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Research Article

Generalized Moisil-Théodoresco Systems and Cauchy Integral Decompositions

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Let $\mathbb{R}_{0,m+1}^{(s)}$ be the space of s-vectors $(0 \leq s \leq m+1)$ in the Clifford algebra $\mathbb{R}_{0,m+1}$ constructed over the quadratic vector space $\mathbb{R}^{0,m+1}$, let $r, p, q \in \mathbb{N}$ with $0 \leq r \leq m+1$, $0 \leq p \leq q$, and $r + 2q \leq m+1$, and let $\mathbb{R}_{0,m+1}^{(r,p,q)} = \sum_{j=p}^{q} \bigoplus \mathbb{R}_{0,m+1}^{(r+2j)}$. Then, an $\mathbb{R}_{0,m+1}^{(r,p,q)}$-valued smooth function W defined in an open subset $\Omega \subset \mathbb{R}^{m+1}$ is said to satisfy the generalized Moisil-Théodoresco system of type (r, p, q) if $\partial_x W = 0$ in Ω, where ∂_x is the Dirac operator in $\mathbb{R}^{m+1}$. A structure theorem is proved for such functions, based on the construction of conjugate harmonic pairs. Furthermore, if Ω is bounded with boundary Γ, where Γ is an Ahlfors-David regular surface, and if W is a $\mathbb{R}_{0,m+1}^{(r,p,q)}$-valued Hölder continuous function on Γ, then necessary and sufficient conditions are given under which W admits on Γ a Cauchy integral decomposition $W = W_+ + W_-$.

1. Introduction

Clifford analysis, a function theory for the Dirac operator in Euclidean space $\mathbb{R}^{m+1}$ $(m \geq 2)$, generalizes in an elegant way the theory of holomorphic functions in the complex plane to a higher dimension and provides at the same time a refinement of the theory of harmonic functions. One of the basic properties relied upon in building up this function theory is the fact that the Dirac operator ∂_x in $\mathbb{R}^{m+1}$ factorizes the Laplacian Δ_x through the relation $\partial_x^2 = -\Delta_x$. The Dirac operator ∂_x is defined by $\partial_x = \sum_{i=0}^{m} e_i \partial_{x_i}$, where $x = (x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$ and $e = (e_i : i = 0, \ldots, m)$ is an orthogonal basis for the quadratic space $\mathbb{R}^{0,m+1}$, the latter being the space $\mathbb{R}^{m+1}$ equipped with a quadratic form of signature $(0, m+1)$. By virtue of the basic multiplication rules

$$e_i^2 = -1, \quad i = 0,1,\ldots,m,$$
$$e_i e_j + e_j e_i = 0, \quad i \neq j;\ i,j = 0,1,\ldots,m, \tag{1.1}$$

valid in the universal Clifford algebra $\mathbb{R}_{0,m+1}$ constructed over $\mathbb{R}^{0,m+1}$, the factorization $\partial_x^2 = -\Delta_x$ is thus obtained.

Notice that $\mathbb{R}_{0,m+1}$ is a real linear associative algebra of dimension 2^{m+1}, having as standard basis the set $(e_A : |A| = s,\ s = 0,1,\ldots,m+1)$, where $A = \{i_1,\ldots,i_s\}$, $0 \le i_1 < i_2 < \cdots < i_s \le m$, $e_A = e_{i_1} e_{i_2} \cdots e_{i_s}$, and $e_\phi = 1$, the identity element in $\mathbb{R}_{0,m+1}$.

Now let $\Omega \subset \mathbb{R}^{m+1}$ be open and let $F : \Omega \mapsto \mathbb{R}_{0,m+1}$ be a C_1-function in Ω. Then, F is said to be left monogenic in Ω if $\partial_x F = 0$ in Ω. The equation $\partial_x F = 0$ gives rise to a first-order linear elliptic system of partial differential equations in the components f_A of $F = \sum_A f_A e_A$. By choosing $\underline{e} = (e_1,\ldots,e_m)$ as an orthogonal basis for the quadratic space $\mathbb{R}^{0,m}$, then inside $\mathbb{R}_{0,m+1}$, $\mathbb{R}^{0,m}$ thus generates the Clifford algebra $\mathbb{R}_{0,m}$. It is then easily seen that

$$\mathbb{R}_{0,m+1} = \mathbb{R}_{0,m} \oplus \overline{e}_0 \mathbb{R}_{0,m}, \tag{1.2}$$

where $\overline{e}_0 = -e_0$. If the $\mathbb{R}_{0,m+1}$-valued C_1-function F in Ω is decomposed following (1.2), that is $F = U + \overline{e}_0 V$ where U and V are $\mathbb{R}_{0,m}$-valued C_1-functions in Ω, then in Ω

$$\partial_x F = 0 \Longleftrightarrow D_x F = 0 \Longleftrightarrow \begin{cases} \partial_{x_0} U + \partial_{\underline{x}} V = 0, \\ \partial_{\underline{x}} U + \partial_{x_0} V = 0, \end{cases} \tag{1.3}$$

where $D_x = \overline{e}_0 \partial_x = \partial_{x_0} + \overline{e}_0 \partial_{\underline{x}}$ is the Cauchy-Riemann operator in $\mathbb{R}^{m+1}$ and $\partial_{\underline{x}} = \sum_{j=1}^m e_j \partial_{x_j}$ is the Dirac operator in $\mathbb{R}^m$.

Obviously the system (1.3) generalizes the classical Cauchy-Riemann system in the plane: it indeed suffices in the case $m = 1$ to take U $\mathbb{R}$-valued and V $\mathbb{R}e_1$-valued.

Left monogenic functions in Ω are real analytic, whence by virtue of $\partial_x^2 = -\Delta_x$, they are in particular $\mathbb{R}_{0,m+1}$-valued and harmonic in Ω.

As the algebra $\mathbb{R}_{0,m+1}$ is noncommutative, one could as well consider right monogenic functions F in Ω, that is F satisfies the equation $F\partial_x = 0$ in Ω. If both $\partial_x F = 0$ and $F\partial_x = 0$ in Ω, then F is said to be two-sided monogenic in Ω.

Notice also that through a natural linear isomorphism $\Theta : \mathbb{R}_{0,m+1} \mapsto \Lambda\mathbb{R}^{m+1}$ (see Section 2), the spaces $\mathcal{E}(\Omega; \mathbb{R}_{0,m+1})$ and $\mathcal{E}(\Omega; \Lambda\mathbb{R}^{m+1})$ of smooth $\mathbb{R}_{0,m+1}$-valued functions and smooth differential forms in Ω may be identified. The left and right actions of ∂_x on $\mathcal{E}(\Omega; \mathbb{R}_{0,m+1})$ then correspond to the actions of $d + d^*$ and $d - d^*$ on $\mathcal{E}(\Omega; \Lambda\mathbb{R}^{m+1})$, where d and d^* denote, respectively, the exterior derivative and the coderivative operators. For the sake of completeness, let us recall the definition of d and d^* on the space $\mathcal{E}(\Omega; \Lambda^s\mathbb{R}^{m+1})$ of smooth s-forms in Ω, $0 \le s \le m+1$ (see [1]).

For $\omega^s \in \mathcal{E}(\Omega; \Lambda^s\mathbb{R}^{m+1})$ with $\omega^s = \sum_{|A|=s} \omega_A^s dx^A$, where $dx^A = dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_s}$, $0 \le i_1 < i_2 < \cdots < i_s \le m$, $d\omega^s$ and $d^*\omega^s$ are defined by

$$d\omega^s = \sum_A \sum_{i=0}^m \partial_{x_i} \omega_A^s dx^i \wedge dx^A,$$
$$d^*\omega^s = \sum_A \sum_{j=1}^s (-1)^j \partial_{x_{i_j}} \omega_A^s dx^{A \setminus \{i_j\}}. \tag{1.4}$$

A smooth differential form ω satisfying $(d - d^*)\omega = 0$ in Ω was called in [2] a self-conjugate differential form.

It thus becomes clear that through the identifications mentioned (see again Section 2) a subsystem of (1.3) corresponds to a subsystem of self-conjugate differential forms and vice versa. For instance, for $0 < s < m+1$ fixed, the study of left monogenic s-vector valued functions W^s thus corresponds to the study of s-forms ω^s satisfying the Hodge-de Rham system $d\omega^s = 0$ and $d^*\omega^s = 0$.

Let us recall that the space $\mathbb{R}_{0,m+1}^{(s)}$ of s-vectors in $\mathbb{R}_{0,m+1}$ $(0 \le s \le m+1)$ is defined by

$$\mathbb{R}_{0,m+1}^{(s)} = \operatorname{span}_{\mathbb{R}}\left(e_A : |A| = s\right). \tag{1.5}$$

For an account on recent investigations on subsystems of (1.3) or, equivalently, on the study of particular systems of self-conjugate differential forms, we refer to [2–10].

