wedge\omega^3;\\
\mathfrak{g}_{5,4}\times\mathbb{C}: B &= \omega^1\otimes\omega^5+\omega^5\otimes\omega^1-\left(\omega^2\otimes\omega^4+\omega^4\otimes\omega^2\right)+\omega^3\otimes\omega^3+\omega^6\otimes\omega^6;\\
I_B &= \omega^1\wedge\omega^2\wedge\omega^3;\\
\mathfrak{g}_{5,4}: B &= \omega^1\otimes\omega^5+\omega^5\otimes\omega^1-\left(\omega^2\otimes\omega^4+\omega^4\otimes\omega^2\right)+\omega^3\otimes\omega^3;\\
I_B &= \omega^1\wedge\omega^2\wedge\omega^3.
\end{aligned} \tag{2.6}$$

3. About the programs

All computations were made by developing programs with the computer algebra system *Reduce*. The adjoint cohomologies have been computed by program 1. Trivial cohomologies were computed twice: by program 2, and by program 3 which computes via harmonic cocycles. Actually, those programs do more than simply compute the dimensions of the

Table 4: Cohomology table for NLAs of dimension 7: rank 1.

Algebra	Adjoint cocycles	Adjoint cohomology	Betti numbers
$\mathfrak{g}_{7,1.01(i)}$	(1,11,51,113,148,112,45,7)	(1,5,13,17,16,15,10,3)	(1,3,4,4,4,4,3,1)
$\mathfrak{g}_{7,1.01(ii)}$	(1,12,53,118,152,113,45,7)	(1,6,16,24,25,20,11,3)	(1,3,5,7,7,5,3,1)
$\mathfrak{g}_{7,1.02}$	(1,11,49,114,148,110,44,7)	(1,5,11,16,17,13,7,2)	(1,2,3,5,5,3,2,1)
$\mathfrak{g}_{7,1.03}$	(1,11,49,115,148,110,44,7)	(1,5,11,17,18,13,7,2)	(1,2,3,4,4,3,2,1)
$\mathfrak{g}_{7,1.1(i_\lambda)}$ generic	(1,10,48,113,147,109,44,7)	(1,4,9,14,15,11,6,2)	(1,2,3,4,4,3,2,1)
$\mathfrak{g}_{7,1.1(i_\lambda)}$ $\lambda = 1$	(1,11,49,113,147,109,44,7)	(1,5, 11, 15,15,11,6,2)	unchanged
$\mathfrak{g}_{7,1.1(i_\lambda)}$ $\lambda = 3$	(1,10,48,114,147,110,44,7)	(1,4,9,15, 16, 12,7,2)	(1,2,4, 5, 5, 4,2,1)
$\mathfrak{g}_{7,1.1(i_\lambda)}$ $\lambda \in \{-\omega, -\omega^2\}$	(1,10,48,113,148,109,44,7)	(1,4,9,14,16, 12,6,2)	(1,2,3,5, 5,3,2,1)
$\mathfrak{g}_{7,1.1(ii)}$	(1,11,48,114,147,109,44,7)	(1,5,10,15,16,11,6,2)	(1,2,3,4,4,3,2,1)
$\mathfrak{g}_{7,1.1(iii)}$	(2,10,48,113,147,109,44,7)	(2,5,9,14,15,11,6,2)	(1,2,3,4,4,3,2,1)
$\mathfrak{g}_{7,1.1(iv)}$	(1,11,48,114,149,110,44,7)	(1,5,10,15,18,14,7,2)	(1,2,3,6,6,3,2,1)
$\mathfrak{g}_{7,1.1(v)}$	(1,10,50,113,148,112,45,7)	(1,4,11,16,16,15,10,3)	(1,3,4,4,4,4,3,1)
$\mathfrak{g}_{7,1.1(vi)}$	(1,11,50,115,149,112,45,7)	(1,5,12,18,19,16,10,3)	(1,3,4,4,4,4,3,1)
$\mathfrak{g}_{7,1.2(i_\lambda)}$ $(\lambda \neq -1)$ generic	(1,12,53,121,154,114,45,7)	(1,6,16,27,30,23,12,3)	(1,3,6,7,7,6,3,1)
$\mathfrak{g}_{7,1.2(i_\lambda)}$ $\lambda \in \{1, -\omega, -\omega^2\}$	(1,12,53,122, 155,114,45,7)	(1,6,16,28, 32, 24,12,3)	(1,3,6,8, 8,6,3,1)
$\mathfrak{g}_{7,1.2(ii)}$	(1,12,53,121,154,114,45,7)	(1,6,16,27,30,23,12,3)	(1,3,6,7,7,6,3,1)
$\mathfrak{g}_{7,1.2(iii)}$	(2,12,53,122,154,114,45,7)	(2,7,16,28,31,23,12,3)	(1,3,6,7,7,6,3,1)
$\mathfrak{g}_{7,1.2(iv)}$	(1,12,53,122,154,114,45,7)	(1,6,16,28,31,23,12,3)	(1,3,6,7,7,6,3,1)
$\mathfrak{g}_{7,1.3(i_\lambda)}$ generic	(1,13,55,122,155,113,45,7)	(1,7,19,30,32,23,11,3)	(1,3,5,7,7,5,3,1)
$\mathfrak{g}_{7,1.3(i_\lambda)}$ $\lambda = 1$	(1,13,55,122,156,113,45,7)	(1,7,19,30,33, 24,11,3)	(1,3,5,8, 8,5,3,1)
$\mathfrak{g}_{7,1.3(ii)}$	(1,14,55,123,155,114,45,7)	(1,8,20,31,33,24,12,3)	(1,3,6,8,8,6,3,1)
$\mathfrak{g}_{7,1.3(iii)}$	(1,13,55,122,155,113,45,7)	(1,7,19,30,32,23,11,3)	(1,3,5,7,7,5,3,1)
$\mathfrak{g}_{7,1.3(iv)}$	(2,13,57,123,157,113,45,7)	(2,8,21,33,35,25,11,3)	(1,3,5,9,9,5,3,1)
$\mathfrak{g}_{7,1.3(v)}$	(1,13,56,123,155,117,46,7)	(1,7,20,32,33,27,16,4)	(1,4,6,7,7,6,4,1)
$\mathfrak{g}_{7,1.4}$	(1,12,50,116,149,110,44,7)	(1,6,13,19,20,14,7,2)	(1,2,4,6,6,4,2,1)
$\mathfrak{g}_{7,1.5}$ ‡	(2,11,49,115,147,109,44,7)	(2,6,11,17,17,11,6,2)	(1,2,3,4,4,3,2,1)
$\mathfrak{g}_{7,1.6}$	(1,12,52,118,149,110,44,7)	(1,6,15,23,22,14,7,2)	(1,2,4,6,6,4,2,1)
$\mathfrak{g}_{7,1.7}$	(2,15,59,128,159,115,45,7)	(2,10,25,40,42,29,13,3)	(1,3,7,11,11,7,3,1)
$\mathfrak{g}_{7,1.8}$	(1,11,53,120,152,113,45,7)	(1,5,15,26,27,20,11,3)	(1,3,5,6,6,5,3,1)
$\mathfrak{g}_{7,1.9}$	(2,14,58,124,156,114,45,7)	(2,9,23,35,35,25,12,3)	(1,3,6,9,9,6,3,1)
$\mathfrak{g}_{7,1.10}$	(1,11,50,116,150,111,44,7)	(1,5,12,19,21,16,8,2)	(1,2,4,7,7,4,2,1)
$\mathfrak{g}_{7,1.11}$	(1,11,52,117,151,113,45,7)	(1,5,14,22,23,19,11,3)	(1,3,5,6,6,5,3,1)
$\mathfrak{g}_{7,1.12}$	(1,12,53,120,153,113,45,7)	(1,6,16,26,28,21,11,3)	(1,3,5,7,7,5,3,1)
$\mathfrak{g}_{7,1.13}$	(2,12,51,117,150,111,44,7)	(2,7,14,21,22,16,8,2)	(1,2,4,7,7,4,2,1)
$\mathfrak{g}_{7,1.14}$	(2,11,49,115,148,110,44,7)	(2,6,11,17,18,13,7,2)	(1,2,3,4,4,3,2,1)
$\mathfrak{g}_{7,1.15}$	(1,13,54,120,153,113,45,7)	(1,7,18,27,28,21,11,3)	(1,3,5,7,7,5,3,1)
$\mathfrak{g}_{7,1.16}$	(2,15,59,128,158,115,45,7)	(2,10,25,40,41,28,13,3)	(1,3,7,10,10,7,3,1)
$\mathfrak{g}_{7,1.17}$	(1,11,48,114,147,110,44,7)	(1,5,10,15,16,12,7,2)	(1,2,3,4,4,3,2,1)
$\mathfrak{g}_{7,1.18}$	(2,13,56,123,155,114,45,7)	(2,8,20,32,33,24,12,3)	(1,3,6,8,8,6,3,1)
$\mathfrak{g}_{7,1.19}$	(2,11,55,120,157,113,45,7)	(2,6,17,28,32,25,11,3)	(1,3,5,9,9,5,3,1)
$\mathfrak{g}_{7,1.20}$	(1,11,50,114,148,112,45,7)	(1,5,12,17,17,15,10,3)	(1,3,4,4,4,4,3,1)
$\mathfrak{g}_{7,1.21}$	(1,11,50,114,148,112,45,7)	(1,5,12,17,17,15,10,3)	(1,3,4,4,4,4,3,1)

