Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2008, Article ID 362409, 6 pages
doi:10.1155/2008/362409

Research Article

Some Estimates of Certain Subnormal and Hyponormal Derivations

Vasile Lauric

Department of Mathematics, Florida A&M University, Tallahassee, FL 32307, USA

Correspondence should be addressed to Vasile Lauric, vasile.lauric@famu.edu

Received 23 October 2007; Accepted 9 January 2008

Recommended by Nils Ackermann

We prove that if A and B^* are subnormal operators and X is a bounded linear operator such that $AX - XB$ is a Hilbert-Schmidt operator, then $f(A)X - Xf(B)$ is also a Hilbert-Schmidt operator and $\|f(A)X - Xf(B)\|_2 \leq L\|AX - XB\|_2$ for f belongs to a certain class of functions. Furthermore, we investigate the similar problem in the case that S, T are hyponormal operators and $X \in \mathcal{L}(\mathcal{H})$ is such that $SX - XT$ belongs to a norm ideal $(J, \|\cdot\|_J)$, and we prove that $f(S)X - Xf(T) \in J$ and $\|f(S)X - Xf(T)\|_J \leq C\|SX - XT\|_J$ for f being in a certain class of functions.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$ and by $\mathcal{C}_2(\mathcal{H})$ the Hilbert-Schmidt class. For $T \in \mathcal{L}(\mathcal{H}), \sigma(T)$ denotes the spectrum of T, and for a compact subset $\Sigma \subset \mathbb{C}$, $\mathrm{Lip}(\Sigma)$ denotes the set of Lipschitz functions on Σ. Furthermore, $\mathrm{Rat}(\Sigma)$ denotes the algebra of rational functions with poles of Σ, and $R(\Sigma)$ denotes the closure of $\mathrm{Rat}(\Sigma)$ in the supremum norm over Σ.

For operators $A, B \in \mathcal{L}(\mathcal{H})$, the mapping $\Delta_{A,B}(X) = AX - XB$ is called a (generalized) derivation. If A, B are normal (subnormal or co-subnormal, hyponormal or co-hyponormal) operators, then $\Delta_{A,B}$ will be called a normal (subnormal, hyponormal) derivation, respectively.

Next, we recall some theorems that involve normal derivations, and then we extend some of these theorems to the case in which A, B^* are subnormal operators and to the case in which $A = S$, $B = T$ are hyponormal operators.

In [1], a generalization of Fuglede-Putnam theorem for normal operators was proved. For further results concerning normal derivations, the reader can see [2, 3].

Theorem 1.1 (see [1]). *If $A, B \in \mathcal{L}(\mathcal{H})$ are normal operators and $X \in \mathcal{L}(\mathcal{H})$ satisfies $AX - XB \in \mathcal{C}_2(\mathcal{H})$, then $A^*X - XB^* \in \mathcal{C}_2(\mathcal{H})$ and*

$$\|AX - XB\|_2 = \|A^*X - XB^*\|_2. \tag{1.1}$$

In [4], Furuta extended the above result to subnormal operators.

Theorem 1.2 (see [4]). *If $A, B^* \in \mathcal{L}(\mathcal{H})$ are subnormal operators and $X \in \mathcal{L}(\mathcal{H})$ satisfies $AX - XB \in \mathcal{C}_2(\mathcal{H})$, then $A^*X - XB^* \in \mathcal{C}_2(\mathcal{H})$ and*

$$\|AX - XB\|_2 \geq \|A^*X - XB^*\|_2. \tag{1.2}$$

In his paper [5], Kittaneh proved the following theorem using a famous result of Voiculescu [6] according to which every normal operator can be written as the sum of a diagonal operator and a Hilbert-Schmidt operator of an arbitrarily small Hilbert-Schmidt norm.

Theorem 1.3 (see [5]). *Let $A, B \in \mathcal{L}(\mathcal{H})$ be normal operators and $X \in \mathcal{L}(\mathcal{H})$ such that $AX - XB \in \mathcal{C}_2(\mathcal{H})$, and let $f \in \mathrm{Lip}(\sigma(A) \cup \sigma(B))$. Then $f(A)X - Xf(B)$ is also a Hilbert-Schmidt operator and*

$$\|f(A)X - Xf(B)\|_2 \leq L\|AX - XB\|_2, \tag{1.3}$$

where L is the Lipschitz constant of the function f.

2. Subnormal derivations

In this section, we investigate the validity of this inequality in the case that A, B^* are subnormal operators, but with a drawback concerning the extent of the class of functions in which f can run.