Now fix $0 \le r \le m+1$, take $p, q \in \mathbb{N}$ such that $0 \le p \le q$ and $r + 2q \le m+1$, and put

$$\mathbb{R}_{0,m+1}^{(r,p,q)} = \sum_{j=p}^{q} \bigoplus \mathbb{R}_{0,m+1}^{(r+2j)}. \tag{1.6}$$

The present paper is devoted to the study of $\mathbb{R}_{0,m+1}^{(r,p,q)}$-valued smooth functions W in Ω which are left monogenic in Ω (i.e., which satisfy $\partial_x W = 0$ in Ω). The space of such functions is henceforth denoted by $\mathrm{MT}(\Omega, \mathbb{R}_{0,m+1}^{(r,p,q)})$. The system $\partial_x W = 0$ defines a subsystem of (1.3), called the generalized Moisil-Théodoresco system of type (r, p, q) in $\mathbb{R}^{m+1}$.

To be more precise, let us first recall the definition of the differential operators ∂_x^+ and ∂_x^- acting on smooth $\mathbb{R}_{0,m+1}^{(s)}$-valued functions W^s in Ω. Call $\mathcal{E}(\Omega; \mathbb{R}_{0,m+1}^{(s)})$ the space of smooth $\mathbb{R}_{0,m+1}^{(s)}$-valued functions in Ω and put for $W^s \in \mathcal{E}(\Omega; \mathbb{R}_{0,m+1}^{(s)})$,

$$\begin{aligned} \partial_x^+ W^s &= \frac{1}{2}\left(\partial_x W^s + (-1)^s W^s \partial_x\right), \\ \partial_x^- W^s &= \frac{1}{2}\left(\partial_x W^s - (-1)^s W^s \partial_x\right). \end{aligned} \tag{1.7}$$

Note that $\partial_x^+ W^s$ is $\mathbb{R}_{0,m+1}^{(s+1)}$-valued while $\partial_x^- W^s$ is $\mathbb{R}_{0,m+1}^{(s-1)}$-valued and that through the isomorphism Θ, the action of ∂_x^+ and ∂_x^- on $\mathcal{E}(\Omega; \mathbb{R}_{0,m+1}^{(s)})$ corresponds to, respectively, the action of d and d^* on the space $\mathcal{E}(\Omega; \Lambda^s \mathbb{R}^{m+1})$.

If $W \in \mathcal{E}(\Omega; \mathbb{R}_{0,m+1}^{(r,p,q)})$ is written as

$$W = \sum_{j=p}^{q} W^{r+2j}, \quad \text{with } W^{r+2j} \in \mathcal{E}\left(\Omega; \mathbb{R}_{0,m+1}^{(r+2j)}\right),\ j = p, \ldots, q, \tag{1.8}$$

we then have that the generalized Moisil-Théodoresco system of type (r, p, q) reads as follows (see also Section 2):

$$\partial_x W = 0 \Longleftrightarrow \begin{cases} \partial_x^- W^{r+2p} = 0, \\ \partial_x^+ W^{r+2j} + \partial_x^- W^{r+2(j+1)} = 0, \quad j = p, \ldots, q-1, \\ \partial_x^+ W^{r+2q} = 0. \end{cases} \tag{1.9}$$

Note that for $p = q = 0$ and $0 < r < m+1$ fixed, the system (1.9) reduces to the generalized Riesz system $\partial_x W^r = 0$. Its solutions are called harmonic multivector fields (see also [11]). We have

$$\partial_x W^r = 0 \Longleftrightarrow \begin{cases} \partial_x^- W^r = 0, \\ \partial_x^+ W^r = 0. \end{cases} \tag{1.10}$$

Furthermore, for $p = 0$, $q = 1$, and $0 \le r \le m+1$ fixed, the system (1.9) reduces to the Moisil-Théodoreco system in $\mathbb{R}^{m+1}$ (see, e.g., [3]):

$$\partial_x^- W^r = 0, \qquad \partial_x^+ W^r + \partial_x^- W^{r+2} = 0, \qquad \partial_x^+ W^{r+2} = 0. \tag{1.11}$$

In the particular case, where $m+1 = 3$, $p = 0, q = 1$, and $r = 0$, the original Moisil-Théodoresco system introduced in [12] is reobtained (see also [4]).

In this paper, two problems are dealt with; we list them as follows.

(i) To characterize the structure of solutions to the system (1.9).

It is proved in Section 4 (see Theorem 3.2) that, under certain geometric conditions upon Ω, each $W \in \mathrm{MT}(\Omega, \mathbb{R}_{0,m+1}^{(r,p,q)})$ corresponds to a harmonic potential L belonging to a particular subspace of the space $\mathscr{H}(\Omega, \mathbb{R}_{0,m}^{(r,p,q)})$ of harmonic $\mathbb{R}_{0,m}^{(r,p,q)}$-valued functions in Ω.

The proof of Theorem 3.2 relies heavily on the construction of conjugate harmonic pairs elaborated in Section 3.

(ii) To characterize those $W \in C^{0,\alpha}(\Gamma; \mathbb{R}_{0,m+1}^{(r,p,q)})$ which admit a Cauchy-type integral decomposition on Γ of the form

$$W = W_+ + W_-, \tag{1.12}$$

where Γ is the boundary of a bounded open domain $\Omega = \Omega_+$ in $\mathbb{R}^{m+1}$ and $C^{0,\alpha}(\Gamma; \mathbb{R}_{0,m+1}^{(r,p,q)})$ denotes the space of $\mathbb{R}_{0,m+1}^{(r,p,q)}$-valued Hölder continuous functions of order α on Γ, $0 < \alpha < 1$. Putting $\Omega_- = \mathbb{R}^{m+1} \setminus (\Omega \cup \Gamma)$, the elements W_+ and W_- should also belong to $C^{0,\alpha}(\Gamma; \mathbb{R}_{0,m+1}^{(r,p,q)})$ and as such should be the boundary values of solutions W_+ and W_- of (1.9) in Ω_+ and Ω_-, respectively.

In Section 5, this problem is solved in terms of the Cauchy transform $\mathcal{C}_\Gamma$ on Γ, Γ being an m-dimensional Ahlfors-David regular surface (see Theorem 4.2).

In order to make the paper self-contained, we include in Section 2 some basic properties of Clifford algebras and Clifford analysis. For a general account of this function theory, we refer, for example, to the monographs [13–15].

2. Clifford analysis: notations and some basic properties

Let again $e = (e_0, e_1, \ldots, e_m)$ be an orthogonal basis for $\mathbb{R}^{0,m+1}$ and let $\mathbb{R}_{0,m+1}$ be the universal Clifford algebra over $\mathbb{R}^{0,m+1}$. As has already been mentioned in Section 1, $\mathbb{R}_{0,m+1}$ is a real linear associative but noncommutative algebra of dimension 2^{m+1}; its standard basis is given by the set $(e_A : |A| = s, 0 \leq s \leq m+1)$ and the basic multiplication rules are governed by (1.1). For $0 \leq s \leq m+1$ fixed, the space $\mathbb{R}^{(s)}_{0,m+1}$ of s-vectors is defined by (1.5), leading to the decomposition

$$\mathbb{R}_{0,m+1} = \sum_{s=0}^{m+1} \bigoplus \mathbb{R}^{(s)}_{0,m+1} \tag{2.1}$$

and the associated projection operators $[]_s : \mathbb{R}_{0,m+1} \mapsto \mathbb{R}^{(s)}_{0,m+1}$.

Note in particular that for $s = 0$, $\mathbb{R}^{(0)}_{0,m+1} \cong \mathbb{R}$ and that for $s = 1$, $\mathbb{R}^{(1)}_{0,m+1} \cong \mathbb{R}^{0,m+1}$.

An element $x = (x_0, x_1, \ldots, x_m) = (x_0, \underline{x}) \in \mathbb{R}^{m+1}$ is therefore usually identified with $x = \sum_{i=0}^{m} e_i x_i \in \mathbb{R}^{0,m+1}$.

For $x, y \in \mathbb{R}^{(1)}_{0,m+1}$, the product xy splits in two parts, namely,

$$xy = x\bullet y + x \wedge y, \tag{2.2}$$

where $x\bullet y = [xy]_0$ is the scalar part of xy and $x \wedge y = [xy]_2$ is the 2-vector or bivector part of xy. They are given by

$$\begin{aligned} x\bullet y &= -\sum_{i=0}^{m} x_i y_i, \\ x \wedge y &= \sum_{i<j} e_i e_j (x_i y_j - x_j y_i). \end{aligned} \tag{2.3}$$

More generally, for $x \in \mathbb{R}^{(1)}_{0,m+1}$ and $v \in \mathbb{R}^{(s)}_{0,m+1}$ $(0 < s < m+1)$, we have that the product xv decomposes into

$$xv = x\bullet v + x \wedge v, \tag{2.4}$$

where

$$\begin{aligned} x\bullet v &= [xv]_{s-1} = \frac{1}{2}\left(xv - (-1)^s vx\right), \\ x \wedge v &= [xv]_{s+1} = \frac{1}{2}\left(xv + (-1)^s vx\right). \end{aligned} \tag{2.5}$$

Another useful decomposition of $\mathbb{R}_{0,m+1}$ may be obtained by splitting it "along the e_0-direction," as indicated in (1.2). This in fact means that we split $\mathbb{R}^{m+1}$ following $\mathbb{R}^{m+1} = \mathbb{R} \times \mathbb{R}^m$ and that within $\mathbb{R}_{0,m+1}$, the Clifford algebra $\mathbb{R}_{0,m}$ is generated by the orthogonal basis $\underline{e} = (e_1, \ldots, e_m)$ of $\mathbb{R}^{0,m}$. $\mathbb{R}^{0,m}$ denotes the space $\mathbb{R}^m$ to which the original quadratic form of signature $(0, m+1)$ on $\mathbb{R}^{0,m+1}$ has been restricted.