Table 5: Cohomology table for NLAs of dimension 7: rank 2. (continues on next page).

Algebra	Adjoint cocycles	Adjoint cohomology	Betti numbers
$\mathfrak{g}_{7,2.1(i_\lambda)}$ generic	(1,14,55,123,155,114,45,7)	(1,8,20,31,33,24,12,3)	(1,3,6,8,8,6,3,1)
$\mathfrak{g}_{7,2.1(i_\lambda)}$ $\lambda = 0$	$(\underline{2},14,\underline{56},\underline{124},155,114,45,7)$	$(\underline{2},\underline{9},\underline{21},\underline{33},\underline{34},24,12,3)$	unchanged
$\mathfrak{g}_{7,2.1(i_\lambda)}$ $\lambda = \frac{1}{2}$	$(1,\underline{15},55,123,155,114,45,7)$	$(1,\underline{9},\underline{21},31,33,24,12,3)$	unchanged
$\mathfrak{g}_{7,2.1(i_\lambda)}$ $\lambda = 2$	$(1,14,55,\underline{125},\underline{157},\underline{115},45,7)$	$(1,8,20,\underline{33},\underline{37},\underline{27},\underline{13},3)$	$(1,3,\underline{7},\underline{9},\underline{9},\underline{7},3,1)$
$\mathfrak{g}_{7,2.1(i_\lambda)}$ $\lambda \in \{-\omega, -\omega^2\}$	$(1,14,55,123,\underline{156},114,45,7)$	$(1,8,20,31,\underline{34},\underline{25},12,3)$	$(1,3,6,\underline{9},\underline{9},6,3,1)$
$\mathfrak{g}_{7,2.1(ii)}$	(1,14,56,123,155,114,45,7)	(1,8,21,32,33,24,12,3)	(1,3,6,8,8,6,3,1)
$\mathfrak{g}_{7,2.1(iii)}$	(1,14,58,125,159,118,46,7)	(1,8,23,36,39,32,17,4)	(1,4,7,8,8,7,4,1)
$\mathfrak{g}_{7,2.1(iv)}$	(1,14,58,124,157,118,46,7)	(1,8,23,35,36,30,17,4)	(1,4,7,8,8,7,4,1)
$\mathfrak{g}_{7,2.1(v)}$	(2,14,57,124,157,114,45,7)	(2,9,22,34,36,26,12,3)	(1,3,6,10,10,6,3,1)
$\mathfrak{g}_{7,2.2}$	(1,15,59,130,159,116,45,7)	(1,9,25,42,44,30,14,3)	(1,3,7,9,9,7,3,1)
$\mathfrak{g}_{7,2.3}$	(1,13,53,119,149,110,44,7)	(1,7,17,25,23,14,7,2)	(1,2,4,6,6,4,2,1)
$\mathfrak{g}_{7,2.4}$ ‡ ■	(2,12,49,115,147,110,44,7)	(2,7,12,17,17,12,7,2)	(1,2,3,4,4,3,2,1)
$\mathfrak{g}_{7,2.5}$	(1,12,49,114,149,110,44,7)	(1,6,12,16,18,14,7,2)	(1,2,3,6,6,3,2,1)
$\mathfrak{g}_{7,2.6}$	(2,12,49,115,148,110,44,7)	(2,7,12,17,18,13,7,2)	(1,2,3,4,4,3,2,1)
$\mathfrak{g}_{7,2.7}$ ‡	(2,13,52,120,150,111,44,7)	(2,8,16,25,25,16,8,2)	(1,2,4,7,7,4,2,1)
$\mathfrak{g}_{7,2.8}$	(2,13,51,119,152,112,44,7)	(2,8,15,23,26,19,9,2)	(1,2,5,9,9,5,2,1)
$\mathfrak{g}_{7,2.9}$	(2,12,50,118,151,112,44,7)	(2,7,13,21,24,18,9,2)	(1,2,5,8,8,5,2,1)
$\mathfrak{g}_{7,2.10}$	(1,12,51,115,149,112,45,7)	(1,6,14,19,19,16,10,3)	(1,3,4,4,4,4,3,1)
$\mathfrak{g}_{7,2.11}$	(2,14,56,124,155,114,45,7)	(2,9,21,33,34,24,12,3)	(1,3,6,8,8,6,3,1)
$\mathfrak{g}_{7,2.12}$	(2,14,57,123,157,113,45,7)	(2,9,22,33,35,25,11,3)	(1,3,5,9,9,5,3,1)
$\mathfrak{g}_{7,2.13}$	(1,12,50,114,148,112,45,7)	(1,6,13,17,17,15,10,3)	(1,3,4,4,4,4,3,1)
$\mathfrak{g}_{7,2.14}$	(1,12,51,113,148,112,45,7)	(1,6,14,17,16,15,10,3)	(1,3,4,4,4,4,3,1)
$\mathfrak{g}_{7,2.15}$	(1,13,54,119,152,113,45,7)	(1,7,18,26,26,20,11,3)	(1,3,5,7,7,5,3,1)
$\mathfrak{g}_{7,2.16}$	(1,14,55,120,153,113,45,7)	(1,8,20,28,28,21,11,3)	(1,3,5,7,7,5,3,1)
$\mathfrak{g}_{7,2.17}$	(2,13,53,122,154,114,45,7)	(2,8,17,28,31,23,12,3)	(1,3,6,7,7,6,3,1)
$\mathfrak{g}_{7,2.18}$	(2,15,58,125,157,115,45,7)	(2,10,24,36,37,27,13,3)	(1,3,6,10,10,6,3,1)
$\mathfrak{g}_{7,2.19}$	(2,15,58,125,156,114,45,7)	(2,10,24,36,36,25,12,3)	(1,3,6,9,9,6,3,1)
$\mathfrak{g}_{7,2.20}$	(2,16,59,128,158,115,45,7)	(2,11,26,40,41,28,13,3)	(1,3,7,10,10,7,3,1)
$\mathfrak{g}_{7,2.21}$	(2,16,60,129,158,115,45,7)	(2,11,27,42,42,28,13,3)	(1,3,7,10,10,7,3,1)
$\mathfrak{g}_{7,2.22}$	(2,14,56,125,156,115,45,7)	(2,9,21,34,36,26,13,3)	(1,3,7,9,9,7,3,1)
$\mathfrak{g}_{7,2.23}$	(1,13,58,123,156,118,46,7)	(1,7,22,34,34,29,17,4)	(1,4,7,7,7,7,4,1)
$\mathfrak{g}_{7,2.24}$	(2,13,54,123,155,114,45,7)	(2,8,18,30,33,24,12,3)	(1,3,6,8,8,6,3,1)
$\mathfrak{g}_{7,2.25}$	(1,14,56,123,155,117,46,7)	(1,8,21,32,33,27,16,4)	(1,4,6,7,7,6,4,1)
$\mathfrak{g}_{7,2.26}$	(2,13,55,124,157,114,45,7)	(2,8,19,32,36,26,12,3)	(1,3,6,10,10,6,3,1)
$\mathfrak{g}_{7,2.27}$	(2,17,65,132,164,119,46,7)	(2,12,33,50,51,38,18,4)	(1,4,8,11,11,8,4,1)
$\mathfrak{g}_{7,2.28}$	(1,16,63,131,163,119,46,7)	(1,10,30,47,49,37,18,4)	(1,4,8,10,10,8,4,1)
$\mathfrak{g}_{7,2.29}$	(2,14,59,126,159,117,46,7)	(2,9,24,38,40,31,16,4)	(1,4,6,9,9,6,4,1)
$\mathfrak{g}_{7,2.30}$	(1,15,56,125,156,117,46,7)	(1,9,22,34,36,28,16,4)	(1,4,6,8,8,6,4,1)
$\mathfrak{g}_{7,2.31}$	(1,13,54,122,155,114,45,7)	(1,7,18,29,32,24,12,3)	(1,3,6,8,8,6,3,1)
$\mathfrak{g}_{7,2.32}$	(1,14,55,122,156,113,45,7)	(1,8,20,30,33,24,11,3)	(1,3,5,8,8,5,3,1)
$\mathfrak{g}_{7,2.33}$	(1,12,53,120,153,113,45,7)	(1,6,16,26,28,21,11,3)	(1,3,5,7,7,5,3,1)
$\mathfrak{g}_{7,2.34}$	(2,12,55,121,157,113,45,7)	(2,7,18,29,33,25,11,3)	(1,3,5,9,9,5,3,1)
$\mathfrak{g}_{7,2.35}$	(2,12,55,121,157,113,45,7)	(2,7,18,29,33,25,11,3)	(1,3,5,9,9,5,3,1)

Table 5: Continued.

Algebra	Adjoint cocycles	Adjoint cohomology	Betti numbers
$\mathfrak{g}_{7,2.36}$	(2,16,62,132,164,119,46,7)	(2,11,29,47,51,38,18,4)	(1,4,8,11,11,8,4,1)
$\mathfrak{g}_{7,2.37}$	(1,13,53,121,154,114,45,7)	(1,7,17,27,30,23,12,3)	(1,3,6,7,7,6,3,1)
$\mathfrak{g}_{7,2.38}$	(2,16,65,131,164,119,46,7)	(2,11,32,49,50,38,18,4)	(1,4,8,11,11,8,4,1)
$\mathfrak{g}_{7,2.39}$	(2,17,63,131,163,115,45,7)	(2,12,31,47,49,33,13,3)	(1,3,7,13,13,7,3,1)
$\mathfrak{g}_{7,2.40}$	(3,16,63,132,159,115,45,7)	(3,12,30,48,46,29,13,3)	(1,3,7,11,11,7,3,1)
$\mathfrak{g}_{7,2.41}$	(1,13,54,122,155,114,45,7)	(1,7,18,29,32,24,12,3)	(1,3,6,8,8,6,3,1)
$\mathfrak{g}_{7,2.42}$	(2,14,58,126,159,114,45,7)	(2,9,23,37,40,28,12,3)	(1,3,6,10,10,6,3,1)
$\mathfrak{g}_{7,2.43}$	(2,16,60,129,159,115,45,7)	(2,11,27,42,43,29,13,3)	(1,3,7,11,11,7,3,1)
$\mathfrak{g}_{7,2.44}$	(2,16,60,130,161,116,45,7)	(2,11,27,43,46,32,14,3)	(1,3,7,11,11,7,3,1)
$\mathfrak{g}_{7,2.45}$	(2,17,65,134,165,119,46,7)	(2,12,33,52,54,39,18,4)	(1,4,8,12,12,8,4,1)
$\mathfrak{g}_{6,12} \times \mathbb{C}$	(2,16,62,132,164,119,46,7)	(2,11,29,47,51,38,18,4)	(1,4,8,11,11,8,4,1)
$\mathfrak{g}_{6,17} \times \mathbb{C}$	(2,14,55,122,154,113,45,7)	(2,9,20,30,31,22,11,3)	(1,3,5,7,7,5,3,1)
$\mathfrak{g}_{6,19} \times \mathbb{C}$	(2,13,53,120,153,113,45,7)	(2,8,17,26,28,21,11,3)	(1,3,5,7,7,5,3,1)
$\mathfrak{g}_{6,20} \times \mathbb{C}$	(2,12,50,115,149,112,45,7)	(2,7,13,18,19,16,10,3)	(1,3,4,4,4,4,3,1)

cohomology: program 1 computes a basis for $H^2(\mathfrak{g},\mathfrak{g})$ and, when the commutation relations of $\mathfrak{g}$ are given in a basis that diagonalizes a maximal torus, characters of the adjoint cohomology under the action of the maximal torus; for trivial cohomology, the programs compute characters and bases of the eigenspaces under the action of the maximal torus.