Following the decomposition (1.2), the element $x = (x_0, \underline{x}) \in \mathbb{R}^{m+1}$ is then often identified with the so-called paravector $x = x_0 + \overline{e}_0 \underline{x} = x_0 + \overline{e}_0 \sum_{j=1}^{m} x_j e_j \in \mathbb{R} \bigoplus \overline{e}_0 \mathbb{R}^{0,m}$.

Let us also recall that if $\Omega \subset \mathbb{R}^{m+1}$ is open and F is an $\mathbb{R}_{0,m+1}$-valued C_1-function in Ω, then F is said to be left monogenic in Ω if $\partial_x F = 0$ in Ω, $\partial_x = \sum_{i=0}^{m} e_i \partial_{x_i}$ being the Dirac operator in $\mathbb{R}^{m+1}$.

As already mentioned in (1.3), by putting $D_x = \overline{e}_0\partial_x = \partial_{x_0} + \overline{e}_0\partial_{\underline{x}}$, $\partial_{\underline{x}}$ being the Dirac operator in $\mathbb{R}^m$, we have for $F = U + \overline{e}_0 V$,

$$\partial_x F = 0 \Longleftrightarrow D_x F = 0 \Longleftrightarrow \begin{cases} \partial_{x_0} U + \partial_{\underline{x}} V = 0, \\ \partial_{\underline{x}} U + \partial_{x_0} V = 0. \end{cases} \tag{2.6}$$

Let us recall that a pair (U, V) of $\mathbb{R}_{0,m}$-valued harmonic functions in Ω is said to be conjugate harmonic if $F = U + \overline{e}_0 V$ is left monogenic in Ω (see [16]).

Notice also that, when defining the conjugate $\overline{D}_x$ of D_x by $\overline{D}_x = \partial_{x_0} - \overline{e}_0\partial_{\underline{x}}$, we have that $D_x\overline{D}_x = \overline{D}_x D_x = \Delta_x$.

If S is a subspace of $\mathbb{R}_{0,m+1}$, then $\mathcal{M}(\Omega, S)$ and $\mathcal{H}(\Omega, S)$ denote, respectively, the spaces of left monogenic and harmonic S-valued functions in Ω. As $\partial_x^2 = -\Delta_x$, we have that $\mathcal{M}(\Omega, S) \subset \mathcal{H}(\Omega, S)$.

In particular, for $r, p, q \in \mathbb{N}$ such that $0 \leq r \leq m+1$, $0 \leq p \leq q$ with $r + 2q \leq m+1$, we have put in Section 1 (see (1.6)), $\mathbb{R}_{0,m+1}^{(r,p,q)} = \sum_{j=p}^{q} \bigoplus \mathbb{R}_{0,m+1}^{(r+2j)}$ and $\mathcal{M}(\Omega, \mathbb{R}_{0,m+1}^{(r,p,q)}) = \mathrm{MT}(\Omega, \mathbb{R}_{0,m+1}^{(r,p,q)})$.

Furthermore, for $0 \leq s \leq m+1$ fixed, a natural isomorphism

$$\Theta : \mathcal{E}\left(\Omega; \mathbb{R}_{0,m+1}^{(s)}\right) \longmapsto \mathcal{E}\left(\Omega; \Lambda^s \mathbb{R}^{m+1}\right) \tag{2.7}$$

may be then defined as follows.

Put for $W^s = \sum_{|A|=s} W_A^s e_A \in \mathcal{E}(\Omega; \mathbb{R}_{0,m+1}^{(s)})$,

$$\Theta W^s = \omega^s \Longleftrightarrow \omega^s = \sum_{|A|=s} \omega_A^s \, dx^A, \tag{2.8}$$

where for each $A = \{i_1, \ldots, i_s\} \subset \{0, \ldots, m\}$ with $0 \leq i_1 < \cdots < i_s \leq m$, $dx^A = dx^{i_1} \wedge \cdots \wedge dx^{i_s}$ and $\omega_A^s = W_A^s$ for all A.

By means of the decomposition (2.1), Θ may be extended by linearity to $\mathbb{R}_{0,m+1}$, thus leading to the isomorphism $\Theta : \mathcal{E}(\Omega; \mathbb{R}_{0,m+1}) \mapsto \mathcal{E}(\Omega; \Lambda\mathbb{R}^{m+1})$, where as usual $\Lambda\mathbb{R}^{m+1} = \sum_{s=0}^{m+1} \bigoplus \Lambda^s \mathbb{R}^{m+1}$.

It may be easily checked that the action of the exterior derivative d and the co-derivative d^* on $\mathcal{E}(\Omega; \Lambda^s\mathbb{R}^{m+1})$ then corresponds through Θ to the left action of ∂_x^+ and ∂_x^- on $\mathcal{E}(\Omega; \mathbb{R}_{0,m+1}^{(s)})$. For the definition of d and d^* (resp., ∂_x^+ and ∂_x^-) we refer to (1.4) and (1.7). In fact, taking into account the relations (2.5), the expressions (1.7) mean that for $W^s \in \mathcal{E}(\Omega; \mathbb{R}_{0,m+1}^{(s)})$,

$$\begin{aligned} \partial_x^- W^s &= \left[\partial_x W^s\right]_{s-1}, \\ \partial_x^+ W^s &= \left[\partial_x W^s\right]_{s+1}. \end{aligned} \tag{2.9}$$

Consequently, for $W^s \in \mathcal{E}(\Omega; \mathbb{R}_{0,m+1}^{(s)})$, $\partial_x W^s$ splits into

$$\partial_x W^s = \left[\partial_x W^s\right]_{s-1} + \left[\partial_x W^s\right]_{s+1} = \partial_x^- W^s + \partial_x^+ W^s. \tag{2.10}$$

It thus follows that for $W \in \mathcal{E}(\Omega; \mathbb{R}_{0,m+1}^{(r,p,q)})$, the system $\partial_x W = 0$ is given by (1.9).

Obviously, for $s = 0$, $\partial_x^- W^0 = 0$, while for $s = m+1$, $\partial_x^+ W^{m+1} = 0$. Finally, notice that $\partial_x = \partial_x^+ + \partial_x^-$ and that hence, as mentioned in Section 1, through Θ, the left action of ∂_x on

$\mathcal{E}(\Omega;\mathbb{R}_{0,m+1})$ corresponds to the action of $d+d^*$ on $\mathcal{E}(\Omega;\Lambda\mathbb{R}^{m+1})$. We thus have on $\mathcal{E}(\Omega;\mathbb{R}_{0,m+1})$ that $\Delta_x = -(\partial_x^+\partial_x^- + \partial_x^-\partial_x^+)$.

The following notations will also be used:

$$\ker^s\partial_x^+ = \left\{W^s \in \mathcal{E}\left(\Omega;\mathbb{R}_{0,m+1}^{(s)}\right) : \partial_x^+W^s = 0\right\},$$
$$\ker^s\partial_x^- = \left\{W^s \in \mathcal{E}\left(\Omega;\mathbb{R}_{0,m+1}^{(s)}\right) : \partial_x^-W^s = 0\right\}. \tag{2.11}$$

Let us recall that if Ω is contractible to a point, a refined version of the inverse Poincaré lemma then implies that

$$\partial_x^+\partial_x^- : \ker^s\partial_x^+ \longmapsto \ker^s\partial_x^+,$$
$$\partial_x^-\partial_x^+ : \ker^s\partial_x^- \longmapsto \ker^s\partial_x^- \tag{2.12}$$

are surjective operators.

For the inverse Poincaré lemma and its refined version we refer to, respectively, [1, 17]. For more information concerning the interplay between differential forms and multivectors, the reader is referred to [17, 18].

Obviously, all notions, notations, and properties introduced above may be easily adapted to the case where $\widetilde{\Omega} \subset \mathbb{R}^m$ is the orthogonal projection of Ω on $\mathbb{R}^m$ and $\partial_{\underline{x}}$ and $\Delta_{\underline{x}}$ are the Dirac and Laplace operators in $\mathbb{R}^m$.

3. Conjugate harmonic pairs

Let $r,p,q \in \mathbb{N}$ be as in Section 1, let $W \in \mathcal{E}(\Omega;\mathbb{R}_{0,m+1}^{(r,p,q)})$ with $W = \sum_{j=p}^{q} W^{r+2j}$, and decompose each $W^{r+2j} \in \mathcal{E}(\Omega;\mathbb{R}_{0,m+1}^{(r+2j)})$ following (1.2), that is

$$W^{r+2j} = U^{r+2j} + \overline{e}_0 V^{r-1+2j}, \tag{3.1}$$

where $U^{r+2j} \in \mathcal{E}(\Omega,\mathbb{R}_{0,m}^{(r+2j)})$ and $V^{r-1+2j} \in \mathcal{E}(\Omega,\mathbb{R}_{0,m}^{(r-1+2j)})$.