Program 2 computes all $H^k(\mathfrak{g},\mathbb{C})$ $(0 \leqslant k \leqslant N)$ and their respective bases and characters independently, making no use of PD. Then PD shows up as a result. There is also a variant program 2′ which does the same, yet offers the option to make use of the computed bases of $H^k(\mathfrak{g},\mathbb{C})$ and $H^{N-k}(\mathfrak{g},\mathbb{C})$ $(2k \leqslant N)$ to get the matrix of the bilinear form in PD and modify the basis of $H^{N-k}(\mathfrak{g},\mathbb{C})$ so as to get the dual basis in PD of the basis of $H^k(\mathfrak{g},\mathbb{C})$.

As to program 3, harmonic cohomology comes naturally in the following way, which can be formulated for unimodular $\mathfrak{g}$: suppose we already computed a basis of $H^k(\mathfrak{g},\mathbb{C})$ *only* for $2k \leqslant N$; how to deduce by PD a basis of $H^{N-k}(\mathfrak{g},\mathbb{C})$? Let $\varrho : C^k(\mathfrak{g},\mathbb{C}) = \bigwedge^k \mathfrak{g}^* \to C_{N-k}(\mathfrak{g},\mathbb{C}) = \bigwedge^{N-k}\mathfrak{g}$ $(0 \leqslant k \leqslant N)$ be the isomorphism defined by $\varrho(f) = \Omega\lfloor f$, where $\Omega = x_1 \wedge \cdots \wedge x_N$ and $\lfloor$ denotes the (right) interior product. As $\varrho(df) = (-1)^{k+1}\partial(\varrho(f))$ for all $f \in C^k(\mathfrak{g},\mathbb{C})$ (∂ boundary operator) [18], ϱ defines an isomorphism $H^k(\mathfrak{g},\mathbb{C}) \to H_{N-k}(\mathfrak{g},\mathbb{C})$, which is actually (up to the factor $(-1)^{k(N-k)}$) the one of (2.4) $(V = \mathbb{C})$. Now, what we look for is an explicit identification algorithm $H_{N-k}(\mathfrak{g},\mathbb{C}) \to H^{N-k}(\mathfrak{g},\mathbb{C})$ to be implemented in programs. For any subset I of $\{1,\dots,N\}$, denote $\omega^I = \omega^{i_1} \wedge \cdots \wedge \omega^{i_k}$ for $I = \{i_1,\dots,i_k\}$ $(1 \leqslant i_1 < \cdots < i_k \leqslant N, 1 \leqslant k \leqslant N)$, $\omega^{\varnothing} = 1$, and similarly for x_I. Let $(\cdot|\cdot)$be the Hermitian scalar product on $\bigwedge \mathfrak{g}^* = \bigoplus_{k=0}^N C^k(\mathfrak{g},\mathbb{C})$ obtained by decreeing the basis (ω^I) to be orthonormal. For $1 \leqslant k \leqslant N$, let $z \mapsto g_z$be the conjugate linear bijective map $C_k(\mathfrak{g},\mathbb{C}) \to C^k(\mathfrak{g},\mathbb{C})$ defined by $(f \mid g_z) = f(z)$ for all $f \in C^k(\mathfrak{g},\mathbb{C})$, $z \in C_k(\mathfrak{g},\mathbb{C})$ (we set $g_1 = 1$ for $k = 0$). Then for any subset I of $\{1,\dots,N\}$, $g_{x_I} = \omega^I$ and $g_{\varrho(\omega^I)} = \rho_{I,I'}\omega^{I'}$, where I' is the complementary subset to I, and $\rho_{I,I'} = (-1)^{N_{I,I'}}$, $N_{I,I'} = \operatorname{card}\{(i,j) \in I \times I'; j < i\}$. Let d^* be the adjoint of d on $\bigwedge \mathfrak{g}^*$. Then $d^* g_z = g_{\partial z}$ for all $z \in \bigwedge \mathfrak{g} = \bigoplus_{k=0}^N C_k(\mathfrak{g},\mathbb{C})$. d and d^* are disjoint on $\bigwedge \mathfrak{g}^*$ in the sense of [19], hence $\ker d/\operatorname{im} d = \bigoplus_{k=0}^N H^k(\mathfrak{g},\mathbb{C})$ is isomorphic to $\ker\Delta = \ker d \cap \operatorname{im} d^*$, where $\Delta = dd^* + d^*d$. Then $\{f \in C^k(\mathfrak{g},\mathbb{C});\ \Delta f = 0\}$ is the kth harmonic cocycle space. It is contained in the kth cocycle space $Z^k(\mathfrak{g},\mathbb{C})$. Each equivalence class of $Z^k(\mathfrak{g},\mathbb{C})$ modulo coboundaries contains exactly one harmonic cocycle.

Table 6: Cohomology table for NLAs of dimension 7: rank 3.

Algebra	Adjoint cocycles	Adjoint cohomology	Betti numbers
$\mathfrak{g}_{7,3.1(i_\lambda)}$ generic	(1,15,59,130,159,116,45,7)	(1,9,25,42,44,30,14,3)	(1,3,7,9,9,7,3,1)
$\mathfrak{g}_{7,3.1(i_\lambda)}$ $\lambda = 0, 1$	$(\underline{2},15,59,130,\underline{161},116,45,7)$	$(\underline{2},\underline{10},25,42,\underline{46},\underline{32},14,3)$	$(1,3,7,\underline{11},\underline{11},7,3,1)$
$\mathfrak{g}_{7,3.1(i_\lambda)}$ $\lambda = -1, 2, \frac{1}{2}$	$(1,\underline{17},59,\underline{131},159,\underline{117},45,7)$	$(1,\underline{11},\underline{27},\underline{43},\underline{45},\underline{31},\underline{15},3)$	$(1,3,\underline{8},\underline{10},\underline{10},\underline{8},3,1)$
$\mathfrak{g}_{7,3.1(i_\lambda)}$ $\lambda \in \{-\omega, -\omega^2\}$	$(1,15,59,130,\underline{160},116,45,7)$	$(1,9,25,42,\underline{45},\underline{31},14,3)$	$(1,3,7,\underline{10},\underline{10},7,3,1)$
$\mathfrak{g}_{7,3.1(iii)}$	(1,15,63,130,161,119,46,7)	(1,9,29,46,46,35,18,4)	(1,4,8,9,9,8,4,1)
$\mathfrak{g}_{7,3.2}$	(2,17,62,130,158,115,45,7)	(2,12,30,45,43,28,13,3)	(1,3,7,10,10,7,3,1)
$\mathfrak{g}_{7,3.3}$	(2,15,56,126,156,115,45,7)	(2,10,22,35,37,26,13,3)	(1,3,7,9,9,7,3,1)
$\mathfrak{g}_{7,3.4}$	(2,13,55,124,157,114,45,7)	(2,8,19,32,36,26,12,3)	(1,3,6,10,10,6,3,1)
$\mathfrak{g}_{7,3.5}$	(2,14,56,125,157,114,45,7)	(2,9,21,34,37,26,12,3)	(1,3,6,10,10,6,3,1)
$\mathfrak{g}_{7,3.6}$	(3,18,64,134,163,117,45,7)	(3,14,33,51,52,35,15,3)	(1,3,8,14,14,8,3,1)
$\mathfrak{g}_{7,3.7}$	(2,15,61,131,164,119,46,7)	(2,10,27,45,50,38,18,4)	(1,4,8,11,11,8,4,1)
$\mathfrak{g}_{7,3.8}$	(2,19,68,139,169,120,46,7)	(2,14,38,60,63,44,19,4)	(1,4,9,14,14,9,4,1)
$\mathfrak{g}_{7,3.9}$	(2,18,65,137,168,120,46,7)	(2,13,34,55,60,43,19,4)	(1,4,9,13,13,9,4,1)
$\mathfrak{g}_{7,3.10}$	(1,15,58,124,157,118,46,7)	(1,9,24,35,36,30,17,4)	(1,4,7,8,8,7,4,1)
$\mathfrak{g}_{7,3.11}$	(2,18,65,136,166,119,46,7)	(2,13,34,54,57,40,18,4)	(1,4,8,13,13,8,4,1)
$\mathfrak{g}_{7,3.12}$	(3,19,66,147,172,122,46,7)	(3,15,36,66,74,49,21,4)	(1,4,11,14,14,11,4,1)
$\mathfrak{g}_{7,3.13}$	(2,16,59,127,159,117,46,7)	(2,11,26,39,41,31,16,4)	(1,4,6,9,9,6,4,1)
$\mathfrak{g}_{7,3.14}$	(2,18,65,135,166,120,46,7)	(2,13,34,53,56,41,19,4)	(1,4,9,12,12,9,4,1)
$\mathfrak{g}_{7,3.15}$	(2,17,63,133,164,119,46,7)	(2,12,31,49,52,38,18,4)	(1,4,8,11,11,8,4,1)
$\mathfrak{g}_{7,3.16}$	(1,14,58,123,157,118,46,7)	(1,8,23,34,35,30,17,4)	(1,4,7,8,8,7,4,1)
$\mathfrak{g}_{7,3.17}$	(1,16,56,127,157,117,46,7)	(1,10,23,36,39,29,16,4)	(1,4,6,9,9,6,4,1)
$\mathfrak{g}_{7,3.18}$	(1,19,70,135,170,125,47,7)	(1,13,40,58,60,50,25,5)	(1,5,10,11,11,10,5,1)
$\mathfrak{g}_{7,3.19}$	(2,19,70,148,171,124,47,7)	(2,14,40,71,74,50,24,5)	(1,5,9,15,15,9,5,1)
$\mathfrak{g}_{7,3.20}$	(2,19,66,133,163,115,45,7)	(2,14,36,52,51,33,13,3)	(1,3,7,13,13,7,3,1)
$\mathfrak{g}_{7,3.21}$	(2,15,58,127,159,114,45,7)	(2,10,24,38,41,28,12,3)	(1,3,6,10,10,6,3,1)
$\mathfrak{g}_{7,3.22}$	(2,15,57,124,157,114,45,7)	(2,10,23,34,36,26,12,3)	(1,3,6,10,10,6,3,1)
$\mathfrak{g}_{7,3.23}$	(3,17,63,132,159,116,45,7)	(3,13,31,48,46,30,14,3)	(1,3,7,11,11,7,3,1)
$\mathfrak{g}_{7,3.24}$	(3,22,75,150,175,122,46,7)	(3,18,48,78,80,52,21,4)	(1,4,11,17,17,11,4,1)
$\mathfrak{g}_{6,5} \times \mathbb{C}$	(3,18,65,135,165,119,46,7)	(3,14,34,53,55,39,18,4)	(1,4,8,11,11,8,4,1)
$\mathfrak{g}_{6,7} \times \mathbb{C}$	(3,20,70,142,170,120,46,7)	(3,16,41,65,67,45,19,4)	(1,4,9,14,14,9,4,1)
$\mathfrak{g}_{6,8} \times \mathbb{C}$	(3,19,66,135,165,119,46,7)	(3,15,36,54,55,39,18,4)	(1,4,8,11,11,8,4,1)
$\mathfrak{g}_{6,10} \times \mathbb{C}$	(2,17,64,134,165,119,46,7)	(2,12,32,51,54,39,18,4)	(1,4,8,11,11,8,4,1)
$\mathfrak{g}_{6,11} \times \mathbb{C}$	(2,17,63,132,164,119,46,7)	(2,12,31,48,51,38,18,4)	(1,4,8,11,11,8,4,1)
$\mathfrak{g}_{6,13} \times \mathbb{C}$	(2,15,58,125,159,118,46,7)	(2,10,24,36,39,32,17,4)	(1,4,7,8,8,7,4,1)
$\mathfrak{g}_{6,14} \times \mathbb{C}$	(3,16,58,126,158,115,45,7)	(3,12,25,37,39,28,13,3)	(1,3,6,10,10,6,3,1)
$\mathfrak{g}_{6,15} \times \mathbb{C}$	(2,14,56,124,157,115,45,7)	(2,9,21,33,36,27,13,3)	(1,3,6,10,10,6,3,1)
$\mathfrak{g}_{6,16} \times \mathbb{C}$	(2,15,57,123,154,113,45,7)	(2,10,23,33,32,22,11,3)	(1,3,5,7,7,5,3,1)
$\mathfrak{g}_{6,18} \times \mathbb{C}$	(2,13,51,115,149,112,45,7)	(2,8,15,19,19,16,10,3)	(1,3,4,4,4,4,3,1)
$\mathfrak{g}_{5,6} \times \mathbb{C}^2$	(3,18,63,132,164,119,46,7)	(3,14,32,48,51,38,18,4)	(1,4,8,11,11,8,4,1)

Table 7: Cohomology table for NLAs of dimension 7: rank 4.