Then, $W = U + \overline{e}_0 V$ with

$$U = \sum_{j=p}^{q} U^{r+2j} \quad \mathbb{R}_{0,m}^{(r,p,q)}\text{-valued},$$
$$V = \sum_{j=p}^{q} V^{r-1+2j} \quad \mathbb{R}_{0,m}^{(r-1,p,q)}\text{-valued}. \tag{3.2}$$

Now suppose that $W \in \mathrm{MT}(\Omega,\mathbb{R}_{0,m+1}^{(r,p,q)})$, that is, (U,V) is a conjugate harmonic pair in Ω in the sense of [16]. Then, as already stated in (1.3),

$$\partial_x W = 0 \Longleftrightarrow \begin{cases} \partial_{x_0}U + \partial_{\underline{x}}V = 0, \\ \partial_{\underline{x}}U + \partial_{x_0}V = 0. \end{cases} \tag{3.3}$$

By virtue of (2.10) and (3.2), the equations in (3.3) lead to the systems

$$\begin{gathered}\partial_{\underline{x}}^{-} V^{r-1+2p} = 0,\\ \partial_{x_0} U^{r+2j} + \partial_{\underline{x}}^{+} V^{r-1+2j} + \partial_{\underline{x}}^{-} V^{r-1+2j+2} = 0, \quad j = p, \dots, q-1,\\ \partial_{x_0} U^{r+2q} + \partial_{\underline{x}}^{+} V^{r-1+2q} = 0,\end{gathered} \tag{3.4}$$

$$\begin{gathered}\partial_{\underline{x}}^{-} U^{r+2p} + \partial_{x_0} V^{r-1+2p} = 0,\\ \partial_{\underline{x}}^{+} U^{r+2j} + \partial_{\underline{x}}^{-} U^{r+2j+2} + \partial_{x_0} V^{r-1+2j+2} = 0, \quad j = p, \dots, q-1,\\ \partial_{\underline{x}}^{+} U^{r+2q} = 0.\end{gathered} \tag{3.5}$$

From (3.5) it thus follows that $W \in \mathrm{MT}(\Omega, \mathbb{R}_{0,m+1}^{(r,p,q)})$ implies that $\partial_{\underline{x}}^{+} U^{r+2q} = 0$ in Ω.

We now claim that, under certain geometric conditions upon Ω, given $U = \sum_{j=p}^{q} U^{r+2j}$, harmonic and $\mathbb{R}_{0,m}^{(r,p,q)}$-valued in Ω, the condition $\partial_{\underline{x}}^{+} U^{r+2q} = 0$ in Ω is sufficient to ensure the existence of a $\widetilde{V}$, harmonic and $\mathbb{R}_{0,m}^{(r-1,p,q)}$-valued in Ω, which is conjugate harmonic to U, that is $\widetilde{W} = U + \overline{e}_0 \widetilde{V} \in \mathrm{MT}(\Omega, \mathbb{R}_{0,m+1}^{(r,p,q)})$.

In proving this statement, we will adapt where necessary the techniques worked out in [16] for constructing conjugate harmonic pairs.

Let again $\widetilde{\Omega}$ denote the orthogonal projection of Ω on $\mathbb{R}^m$. Then, we suppose henceforth that Ω satisfies the following conditions (C1) and (C2):

(C1) Ω is normal with respect to the $\overline{e}_0$ direction, that is, there exists $x_0^* \in \mathbb{R}$ such that for all $\underline{x} \in \widetilde{\Omega}$, $\Omega \cap \{\underline{x} + t e_0 : t \in \mathbb{R}\}$ is connected and it contains the element $(x_0^*, \underline{x})$;

(C2) $\widetilde{\Omega}$ is contractible to a point.

The condition (C1) is sufficient for constructing harmonic conjugates to U (see [16]), while the condition (C2) ensures the applicability of the inverse Poincaré lemma and its consequences in $\widetilde{\Omega}$ (see [17]).

As is well known, classical results of cohomology theory provide necessary and sufficient conditions for the validity of the inverse Poincaré lemma in $\widetilde{\Omega}$. For convenience of the reader, we restrict ourselves to the condition (C2), thus making the inverse Poincaré lemma applicable for any closed or coclosed form ω^s in $\widetilde{\Omega}$ $(0 < s < m)$.

Now assume that $U = \sum_{j=p}^{q} U^{r+2j}$ harmonic and that $\mathbb{R}_{0,m}^{(r,p,q)}$-valued in Ω satisfies the condition $\partial_{\underline{x}}^{+} U^{r+2q} = 0$ in Ω.

Put

$$\widetilde{H}(x_0, \underline{x}) = \int_{x_0^*}^{x_0} U(t, \underline{x}) dt - \widetilde{h}(\underline{x}), \tag{3.6}$$

where $\widetilde{h} = \sum_{j=p}^{q} \widetilde{h}^{r+2j}$ is a smooth $\mathbb{R}_{0,m}^{(r,p,q)}$-valued solution in $\widetilde{\Omega}$ of the equation

$$\Delta_{\underline{x}} \widetilde{h}(\underline{x}) = \partial_{x_0} U(x_0^*, \underline{x}). \tag{3.7}$$

As $\Delta_{\underline{x}} : \mathcal{E}(\widetilde{\Omega}; \mathbb{R}_{0,m}^{(r,p,q)}) \mapsto \mathcal{E}(\widetilde{\Omega}; \mathbb{R}_{0,m}^{(r,p,q)})$ is surjective (see [19]), such $\widetilde{h}$ indeed exists and any other similar solution of (3.7) has the form $\widetilde{h} + h$, where $h \in \mathscr{H}(\widetilde{\Omega}; \mathbb{R}_{0,m}^{(r,p,q)})$.

Fix a solution $\widetilde{h}$ of (3.7). Then by construction, the corresponding $\widetilde{H}$ determined by (3.6) belongs to $\mathscr{H}(\Omega; \mathbb{R}_{0,m}^{(r,p,q)})$ (see [16]).

We now prove that there exists $h^{r+2q} \in \mathscr{H}(\widetilde{\Omega}; \mathbb{R}_{0,m}^{(r+2q)})$ such that in $\widetilde{\Omega}$,

$$\partial_{\underline{x}}^{+}\left(\widetilde{h}^{r+2q} + h^{r+2q}\right) = 0. \tag{3.8}$$

To this end, first notice that, as by the assumption $\partial_{\underline{x}}^{+} U^{r+2q} = 0$ in the Ω, we have that $\partial_{\underline{x}}^{+}(\partial_{x_0} U^{r+2q})(x_0^*, \underline{x}) = 0$ in $\widetilde{\Omega}$, whence $\partial_{x_0} U^{r+2q}(x_0^*, \underline{x}) \in \ker^{r+2q} \partial_{\underline{x}}^{+}$.

As $\partial_{\underline{x}}^{+}\partial_{\underline{x}}^{-} : \ker^{r+2q}\partial_{\underline{x}}^{+} \mapsto \ker^{r+2q}\partial_{\underline{x}}^{+}$ is surjective (see also (2.12)) there exists $\widetilde{W}^{r+2q} \in \ker^{r+2q}\partial_{\underline{x}}^{+}$ such that $\partial_{\underline{x}}^{+}\partial_{\underline{x}}^{-}\widetilde{W}^{r+2q}(\underline{x}) = -\partial_{x_0} U^{r+2q}(x_0^*, \underline{x})$, that is, $\widetilde{W}^{r+2q}$ satisfies in $\widetilde{\Omega}$ the relations

$$\partial_{\underline{x}}^{+}\widetilde{W}^{r+2q} = 0, \qquad \partial_{\underline{x}}^{+}\partial_{\underline{x}}^{-}\widetilde{W}^{r+2q}(\underline{x}) = -\partial_{x_0} U^{r+2q}(x_0^*, \underline{x}). \tag{3.9}$$

Furthermore, put $h^{r+2q} = \widetilde{W}^{r+2q} - \widetilde{h}^{r+2q}$. Then, on the one hand,

$$\begin{aligned} \Delta_{\underline{x}}\left(\widetilde{h}^{r+2q} + h^{r+2q}\right)(\underline{x}) &= -\left(\partial_{\underline{x}}^{-}\partial_{\underline{x}}^{+} + \partial_{\underline{x}}^{+}\partial_{\underline{x}}^{-}\right)\widetilde{W}^{r+2q}(\underline{x}) \\ &= -\partial_{\underline{x}}^{+}\partial_{\underline{x}}^{-}\widetilde{W}^{r+2q}(\underline{x}) \\ &= \partial_{x_0} U^{r+2q}(x_0^*, \underline{x}), \end{aligned} \tag{3.10}$$

while on the other hand

$$\begin{aligned} \Delta_{\underline{x}}\left(\widetilde{h}^{r+2q} + h^{r+2q}\right)(\underline{x}) &= \Delta_{\underline{x}}\widetilde{h}^{r+2q}(\underline{x}) + \Delta_{\underline{x}} h^{r+2q}(\underline{x}) \\ &= \partial_{x_0} U^{r+2q}(x_0^*, \underline{x}) + \Delta_{\underline{x}} h^{r+2q}(\underline{x}). \end{aligned} \tag{3.11}$$

Consequently, $\Delta_{\underline{x}} h^{r+2q} = 0$ in $\widetilde{\Omega}$ and $\partial_{\underline{x}}(\widetilde{h}^{r+2q} + h^{r+2q})$ is $\mathbb{R}_{0,m}^{(r-1+2q)}$-valued in $\widetilde{\Omega}$.

Now define H by

$$H(x_0, \underline{x}) = \widetilde{H}(x_0, \underline{x}) - h^{r+2q}(\underline{x}). \tag{3.12}$$

Then by construction, $H \in \mathscr{H}(\Omega; \mathbb{R}_{0,m}^{(r,p,q)})$ and clearly in Ω, $\partial_{x_0} H = U$.

Furthermore, as $\partial_{\underline{x}}^{+} U^{r+2q}(x_0, \underline{x}) = 0$ in Ω, $\int_{x_0^*}^{x_0} \partial_{\underline{x}} U(t, \underline{x})dt$ is $\mathbb{R}_{0,m}^{(r-1,p,q)}$-valued and obviously $\partial_{\underline{x}}(\sum_{j=p}^{q-1} \widetilde{h}^{r+2j})$ is $\mathbb{R}_{0,m}^{(r-1,p,q)}$-valued. As moreover $\partial_{\underline{x}}(\widetilde{h}^{r+2q} + h^{r+2q}) = 0$, we get that $\widetilde{V}(x_0, \underline{x}) = -\partial_{\underline{x}} H(x_0, \underline{x})$ is $\mathbb{R}_{0,m}^{(r-1,p,q)}$-valued.

Consequently, as $D_x \overline{D}_x = \Delta_x$, $\widetilde{W} = \overline{D_x} H = U + \overline{e}_0 \widetilde{V} \in \mathrm{MT}(\Omega, \mathbb{R}_{0,m+1}^{(r,p,q)})$, that is, $(U, \widetilde{V})$ is a conjugate harmonic pair in Ω.

We have thus proved the following theorem.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^{m+1}$ be open and normal with respect to the $\overline{e}_0$-direction and let Ω be contractible to a point. Furthermore, let $U \in \mathscr{H}(\Omega; \mathbb{R}_{0,m}^{(r,p,q)})$ be given. Then, U admits a conjugate harmonic $\widetilde{V} \in \mathscr{H}(\Omega; \mathbb{R}_{0,m}^{(r-1,p,q)})$ if and only if $\partial_{\underline{x}}^{+} U^{r+2q} = 0$ in Ω.*

Remarks

(1) If $r+2q=m+1$, then $U_{0,m}^{r+2q}=U_{0,m}^{m+1}\equiv 0$ in Ω, thus implying that the condition $\partial_{\underline{x}}^{+}U^{r+2q}=0$ is automatically satisfied and that in constructing H, no correction term h^{r+2q} should be added to $\widetilde{H}$ (i.e., we may take $H=\widetilde{H}$ in (3.12)).

(2) It is of course tacitly understood that if $r=p=0$, then in the expression of $V=\sum_{j=0}^{q}V^{-1+2j}$ (see (3.2)), the first term V^{-1} is taken to be identically zero in Ω.

(3) The systems (3.4) and (3.5) show a lot of symmetry.

The following theorem (Theorem 3.2) holds, the proof of which is omitted.

Theorem 3.2. *Let $\Omega\subset\mathbb{R}^{m+1}$ be open and normal with respect to the $\overline{e}_0$-direction and let Ω be contractible to a point. Furthermore, let $V\in\mathscr{H}(\Omega;\mathbb{R}_{0,m+1}^{(r-1),p,q})$ be given. Then, V admits a conjugate harmonic $\widetilde{U}\in\mathscr{H}(\Omega;\mathbb{R}_{0,m+1}^{(r,p,q)})$ if and only if $\partial_{\underline{x}}^{-}V^{r-1+2p}=0$ in Ω.*

4. Structure theorems

Assume that $r,p,q\in\mathbb{N}$ are such that $0\leq r<m+1$ and that $0\leq p<q$ with $r+2q\leq m+1$.

This section essentially deals with the construction of harmonic potentials corresponding to solutions of the generalized Moisil-Théodoresco system.

We start with the following lemma.

Lemma 4.1. *Let $\Omega\subset\mathbb{R}^{m+1}$ be open and contractible to a point and let $W\in\mathcal{E}(\Omega,\mathbb{R}_{0,m+1}^{(r,p,q)})$. The following properties are equivalent:*

(i) $W\in\mathrm{MT}(\Omega,\mathbb{R}_{0,m+1}^{(r,p,q)})$,

(ii) *there exists $H\in\mathscr{H}(\Omega,\mathbb{R}_{0,m+1}^{(r+1,p,q-1)})$ such that $W=\partial_x H$.*

Proof. It is clear that if $H\in\mathscr{H}(\Omega,\mathbb{R}_{0,m+1}^{(r+1,p,q-1)})$, then $W=\partial_x H$ is $\mathbb{R}_{0,m+1}^{(r,p,q)}$-valued. As moreover $\partial_x^2=-\Delta_x$, $W\in\mathrm{MT}(\Omega,\mathbb{R}_{0,m+1}^{(r,p,q)})$; whence (ii)$\Rightarrow$(i) is proved.

Conversely, assume that $W\in\mathrm{MT}(\Omega,\mathbb{R}_{0,m+1}^{(r,p,q)})$ and put $W=\sum_{j=p}^{q}W^{r+2j}$. From $\partial_x W=0$ it follows that

$$\begin{gathered}\partial_x^{-}W^{r+2p}=0,\\ \partial_x^{+}W^{r+2j}+\partial_x^{-}W^{r+2(j+1)}=0,\quad j=p,\ldots,q-1,\\ \partial_x^{+}W^{r+2q}=0.\end{gathered}\tag{4.1}$$

By a refined version of the inverse Poincaré lemma (see [17]) we obtain from the first equation in (4.1) that there exists $W_{-}^{r+2p+1}\in\mathcal{E}(\Omega;\mathbb{R}_{0,m+1}^{(r+2p+1)})$ such that in Ω

$$W^{r+2p}=\partial_x^{-}W_{-}^{r+2p+1},\qquad \partial_x^{+}W_{-}^{r+2p+1}=0.\tag{4.2}$$

Analogously, the third equation in (4.1) implies the existence of

$$W_{+}^{r+2q-1}\in\mathcal{E}\left(\Omega;\mathbb{R}_{0,m+1}^{(r+2q-1)}\right)\tag{4.3}$$

such that in Ω

$$W^{r+2q} = \partial_x^+ W_+^{r+2q-1}, \qquad \partial_x^- W_+^{r+2q-1} = 0. \tag{4.4}$$

Put $H' = W_-^{r+2p+1} + W_+^{r+2q-1}$. Then,

$$H' \in \mathcal{E}\left(\Omega; \mathbb{R}_{0,m+1}^{(r+1+2p)} \oplus \mathbb{R}_{0,m+1}^{(r+1+2(q-1))}\right) \tag{4.5}$$

and by virtue of (4.2) and (4.4),

$$\begin{aligned}\partial_x H' &= \partial_x^+\left(W_-^{r+2p+1} + W_+^{r+2q-1}\right) + \partial_x^-\left(W_-^{r+2p+1} + W_+^{r+2q-1}\right) \\ &= W^{r+2q} + W^{r+2p}.\end{aligned} \tag{4.6}$$

But $W = W^{2r+p} + W^* + W^{r+2q}$, where $W^* = \sum_{j=p+1}^{q-1} W^{r+2j}$ is $\mathbb{R}_{0,m+1}^{(r,p+1,q-1)}$-valued and harmonic in Ω. As $\Delta_x : \mathcal{E}(\Omega; \mathbb{R}_{0,m+1}^{(r,p+1,q-1)}) \mapsto \mathcal{E}(\Omega, \mathbb{R}_{0,m+1}^{(r,p+1,q-1)})$ is surjective (see [19]), there exists that $H^* \in \mathcal{E}(\Omega, \mathbb{R}_{0,m+1}^{(r,p+1,q-1)})$ such that $\Delta_x H^* = W^*$.

Put $H'' = -\partial_x H^*$. Then, clearly $H'' \in \mathcal{E}(\Omega; \mathbb{R}_{0,m+1}^{(r+1,p,q-1)})$ and

$$\partial_x H'' = -\partial_x^2 H^* = \Delta_x H^* = W^*. \tag{4.7}$$

Finally, put $H = H' + H''$. Then, H is $\mathbb{R}_{0,m+1}^{(r+1,p,q-1)}$-valued and

$$\partial_x H = \partial_x H' + \partial_x H'' = \sum_{j=p}^{q} W^{r+2j} = W. \tag{4.8}$$

As H is obviously harmonic in Ω, the proof is done. $\square$

Remarks

(1) In the case where $r = p = 0$ and $2q < m+1$, we have that in (4.1) the equation $\partial_x^- W^0 = 0$ is automatically satisfied. Putting $\widetilde{W} = \sum_{j=0}^{q-1} W^{2j}$, take $\widetilde{H} \in \mathcal{E}(\Omega; \mathbb{R}_{0,m+1}^{(0,0,q-1)})$ such that $\Delta_x \widetilde{H} = \widetilde{W}$ and define H by $H = \partial_x(-\widetilde{H} + W_+^{2q-1})$. Then, $W = \partial_x H$.