Algebra	Adjoint cocycles	Adjoint cohomology	Betti numbers
$\mathfrak{g}_{7,4.1}$	(3,20,71,149,174,122,46,7)	(3,16,42,73,78,51,21,4)	(1,4,11,16,16,11,4,1)
$\mathfrak{g}_{7,4.2}$	(3,25,78,157,178,123,46,7)	(3,21,54,88,90,56,22,4)	(1,4,12,18,18,12,4,1)
$\mathfrak{g}_{7,4.3}$	(2,21,72,150,175,125,47,7)	(2,16,44,75,80,55,25,5)	(1,5,10,16,16,10,5,1)
$\mathfrak{g}_{7,4.4}$	(1,28,91,140,189,134,48,7)	(1,22,70,84,84,78,35,6)	(1,6,14,14,14,14,6,1)
$\mathfrak{g}_{6,1} \times \mathbb{C}$	(3,24,81,157,184,127,47,7)	(3,20,56,91,96,66,27,5)	(1,5,12,18,18,12,5,1)
$\mathfrak{g}_{6,2} \times \mathbb{C}$	(2,20,70,138,171,125,47,7)	(2,15,41,61,64,51,25,5)	(1,5,10,12,12,10,5,1)
$\mathfrak{g}_{6,3} \times \mathbb{C}$ ‡ ■	(4,25,80,153,178,123,46,7)	(4,22,56,86,86,56,22,4)	(1,4,11,20,20,11,4,1)
$\mathfrak{g}_{6,4} \times \mathbb{C}$	(3,19,68,140,169,120,46,7)	(3,15,38,61,64,44,19,4)	(1,4,9,14,14,9,4,1)
$\mathfrak{g}_{6,6} \times \mathbb{C}$	(3,21,72,143,170,120,46,7)	(3,17,44,68,68,45,19,4)	(1,4,9,14,14,9,4,1)
$\mathfrak{g}_{6,9} \times \mathbb{C}$	(2,16,63,132,163,119,46,7)	(2,11,30,48,50,37,18,4)	(1,4,8,11,11,8,4,1)
$\mathfrak{g}_{5,3} \times \mathbb{C}^2$	(3,22,75,149,179,126,47,7)	(3,18,48,77,83,60,26,5)	(1,5,11,15,15,11,5,1)
$\mathfrak{g}_{5,4} \times \mathbb{C}^2$ ‡ ■	(4,22,68,135,166,120,46,7)	(4,19,41,56,56,41,19,4)	(1,4,8,11,11,8,4,1)
$\mathfrak{g}_{5,5} \times \mathbb{C}^2$	(3,19,65,133,164,119,46,7)	(3,15,35,51,52,38,18,4)	(1,4,8,11,11,8,4,1)
$\mathfrak{g}_4 \times \mathfrak{g}_3$	(2,17,63,132,164,119,46,7)	(2,12,31,48,51,38,18,4)	(1,4,8,11,11,8,4,1)

Table 8: Cohomology table for NLAs of dimension 7: rank 5.

Algebra	Adjoint cocycles	Adjoint cohomology	Betti numbers
$\mathfrak{g}_{5,1} \times \mathbb{C}^2$	(3,29,91,166,194,134,48,7)	(3,25,71,110,115,83,35,6)	(1,6,14,19,19,14,6,1)
$\mathfrak{g}_{5,2} \times \mathbb{C}^2$	(4,27,89,168,190,128,47,7)	(4,24,67,110,113,73,28,5)	(1,5,13,21,21,13,5,1)
$\mathfrak{g}_4 \times \mathbb{C}^3$	(4,25,78,150,179,126,47,7)	(4,22,54,81,84,60,26,5)	(1,5,11,15,15,11,5,1)

Table 9: Cohomology table for NLAs of dimension 7: rank 6.

Algebra	Adjoint cocycles	Adjoint cohomology	Betti numbers
$\mathfrak{g}_3 \times \mathbb{C}^4$	(5,34,106,191,209,136,48,7)	(5,32,91,150,155,100,37,6)	(1,6,16,25,25,16,6,1)

Table 10: Cohomology table for NLAs of dimension 7: rank 7 (Abelian).

Algebra	Adjoint cocycles	Adjoint cohomology	Betti numbers
$\mathbb{C}^7$ ‡ ■	(7,49,147,245,245,147,49,7)	(7,49,147,245,245,147,49,7)	(1,7,21,35,35,21,7,1)

Lemma 3.1. *Suppose that $\mathfrak{g}$ is unimodular and let $f \in Z^k(\mathfrak{g},\mathbb{C})$ $(0 \leqslant k \leqslant N)$. Then $g_{\varrho(f)} \in Z^{N-k}(\mathfrak{g},\mathbb{C})$ if and only if f is harmonic; in that case, $g_{\varrho(f)}$ is also harmonic.*

Proof. It is enough to prove $d(g_{\varrho(f)}) = (-1)^k g_{\varrho(d^*f)}$, that is, $d\Phi f = (-1)^k \Phi d^* f$ for all $f \in C^k(\mathfrak{g},\mathbb{C})$, with Φ the Hodge operator on $\bigwedge \mathfrak{g}^*$ defined by $\Phi f = g_{\varrho(f)}$. For I subset of cardinality k of $\{1,\ldots,N\}$, $\Phi^2(\omega^I) = \rho_{I,I'}\,\Phi(\omega^{I'}) = \rho_{I,I'}\,\rho_{I',I}\,\omega^I = (-1)^{k(N-k)}\,\omega^I$, hence $\Phi^2 = \bigoplus_{k=0}^N (-1)^{k(N-k)} \times \mathrm{Id}_{C^k(\mathfrak{g},\mathbb{C})}$. Now, $f \in C^k(\mathfrak{g},\mathbb{C}), d^*\Phi f = d^*(g_{\varrho(f)}) = g_{\partial\varrho(f)} = (-1)^{k+1} g_{\varrho(df)} = (-1)^{k+1}\Phi(df)$ implies successively $d^*\Phi^2 f = (-1)^{N-k+1}\Phi d\Phi f$ and $\Phi d^*\Phi^2 f = (-1)^{N-k+1}\Phi^2 d\Phi f = (-1)^{k(N-k+1)} d\Phi f$, which reads $\Phi d^* f = (-1)^k\, d\Phi f$. □

Then $H^k(\mathfrak{g},\mathbb{C})$ is isomorphic to the space of harmonic cocycles and the map $[f] \mapsto [g_{\varrho(f)}]$ which assigns to the class of the harmonic cocycle f the class of the harmonic cocycle

Table 11: Cohomology table for NLAs of dimension 6.