In the case where $r + 2q = m+1$, the equation $\partial_x^+ W^{m+1} = 0$ is automatically satisfied. An analogous reasoning to the one just made then leads to an appropriate $H \in \mathscr{H}(\Omega; \mathbb{R}_{0,m+1}^{(r+1,p,q-1)})$ such that $W = \partial_x H$.

(2) Obviously, in the case where as well $r = p = 0$ as $2q = m+1$, the technique suggested in Remark (1) then produces $H \in \mathscr{H}(\Omega; \mathbb{R}_{0,m+1}^{(1,0,(m+1)/2-1)})$ such that $W = \partial_x H$.

(3) A particularly important example where as well $r = p = 0$ as $2q = m+1$ occurs when $m+1 = 4$. Indeed, put for given real valued smooth functions f_i in $\Omega \subset \mathbb{R}^4$, $i = 0, 1, 2, 3$,

$$\begin{aligned}W^0 &= f_0, \\ W^2 &= f_1(e_2e_3 - e_0e_1) + f_2(e_3e_1 - e_0e_2) + f_3(e_1e_2 - e_0e_3), \\ W^4 &= f_0e_0e_1e_2e_3.\end{aligned} \tag{4.9}$$

Then, for $W = W^0 + W^2 + W^4 \in \mathcal{E}(\Omega; \mathbb{R}_{0,4}^{(0,0,2)})$,

$$\partial_x W = 0 \Longleftrightarrow \begin{cases} \partial_{\underline{x}}^+ W^0 + \partial_{\underline{x}}^- W^2 = 0, \\ \partial_{\underline{x}}^+ W^2 + \partial_{\underline{x}}^- W^4 = 0. \end{cases} \tag{4.10}$$

Both equations in (4.10) give rise to the same system to be satisfied by $f = (f_0, f_1, f_2, f_3)$, namely

$$\begin{aligned} &\frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} = 0, \\ &\frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0} - \frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2} = 0, \\ &\frac{\partial f_0}{\partial x_2} + \frac{\partial f_1}{\partial x_3} + \frac{\partial f_2}{\partial x_0} - \frac{\partial f_3}{\partial x_1} = 0, \\ &\frac{\partial f_0}{\partial x_3} - \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} + \frac{\partial f_3}{\partial x_0} = 0. \end{aligned} \tag{4.11}$$

The system (4.11) is the Fueter system in $\mathbb{R}^4$ for so-called left regular functions of a quaternion variable; it lies at the basis of quaternionic analysis (see [20, 21]).

We have taken this example from [2], where it was proved in the framework of self-conjugate differential forms. We have inserted it here because it demonstrates how quaternionic analysis can be viewed upon as part of Clifford analysis in $\mathbb{R}^4$, namely as the theory of special solutions to a generalized Moisil-Théodoresco system in $\mathbb{R}^4$ of type $(0, 0, 2)$.

(4) In the case where $p = 0$ and $q = 1$, Lemma 4.1 tells us that, given $W = W^r + W^{r+2} \in \mathrm{MT}(\Omega; \mathbb{R}_{0,m+1}^{(r)} \oplus \mathbb{R}_{0,m+1}^{(r+2)})$, there exists $H \in \mathcal{H}(\Omega; \mathbb{R}_{0,m+1}^{(r+1)})$ such that $W = \partial_x H$. This result was already obtained in [3, Lemma 3.1].

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^{m+1}$ be open and normal with respect to the $\overline{e}_0$-direction, let $\widetilde{\Omega}$ be contractible to a point, and let $W \in \mathcal{E}(\Omega; \mathbb{R}_{0,m+1}^{(r,p,q)})$. The following properties are equivalent:*

(i) $W \in \mathrm{MT}(\Omega; \mathbb{R}_{0,m+1}^{(r,p,q)})$,

(ii) *there exists $L \in \mathcal{H}(\Omega; \mathbb{R}_{0,m}^{(r,p,q)})$ with $\partial_{\underline{x}}^+ L^{r+2q} = 0$ in Ω such that $W = \overline{D}_x L$.*

Proof. (i)→(ii). Let $W \in \mathrm{MT}(\Omega; \mathbb{R}_{0,m+1}^{(r,p,q)})$ and put, following (1.2), $W = U + \overline{e}_0 V$. Then, the pair (U, V) is conjugate harmonic in Ω with $U \in \mathcal{H}(\Omega; \mathbb{R}_{0,m}^{(r,p,q)})$ and $V \in \mathcal{H}(\Omega; \mathbb{R}_{0,m}^{(r-1,p,q)})$.

Associate with U the harmonic $\mathbb{R}_{0,m}^{(r,p,q)}$-valued potential H given by (3.12), that is,

$$H(x_0, \underline{x}) = \int_{x_0^*}^{x_0} U(t, \underline{x})\,dt - (\widetilde{h} + h^{r+2q})(\underline{x}), \tag{4.12}$$

where in $\widetilde{\Omega}$, $\Delta_{\underline{x}} \widetilde{h}(\underline{x}) = \partial_{x_0} U(x_0^*, \underline{x})$, $h^{r+2q} \in \mathcal{H}(\widetilde{\Omega}; \mathbb{R}_{0,m}^{(r+2q)})$, and $\partial_{\underline{x}}^+(\widetilde{h}^{r+2q} + h^{r+2q}) = 0$.

As moreover $\partial_{\underline{x}}^+ U^{r+2q} = 0$ in Ω (see Theorem 3.1), it thus follows from (4.12) that $\partial_{\underline{x}}^+ H^{r+2q} = 0$ in Ω. Consequently, $\widetilde{W} = \overline{D}_x H = \widetilde{U} + \overline{e}_0 \widetilde{V} \in \mathrm{MT}(\Omega; \mathbb{R}_{0,m+1}^{(r,p,q)})$ with $\widetilde{U} = \partial_{x_0} H = U$ and $\widetilde{V} = -\partial_{\underline{x}} H$.

From $\partial_x(\widetilde{W} - W) = 0$, it is then easily obtained that $\widetilde{V} - V$ is independent of x_0 and that in $\widetilde{\Omega}$, $\partial_{\underline{x}}(\widetilde{V} - V) = 0$, that is, $\widetilde{V} - V \in \mathrm{MT}(\widetilde{\Omega}; \mathbb{R}_{0,m}^{(r-1,p,q)})$. By virtue of Lemma 4.1, there exists $H^* \in \mathcal{H}(\widetilde{\Omega}; \mathbb{R}_{0,m}^{(r,p,q-1)})$ such that $\widetilde{V} - V = \partial_{\underline{x}} H^*$; whence $V = -\partial_{\underline{x}}(H + H^*)$.

Put $L = H + H^*$. Then by construction,

(i) $L \in \mathcal{H}(\Omega; \mathbb{R}_{0,m}^{(r,p,q)})$,

(ii) $\partial_{\underline{x}}^{+} L^{r+2q} = 0$,

(iii) $W = \overline{D}_x L$;

whence (i)→(ii) is proved.

Conversely, let $L \in \mathcal{H}(\Omega; \mathbb{R}_{0,m}^{(r,p,q)})$ with $\partial_{\underline{x}}^{+} L^{r+2q} = 0$. Then clearly $W = \overline{D}_x L \in \mathrm{MT}(\Omega; \mathbb{R}_{0,m+1}^{(r,p,q)})$. □

Remarks

(1) Theorem 4.2 tells us that each $W \in \mathrm{MT}(\Omega; \mathbb{R}_{0,m+1}^{(r,p,q)})$ admits an $\mathbb{R}_{0,m}^{(r,p,q)}$- valued harmonic potential L in Ω satisfying $\partial_{\underline{x}}^{+} L^{r+2q} = 0$.

(2) Let $W = W^r + W^{r+2} \in \mathcal{E}(\Omega; \mathbb{R}_{0,m+1}^{(r)} \oplus \mathbb{R}_{0,m+1}^{(r+2)})$, that is, we take $p = 0$ and $q = 1$. Then from Theorem 4.2, it follows that the following properties are equivalent:

(i) $W \in \mathrm{MT}(\Omega; \mathbb{R}_{0,m+1}^{(r)} \oplus \mathbb{R}_{0,m+1}^{(r+2)})$,

(ii) there exists $L \in \mathcal{H}(\Omega; \mathbb{R}_{0,m}^{(r)} \oplus \mathbb{R}_{0,m}^{(r+2)})$ with $\partial_{\underline{x}}^{+} L^{r+2} = 0$ such that $W = \overline{D}_x L$.

This characterization was already obtained in [3, Theorem 3.1].

5. Cauchy integral decompositions

Let $\Omega = \Omega_+ \subset \mathbb{C}$ be a bounded open subset with boundary γ, where γ is a rectifiable closed Jordan curve such that for some constant $c > 0$, $H^1(\gamma \cap B(z,p)) \leq cp$ and this for all $z \in \gamma$ and $p > 0$, where $B(z,p)$ is the closed disc with center z and radius p and H^1 is the 1-dimensional Hausdorff measure on γ. Furthermore, let $\Omega_- = \mathbb{C} \setminus (\Omega \cup \gamma)$ and let $f \in C^{0,\alpha}(\gamma)$, $0 < \alpha < 1$.

In classical complex analysis, the following jump problem (5.1) is solved by means of the Cauchy transform:

"Find a pair of functions f_+ and f_-, holomorphic in Ω_+ and Ω_- with $f_-(\infty) = 0$, such that $f_\pm$ are continuously extendable to γ and that on γ

$$f = f_+ + f_-, \tag{5.1}$$

where in (5.1), $f_\pm(u) = \lim_{\Omega_\pm \ni z \to u} f_\pm(z), u \in \gamma$."

Let $\mathcal{C}_\gamma$ be the Cauchy transform on $C^{0,\alpha}(\gamma)$, that is, for $f \in C^{0,\alpha}(\gamma)$,

$$\mathcal{C}_\gamma f(z) = \frac{1}{2\pi} \int_\gamma \frac{1}{t - z} \nu(t) f(t) dH^1(t), \quad z \in \mathbb{C} \setminus \gamma, \tag{5.2}$$

where $\nu(t)$ is the outward pointing unit normal at $t \in \gamma$ and ds is the elementary Lebesgue measure on γ.

Then, the following fundamental properties hold (see, e.g., [22]):

(i) $\mathcal{C}_\gamma f$ is holomorphic and of the class $C^{0,\alpha}$ on $\Omega_+ \cup \Omega_-$ with $\mathcal{C}_\gamma f(\infty) = 0$;

(ii) Plemelj-Sokhotzki formulae:

$$\mathcal{C}_\gamma^\pm f(u) = \lim_{\Omega_\pm \ni z \to u} \mathcal{C}_\gamma f(z) = \frac{1}{2}\big(\pm f(u) + \mathcal{S}_\gamma f(u)\big), \quad u \in \gamma, \tag{5.3}$$

where for $u \in \gamma$,

$$\mathcal{S}_\gamma f(u) = \frac{1}{\pi} \mathrm{PV} \int_\gamma \frac{1}{t-u} \nu(t)\big(f(t) - f(u)\big) dH^1(t) + f(u) \tag{5.4}$$

define the Hilbert transform $\mathcal{S}_\gamma$ on $C^{0,\alpha}(\gamma)$;

(iii) $f = \mathcal{C}_\gamma^+ f - \mathcal{C}_\gamma^- f$ on γ.

It thus follows that the answer to the jump problem (5.1) is indeed given by $\mathcal{C}_\gamma f$.

The decomposition (iii) thus obtained is known as the Cauchy integral decomposition of f on γ.

Now let $\Omega = \Omega_+$ be a bounded and open subset of $\mathbb{R}^{m+1}$ with boundary $\Gamma = \partial\Omega$. Then, in Clifford analysis, for suitable pairs (Γ, f) of boundaries Γ and $\mathbb{R}_{0,m+1}$-valued functions f on Γ, the Cauchy transform $\mathcal{C}_\Gamma f$ is defined by

$$\mathcal{C}_\Gamma f(x) = \int_\Gamma E(y-x)\nu(y) f(y) d\mathcal{H}^m(y), \quad x \in \mathbb{R}^{m+1} \setminus \Gamma, \tag{5.5}$$

where the following conditions hold.

(i) $E(x) = (-1/A_{m+1})(x/|x|^{m+1})$, $x \in \mathbb{R}^{m+1} \setminus \{0\}$, is the fundamental solution of the Dirac operator ∂_x, where A_{m+1} is the area of the unit sphere in $\mathbb{R}^{m+1}$. $E(x)$ is $\mathbb{R}_{0,m+1}^{(1)}$-valued and monogenic in $\mathbb{R}^{m+1} \setminus \{0\}$.

(ii) $\nu(y) = \sum_{i=0}^m e_i \nu_i(y)$ is the outward pointing unit normal at $y \in \Gamma$.

(iii) $\mathcal{H}^m$ is the m-dimensional Hausdorff measure on Γ. For the definition of $\mathcal{H}^m$, see, for example, [23, 24].

In what follows we restrict ourselves to the following conditions on the pair (Γ, f) (see also the remarks made at the end of this section).

(C1) Γ is an m-dimensional Ahlfors-David regular surface, that is, there exists a constant $c > 0$ such that for all $y \in \Gamma$ and $0 < \rho \le \operatorname{diam}\Gamma$,

$$c^{-1}\rho^m \le \mathcal{H}^m\big(\Gamma \cap B(y,\rho)\big) \le c\rho^m, \tag{5.6}$$

where $B(y,\rho)$ is the closed ball in $\mathbb{R}^{m+1}$ with center y and radius ρ and $\operatorname{diam}\Gamma$ is the diameter of Γ.

For the definition of AD-regular surfaces, see, for example, [24, 25].

(C2) $f \in C^{0,\alpha}(\Gamma; \mathbb{R}_{0,m+1})$, $0 < \alpha < 1$, $C^{0,\alpha}(\Gamma; \mathbb{R}_{0,m+1})$ being the space of $\mathbb{R}_{0,m+1}$-valued Hölder continuous functions of order α on Γ.

Under the conditions (C1) and (C2), the following properties hold (see, e.g., [26–28]):

(i) $\mathcal{C}_\Gamma f$ is left monogenic in $\mathbb{R}^{m+1} \setminus \Gamma$ and $\mathcal{C}_\Gamma f(\infty) = 0$;

(ii) (Plemelj-Sokhotzki formulae) the functions $\mathcal{C}_\Gamma^{\pm} f$ determined by

$$\mathcal{C}_\Gamma^{\pm} f(u) = \lim_{\Omega_\pm \ni x \to u} \mathcal{C}_\Gamma f(x) = \frac{1}{2}\left(\mathcal{S}_\Gamma f(u) \pm f(u)\right), \quad u \in \Gamma, \tag{5.7}$$

belong to $C^{0,\alpha}(\Gamma; \mathbb{R}_{0,m+1})$, where

$$\mathcal{S}_\Gamma f(u) = 2\int_\Gamma E(y-u)\nu(y)\left[f(y) - f(u)\right] d\mathcal{H}^m(y) + f(u), \quad u \in \Gamma, \tag{5.8}$$

the integral being taken in the sense of principal values;

(iii) $f(u) = \mathcal{C}_\Gamma^{+} f(u) - \mathcal{C}_\Gamma^{-} f(u)$, $u \in \Gamma$.

It thus follows that, given a Hölder continuous $\mathbb{R}_{0,m+1}$-valued density f on Γ, the jump problem (5.9)

"Find f_+ and f_-, belonging to $C^{0,\alpha}(\Gamma; \mathbb{R}_{0,m+1})$ and which are the boundary values of left monogenic functions f_+ and f_- in, respectively, Ω_+ and Ω_- with $f_-(\infty) = 0$ such that on Γ

$$f = f_+ + f_- \text{"} \tag{5.9}$$

is solved by considering the Cauchy transform $\mathcal{C}_\Gamma f$. Indeed, we can take $f_+ = \mathcal{C}_\Gamma f$ in Ω_+ and $f_- = -\mathcal{C}_\Gamma f$ in Ω_-.

Now let again $r, p, q \in \mathbb{N}$ be a triplet satisfying $0 \le r \le m+1$ and $0 \le p < q$ with $r + 2q \le m+1$, and let $W \in C^{0,\alpha}(\Gamma; \mathbb{R}_{0,m+1}^{(r,p,q)})$.

As $E(y-x)\nu(y)$ is $\mathbb{R}_{0,m+1}^{(0)} \oplus \mathbb{R}_{0,m+1}^{(2)}$-valued, it is easily seen that $\mathcal{C}_\Gamma W$ is $\mathbb{R}_{0,m+1}^{(r-2+2p)} \oplus \mathbb{R}_{0,m+1}^{(r,p,q)} \oplus \mathbb{R}_{0,m+1}^{(r+2+2q)}$-valued. Consequently, if the jump problem (5.9) is formulated in terms of $\mathbb{R}_{0,m+1}^{(r,p,q)}$-valued Hölder continuous functions W, W_+, and W_- on Γ, then if we wish to solve it by means of the Cauchy transform $\mathcal{C}_\Gamma$, restrictions on $\mathcal{C}_\Gamma W$ have to be imposed, namely, in $\mathbb{R}^{m+1} \setminus \Gamma$ we should have

$$\begin{gathered} \left[\mathcal{C}_\Gamma W\right]_{r-2+2p} \equiv 0, \\ \left[\mathcal{C}_\Gamma W\right]_{r+2+2q} \equiv 0. \end{gathered} \tag{5.10}$$

The very heart of the following theorem (Theorem 5.1) tells us that the conditions (5.10) are necessary and sufficient. Although the arguments used in proving Theorem 5.1 are similar to the ones given in the proof of [29, Theorem 4.1], for convenience of the reader we write them out in full detail.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^{m+1}$ be open and bounded such that $\Gamma = \partial\Omega$ is an m-dimensional Ahlfors-David regular surface and let $W \in C^{0,\alpha}(\Gamma; \mathbb{R}_{0,m+1}^{(r,p,q)})$ with $0 < \alpha < 1$ and $p < q$. The following properties are equivalent:*

(i) *W admits on Γ a decomposition $W = W_+ + W_-$, where $W_\pm$ belong to $C^{0,\alpha}(\Gamma; \mathbb{R}_{0,m+1}^{(r,p,q)})$ and moreover are the boundary values of functions $W_\pm \in \mathrm{MT}(\Omega_\pm; \mathbb{R}_{0,m+1}^{(r,p,q)})$ with $W_-(\infty) = 0$,*

(ii) $\mathcal{C}_\Gamma W \in \mathrm{MT}(\mathbb{R}^{m+1} \setminus \Gamma; \mathbb{R}_{0,m+1}^{(r,p,q)})$,

(iii) $\sum_{j=p}^{q} [\mathcal{C}_\Gamma W]_{r+2j} \in \mathrm{MT}(\mathbb{R}^{m+1} \setminus \Gamma; \mathbb{R}_{0,m+1}^{(r,p,q)})$,

(iv) $[\mathcal{C}_\Gamma W]_{r-2+2p} \equiv 0$ *and* $[\mathcal{C}_\Gamma W]_{r+2+2q} \equiv 0$ *in* $\mathbb{R}^{m+1}$.