Algebra	Adjoint cocycles	Adjoint cohomology	Betti numbers
Rank 1			
$\mathfrak{g}_{6,12}$	(1,11,40,71,68,33,6)	(1,6,15,21,19,11,3)	(1,3,5,6,5,3,1)
$\mathfrak{g}_{6,17}$	(1,10,36,67,64,32,6)	(1,5,10,13,11,6,2)	(1,2,3,4,3,2,1)
$\mathfrak{g}_{6,19}$	(1,9,35,66,64,32,6)	(1,4,8,11,10,6,2)	(1,2,3,4,3,2,1)
$\mathfrak{g}_{6,20}$	(1,8,34,64,63,32,6)	(1,3,6,8,7,5,2)	(1,2,2,2,2,2,1)
Rank 2			
$\mathfrak{g}_{6,5}$	(2,12,42,72,68,33,6)	(2,8,18,24,20,11,3)	(1,3,5,6,5,3,1)
$\mathfrak{g}_{6,7}$	(2,14,44,75,69,33,6)	(2,10,22,29,24,12,3)	(1,3,6,8,6,3,1)
$\mathfrak{g}_{6,8}$	(2,13,42,72,68,33,6)	(2,9,19,24,20,11,3)	(1,3,5,6,5,3,1)
$\mathfrak{g}_{6,10}$	(1,12,41,72,68,33,6)	(1,7,17,23,20,11,3)	(1,3,5,6,5,3,1)
$\mathfrak{g}_{6,11}$	(1,12,40,71,68,33,6)	(1,7,16,21,19,11,3)	(1,3,5,6,5,3,1)
$\mathfrak{g}_{6,13}$	(1,10,38,68,67,33,6)	(1,5,12,16,15,10,3)	(1,3,4,4,4,3,1)
$\mathfrak{g}_{6,14}$	(2,11,37,68,66,32,6)	(2,7,12,15,14,8,2)	(1,2,4,6,4,2,1)
$\mathfrak{g}_{6,15}$	(1,10,36,67,66,32,6)	(1,5,10,13,13,8,2)	(1,2,4,6,4,2,1)
$\mathfrak{g}_{6,16}$	(1,11,37,67,64,32,6)	(1,6,12,14,11,6,2)	(1,2,3,4,3,2,1)
$\mathfrak{g}_{6,18}$	(1,9,34,64,63,32,6)	(1,4,7,8,7,5,2)	(1,2,2,2,2,2,1)
$\mathfrak{g}_{5,6} \times \mathbb{C}$	(2,12,40,71,68,33,6)	(2,8,16,21,19,11,3)	(1,3,5,6,5,3,1)
Rank 3			
$\mathfrak{g}_{6,1}$	(2,17,50,82,74,34,6)	(2,13,31,42,36,18,4)	(1,4,8,10,8,4,1)
$\mathfrak{g}_{6,2}$	(1,14,44,73,72,34,6)	(1,9,22,27,25,16,4)	(1,4,6,6,6,4,1)
$\mathfrak{g}_{6,3}$ ‡■	(3,18,48,78,72,33,6)	(3,15,30,36,30,15,3)	(1,3,8,12,8,3,1)
$\mathfrak{g}_{6,4}$	(2,13,43,74,69,33,6)	(2,9,20,27,23,12,3)	(1,3,6,8,6,3,1)
$\mathfrak{g}_{6,6}$	(2,15,45,75,69,33,6)	(2,11,24,30,24,12,3)	(1,3,6,8,6,3,1)
$\mathfrak{g}_{6,9}$	(1,11,41,70,68,33,6)	(1,6,16,21,18,11,3)	(1,3,5,6,5,3,1)
$\mathfrak{g}_{5,3} \times \mathbb{C}$	(2,15,47,79,76,34,6)	(2,11,26,36,32,17,4)	(1,4,7,8,7,4,1)
$\mathfrak{g}_{5,4} \times \mathbb{C}$ ‡■	(3,15,42,72,69,33,6)	(3,12,21,24,21,12,3)	(1,3,5,6,5,3,1)
$\mathfrak{g}_{5,5} \times \mathbb{C}$	(2,13,41,71,68,33,6)	(2,9,18,22,19,11,3)	(1,3,5,6,5,3,1)
Rank 4			
$\mathfrak{g}_{5,1} \times \mathbb{C}$	(2,21,55,86,79,35,6)	(2,17,40,51,45,24,5)	(1,5,9,10,9,5,1)
$\mathfrak{g}_{5,2} \times \mathbb{C}$	(3,19,55,86,75,34,6)	(3,16,38,51,41,19,4)	(1,4,9,12,9,4,1)
$\mathfrak{g}_4 \times \mathbb{C}^2$	(3,17,48,79,73,34,6)	(3,14,29,37,32,17,4)	(1,4,7,8,7,4,1)
$\mathfrak{g}_3 \times \mathfrak{g}_3$	(2,16,50,82,74,34,6)	(2,12,30,42,36,18,4)	(1,4,8,10,8,4,1)
Rank 5			
$\mathfrak{g}_3 \times \mathbb{C}^3$	(4,24,65,97,81,35,6)	(4,22,53,72,58,26,5)	(1,5,11,14,11,5,1)
Rank 6 (abelian)			
$\mathbb{C}^6$ ‡■	(6,36,90,120,90,36,6)	(6,36,90,120,90,36,6)	(1,6,15,20,15,6,1)

$g_{\varrho(f)}$ is a conjugate-linear isomorphism from $H^k(\mathfrak{g},\mathbb{C})$ onto $H^{N-k}(\mathfrak{g},\mathbb{C})$. If $([\varphi_j])$ is a basis for $H^k(\mathfrak{g},\mathbb{C})$ $(2k \leqslant N)$ consisting of harmonic cocycles, then $([g_{\varrho(\varphi_j)}])$ is a basis for $H^{N-k}(\mathfrak{g},\mathbb{C})$ consisting of harmonic cocycles. Hence we see that the price to be paid for computing $H^k(\mathfrak{g},\mathbb{C})$ and their base *only* for $2k \leqslant N$, yet get bases for the whole cohomology, is to go to harmonic cohomology. That was implemented as an option in the variant program $3'$: with that option on, the basis of $H^{N-k}(\mathfrak{g},\mathbb{C})$ is computed as explained, then modified into the dual basis in PD of the basis of $H^k(\mathfrak{g},\mathbb{C})$ $(2k \leqslant N)$. With the option off, no use of PD occurs: harmonic cocycles and bases are computed independently for each k $(0 \leqslant k \leqslant N)$.

Table 12: Cohomology table for NLAs of dimension 5.

Algebra	Adjoint cocycles	Adjoint cohomology	Betti numbers
Rank 1			
$\mathfrak{g}_{5,6}$	(1,8,24,34,22,5)	(1,4,7,8,6,2)	(1,2,3,3,2,1)
Rank 2			
$\mathfrak{g}_{5,3}$	(1,10,28,37,23,5)	(1,6,13,15,10,3)	(1,3,4,4,3,1)
$\mathfrak{g}_{5,4}$ ‡ ■	(2,10,24,35,22,5)	(2,7,9,9,7,2)	(1,2,3,3,2,1)
$\mathfrak{g}_{5,5}$	(1,9,24,34,22,5)	(1,5,8,8,6,2)	(1,2,3,3,2,1)
Rank 3			
$\mathfrak{g}_{5,1}$	(1,15,30,41,24,5)	(1,11,20,21,15,4)	(1,4,5,5,4,1)
$\mathfrak{g}_{5,2}$	(2,13,31,39,23,5)	(2,10,19,20,12,3)	(1,3,6,6,3,1)
$\mathfrak{g}_4 \times \mathbb{C}$	(2,11,28,37,23,5)	(2,8,14,15,10,3)	(1,3,4,4,3,1)
Rank 4			
$\mathfrak{g}_3 \times \mathbb{C}^2$	(3,16,37,43,24,5)	(3,14,28,30,17,4)	(1,4,7,7,4,1)
Rank 5			
$\mathbb{C}^5$ ‡ ■	(5,25,50,50,25,5)	(5,25,50,50,25,5)	(1,5,10,10,5,1)

Table 13: Cohomology table for NLAs of dimension $\leqslant 4$.

Algebra	Adjoint cocycles	Adjoint cohomology	Betti numbers
$\mathfrak{g}_4$ (dim 4, rk 2)	(1,7,15,14,4)	(1,4,6,5,2)	(1,2,2,2,1)
$\mathfrak{g}_3 \times \mathbb{C}$ (dim 4, rk 3)	(2,10,19,15,4)	(2,8,13,10,3)	(1,3,4,3,1)
$\mathbb{C}^4$ (abel.) ‡ ■ (dim 4, rk 4)	(4,16,24,16,4)	(4,16,24,16,4)	(1,4,6,4,1)
$\mathfrak{g}_3$ (dim 3, rk 2)	(1,6,8,3)	(1,4,5,2)	(1,2,2,1)
$\mathbb{C}^3$ (abel.) ‡ ■ (dim 3, rk 3)	(3,9,9,3)	(3,9,9,3)	(1,3,3,1)
$\mathbb{C}^2$ (abel.) ‡ ■ (dim 2, rk 2)	(2,4,2)	(2,4,2)	(1,2,1)
$\mathbb{C}$ (abel.) ‡ ■ (dim 1, rk 1)	(1,1)	(1,1)	(1,1)

All programs handle dimensions up to 7 and (if necessary) one continuous parameter L. Though they are meant for nilpotent Lie algebras whose commutation relations are given in a basis that diagonalizes a maximal torus, they can be directly applied to any Lie algebra of dimension $\leqslant$7 as well, not necessarily nilpotent nor unimodular (except for the variants), giving explicit calculation of cohomology: in that case, all material involving weights has simply to be skipped. As to trivial cohomology, note that program 2 which computes $H^k(\mathfrak{g}, \mathbb{C})$ up to $k = 7$ makes it possible, thanks to PD, to write down the dimensions of trivial cohomology for unimodular Lie algebras of dimensions up to 15. In the same way, program 3 can handle higher dimensions. However, restrictions may come on one hand from the amount of dynamic storage space available, and on the other from the running time, which increases steeply as the dimension N of the Lie algebra gets higher, typically for the computation of the dimensions of trivial cohomology, with 4 GB RAM: 1 second for $N = 7$, 10 seconds for $N = 9$, 3 minutes for $N = 11$, 3 hours for $N = 13$, up to 10 days for $N = 15$.

The programs are downloadable in the companion archive [20] (programs 1,2,3 are, resp., `geneLplus2007.red`, `ncl2007.red`, `nclhar2007.red`, and the variants program2′, program 3′ are, resp., `ncld2007.red`, `ncldhar2007.red` in [20]) hence we will not enter technicalities about procedures here. Let us simply mention the following concerning the continuous parameter. In the presence of the continuous parameter L, cocycles equations depend on certain unknowns and on L. They are linear with respect to the unknowns. The programs define an algorithm which solves the equations over the rational function field generated by the parameter, while keeping track of the divisions that have been done. If the parameter is a zero of one of the polynomials by which a division occurred, it may very well not be a singular value: one has to compute again the cohomology for all such values.

References

[1] J. M. Ancochea Bermudez and M. Goze, "Classification des algèbres de Lie nilpotentes complexes de dimension 7," *Archiv der Mathematik*, vol. 52, no. 2, pp. 175–185, 1989.

[2] R. Carles, "Weight systems for complex nilpotent Lie algebras and application to the varieties of Lie algebras," Tech. Rep. 96, Département de Mathématiques, Université de Poitiers, Poitiers, France, 1996.

[3] M.-P. Gong, *Classification of nilpotent Lie algebras of dimension 7 (over algebraically closed fields and* $\mathbb{R}$*)*, Ph.D. thesis, University of Waterloo, Waterloo, Canada, 1998.

[4] M. Goze and Y. Khakimdjanov, *Nilpotent Lie Algebras*, vol. 361 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[5] M. Goze and E. Remm, "Classification of complex nilpotent Lie algebras," http://www.math.uha.fr/~algebre/goze/LIE/Lie.html.

[6] M. Romdhani, "Classification of real and complex nilpotent Lie algebras of dimension 7," *Linear and Multilinear Algebra*, vol. 24, no. 3, pp. 167–189, 1989.

[7] C. Seeley, "7-dimensional nilpotent Lie algebras," *Transactions of the American Mathematical Society*, vol. 335, no. 2, pp. 479–496, 1993.

[8] L. Magnin, "Determination of 7-dimensional indecomposable nilpotent complex Lie algebras by adjoining a derivation to 6-dimensional Lie algebras," to appear in *Algebras and Representation Theory*, http://www.u-bourgogne.fr/monge/l.magnin or http://math.u-bourgogne.fr/IMB/magnin/public_html/index.html.

[9] J. Dixmier, "Sur les représentations unitaires des groupes de Lie nilpotents. III," *Canadian Journal of Mathematics*, vol. 10, pp. 321–348, 1958.

[10] M. Vergne, *Variété des algèbres de Lie nilpotentes*, Thèse de 3ème-cycle, Université Paris, Paris, France, 1966.

[11] L. Magnin, *Adjoint and Trivial Cohomology Tables for Indecomposable Nilpotent Lie Algebras of Dimension* ≤ 7 *over* $\mathbb{C}$, 2nd edition, 2007, http://www.u-bourgogne.fr/monge/l.magnin or http://math.u-bourgogne.fr/IMB/magnin/public_html/public-html/index.html.

[12] L. Magnin, "Remarks on weight systems on cohomology of nilpotent Lie algebras," *Algebras, Groups and Geometries*, vol. 9, no. 2, pp. 111–135, 1992.

[13] G. Cairns, B. Jessup, and J. Pitkethly, "On the Betti numbers of nilpotent Lie algebras of small dimension," in *Integrable Systems and Foliations: Feuilletages Et Systèmes Intégrables (Montpellier, 1995)*, C. Albert, R. Brouzet, and J. P. Dufons, Eds., vol. 145 of *Progress in Mathematics*, pp. 19–31, Birkhäuser, Boston, Mass, USA, 1997.

[14] L. Magnin, "Cohomologie adjointe des algèbres de Lie de Heisenberg," *Communications in Algebra*, vol. 21, no. 6, pp. 2101–2129, 1993.

[15] L. J. Santharoubane, "Cohomology of Heisenberg Lie algebras," *Proceedings of the American Mathematical Society*, vol. 87, no. 1, pp. 23–28, 1983.

[16] A. W. Knapp, *Lie Groups, Lie Algebras, and Cohomology*, vol. 34 of *Mathematical Notes*, Princeton University Press, Princeton, NJ, USA, 1988.

[17] G. Pinczon and R. Ushirobira, "New applications of graded Lie algebras to Lie algebras, generalized Lie algebras, and cohomology," *Journal of Lie Theory*, vol. 17, no. 3, pp. 633–667, 2007.

[18] J.-L. Koszul, "Homologie et cohomologie des algèbres de Lie," *Bulletin de la Société Mathématique de France*, vol. 78, pp. 65–127, 1950.

[19] B. Kostant, "Lie algebra cohomology and the generalized Borel-Weil theorem," *The Annals of Mathematics*, vol. 74, no. 2, pp. 329–387, 1961.

[20] http://www.u-bourgogne.fr/monge/l.magnin/CohomArch/CAindex.html or http://math.u-bourgogne.fr/IMB/magnin/public_html/CohomArch/CAindex.html.