Proof. (i)→(ii). Assume that $W = W_+ + W_-$, where W_+ and W_- satisfy the conditions given in (i). Then

$$\mathcal{C}_\Gamma W = \mathcal{C}_\Gamma W_+ + \mathcal{C}_\Gamma W_-. \tag{5.11}$$

In view of the assumptions made on $W_\pm$, we have that $\mathcal{C}_\Gamma W_+ = 0$ in Ω_-, $\mathcal{C}_\Gamma W_- = 0$ in Ω_+ and that $\mathcal{C}_\Gamma W_\pm = W_\pm$ in $\Omega_\pm$.

Consequently,

$$\mathcal{C}_\Gamma W = \begin{cases} W_+ & \text{in } \Omega_+, \\ W_- & \text{in } \Omega_-. \end{cases} \tag{5.12}$$

As $W_\pm \in \mathrm{MT}(\Omega_\pm; \mathbb{R}_{0,m+1}^{(r,p,q)})$ with $W_-(\infty) = 0$, (ii) is proved.

(ii)→(iii): Trivial.

(iii)→(iv). Let us first recall that $\mathcal{C}_\Gamma W$ is left monogenic in $\mathbb{R}^{m+1} \setminus \Gamma$ with $\mathcal{C}_\Gamma W(\infty) = 0$.

According to the decomposition

$$\mathcal{C}_\Gamma W = [\mathcal{C}_\Gamma W]_{r-2+2p} + \sum_{j=p}^{q} [\mathcal{C}_\Gamma W]_{r+2j} + [\mathcal{C}_\Gamma W]_{r+2+2q} \tag{5.13}$$

and by the assumption made on $\sum_{j=p}^{q} [\mathcal{C}_\Gamma W]_{r+2j}$, it follows from [1] that

$$\partial_x \left([\mathcal{C}_\Gamma W]_{r-2+2p} \right) + \partial_x \left([\mathcal{C}_\Gamma W]_{r+2+2q} \right) = 0 \text{ in } \mathbb{R}^{m+1} \setminus \Gamma. \tag{5.14}$$

Furthermore, as $\partial_x([\mathcal{C}_\Gamma W]_{r-2+2p})$ and $\partial_x([\mathcal{C}_\Gamma W]_{r+2+2q})$ split into an $(r-3+2p)$ and an $(r-1+2p)$, respectively, into an $(r+1+2q)$ and an $(r+3+2q)$ multivector, we obtain from [22] that in $\mathbb{R}^{m+1} \setminus \Gamma$

$$\begin{aligned} \partial_x \left([\mathcal{C}_\Gamma W]_{r-2+2p} \right) &= 0, \\ \partial_x \left([\mathcal{C}_\Gamma W]_{r+2+2q} \right) &= 0. \end{aligned} \tag{5.15}$$

Moreover, as by assumption W is $\mathbb{R}_{0,m+1}^{(r,p,q)}$-valued, by virtue of the Plemelj-Sokhotzki formulae, we obtain that on Γ

$$\begin{aligned} [\mathcal{C}_\Gamma^+ W]_{r-2+2p} &= [\mathcal{C}_\Gamma^- W]_{r-2+2p}, \\ [\mathcal{C}_\Gamma^+ W]_{r+2+2q} &= [\mathcal{C}_\Gamma^- W]_{r+2+2q}. \end{aligned} \tag{5.16}$$

Furthermore, $[\mathcal{C}_\Gamma^\pm W]_{r-2+2p}$ and $[\mathcal{C}_\Gamma^\pm W]_{r+2+2q}$ are Hölder continuous on Γ.

It thus follows that $[\mathcal{C}_\Gamma W]_{r-2+2p}$ and $[\mathcal{C}_\Gamma W]_{r+2+2q}$ are left monogenic in $\mathbb{R}^{m+1} \setminus \Gamma$ and continuously extendable to Γ. Painlevé's theorem (see [30]) then implies that $[\mathcal{C}_\Gamma W]_{r-2+2p}$ and $[\mathcal{C}_\Gamma W]_{r+2+2q}$ are left monogenic in $\mathbb{R}^{m+1}$.

Finally, as $[\mathcal{C}_\Gamma W]_{r-2+2p}(\infty) = [\mathcal{C}_\Gamma W]_{r+2+2q}(\infty) = 0$, we obtain by virtue of Liouville's theorem (see [31]) that $[\mathcal{C}_\Gamma W]_{r-2+2p} \equiv 0$ and $[\mathcal{C}_\Gamma W]_{r+2+2q} \equiv 0$ in $\mathbb{R}^{m+1}$.

(iv)→(i). First note that, as $W \in C^{0,\alpha}(\Gamma; \mathbb{R}_{0,m+1}^{(r,p,q)})$, by means of the Plemelj-Sokhotzki formulae, we have on Γ that

$$W = \mathcal{C}_\Gamma^+ W - \mathcal{C}_\Gamma^- W. \tag{5.17}$$

In view of the assumption (iv) made, the functions $W_\pm$ defined in $\Omega_\pm$ by $W_\pm = \pm \mathcal{C}_\Gamma W$ obviously belong to $\mathrm{MT}(\Omega_\pm; \mathbb{R}_{0,m+1}^{(r,p,q)})$ and they satisfy all required properties. □

Remarks

(1) In the last decades, intensive research has been done in studying the Cauchy integral transform and the associated singular integral operator on curves γ in the plane or on hypersurfaces Γ in $\mathbb{R}^{m+1}$ ($m \geq 2$). Two types of boundary data are usually considered, namely a Hölder continuous density or an L_p-density ($1 < p < +\infty$).

In this section, we have formulated the jump problems (5.1) and (5.9) in terms of Hölder continuous densities. The reason for this is that in proving some of the equivalences stated in Theorem 5.1, the continuous extendability of the Cauchy integral up to the boundary plays a crucial role. This becomes clear for instance when use is made of Painlevé's theorem in proving the implication "(iii)→(iv)."

Note that for $f \in C^{0,\alpha}(\Gamma; \mathbb{R}_{0,m+1})$ $(0 < \alpha < 1)$, the continuous extendability of $\mathcal{C}_\Gamma f$ was already obtained in 1965 by V. Iftimie in the case where Γ is a compact Liapunov surface (see [32]). For an overview of recent investigations on conditions which can be put on the pair (Γ, f), f being a continuous density on Γ, we refer the reader to [28, 30, 33–38]. In particular, we wish to point out that the introduction and the references in [35] contain a detailed account of the historical background of the jump problems (5.1) and (5.9).

(2) The case $p = q = 0$ and $0 < r < m + 1$ was dealt with in [39]. For $\Omega \subset \mathbb{R}^{m+1}$ open, bounded and connected with C_∞-boundary Γ such that $\mathbb{R}^{m+1} \setminus (\Omega \cup \Gamma)$ is also connected, a set of equivalent properties was obtained ensuring the validity of the Cauchy integral decomposition for $W^r \in \mathcal{E}(\Gamma, \mathbb{R}_{0,m+1}^{(r)})$ given.

(3) If $W \in C^{0,\alpha}(\Gamma; \mathbb{R}_{0,m+1}^+)$, where

$$\mathbb{R}_{0,m+1}^+ = \sum_{s \text{ even}} \oplus \mathbb{R}_{0,m+1}^{(s)}, \tag{5.18}$$

then $\mathcal{C}_\Gamma W \in \mathrm{MT}(\mathbb{R}^{m+1} \setminus \Gamma; \mathbb{R}_{0,m+1}^+)$, that is the condition (ii) in Theorem 5.1 is satisfied.

Analogously, if $W \in C^{0,\alpha}(\Gamma; \mathbb{R}_{0,m+1}^-)$ where

$$\mathbb{R}_{0,m+1}^- = \sum_{s \text{ odd}} \oplus \mathbb{R}_{0,m+1}^{(s)}, \tag{5.19}$$

then $\mathcal{C}_\Gamma W \in \mathrm{MT}(\mathbb{R}^{m+1} \setminus \Gamma; \mathbb{R}_{0,m+1}^-)$ and so the condition (ii) in Theorem 5.1 is again satisfied.

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