The following lemma is elementary and can be easily established making use of the *minimal normal extension* of a subnormal operator. Its proof is left for the reader.

Lemma 2.1. *If $S_1, S_2 \in \mathcal{L}(\mathcal{H})$ are subnormal operators, then there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and normal operators $N_1, N_2 \in \mathcal{L}(\mathcal{K})$ that are extensions of S_1, S_2, respectively, and $\sigma(N_i) \subseteq \sigma(S_i)$, $i = 1, 2$*

For a subnormal operator $S \in \mathcal{L}(\mathcal{H})$ and a function $f \in R(\sigma(S))$, one can associate an operator $f(S) \in \mathcal{L}(\mathcal{H})$ as follows. Let $r_n \in \mathrm{Rat}(\sigma(S))$, $n \in \mathbb{N}$, such that

$$\|f - r_n\|_{\sigma(S),\infty} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{2.1}$$

and let $N_S \in \mathcal{L}(\mathcal{K})$, where $\mathcal{K} \supset \mathcal{H}$, be the minimal normal extension of S. Since $\sigma(N_S) \subseteq \sigma(S)$, we have

$$r_n(N_S) = \begin{pmatrix} r_n(S) & S'_{12} \\ 0 & S'_{22} \end{pmatrix}, \tag{2.2}$$

and $r_n(N_S) \to f(N_S)$ in the operator norm of $\mathcal{L}(\mathcal{K})$. Therefore, $r_n(S)$ converges in the operator norm of $\mathcal{L}(\mathcal{H})$ to an operator that will be denoted by $f(S)$. It is obvious that this operator does not depend on the sequence $\{r_n\}$. In a similar way, for $f \in R(\sigma(T))$, one can define $f(T)$, when $T^* \in \mathcal{L}(\mathcal{H})$ is a subnormal operator.

Theorem 2.2. *Let $A, B^* \in \mathcal{L}(\mathscr{H})$ be subnormal operators and $X \in \mathcal{L}(\mathscr{H})$ such that $AX - XB \in \mathcal{C}_2(\mathscr{H})$, and let $\Sigma = \sigma(A) \cup \sigma(B)$ and $f \in \mathrm{Lip}(\Sigma) \cap R(\Sigma)$. Then $f(A)X - Xf(B)$ is also a Hilbert-Schmidt operator and*

$$\left\| f(A)X - Xf(B) \right\|_2 \leq L\|AX - XB\|_2, \tag{2.3}$$

where L is the Lipschitz constant of the function f.

Proof. For subnormal operators $A, B^* \in \mathcal{L}(\mathscr{H})$, according to Lemma 2.1, there exists a Hilbert space $\mathscr{K} \supset \mathscr{H}$ and there are some normal operators $N_A, N_{B^*} \in \mathcal{L}(\mathscr{K})$ such that relative to the decomposition of $\mathscr{K} = \mathscr{H} \oplus \mathscr{H}^\perp$, we have

$$N_A = \begin{pmatrix} A & A_{12} \\ 0 & A_{22} \end{pmatrix}, \qquad N_{B^*} = \begin{pmatrix} B^* & B_{12} \\ 0 & B_{22} \end{pmatrix}, \tag{2.4}$$

and $\sigma(N_A) \subseteq \sigma(A)$, $\sigma(N_{B^*}) \subseteq \sigma(B^*)$.

If we put $\widetilde{X} = X \oplus 0$ on $\mathscr{H} \oplus \mathscr{H}^\perp$, then we have $N_A\widetilde{X} - \widetilde{X}N_{B^*}^* = (AX - XB) \oplus 0$, and therefore $N_A\widetilde{X} - \widetilde{X}N_{B^*}^* \in \mathcal{C}_2(\mathscr{K})$.

For $r \in \mathrm{Rat}(\Sigma)$, where $\Sigma = \sigma(A) \cup \sigma(B)$, a simple calculation shows that

$$r(N_A) = \begin{pmatrix} r(A) & A'_{12} \\ 0 & A'_{22} \end{pmatrix}, \qquad r(N_{B^*}^*) = \begin{pmatrix} r(B) & 0 \\ B'_{21} & B'_{22} \end{pmatrix}. \tag{2.5}$$

Thus, if $f \in \mathrm{Lip}(\Sigma) \cap R(\Sigma)$, using a limiting argument, one can see that $f(N_A)$ and $f(N_{B^*}^*)$ have similar matrix representation as in (2.5), but with f replacing r. According to Theorem 1.3,

$$\begin{gathered} f(N_A)\widetilde{X} - \widetilde{X}f(N_{B^*}^*) \in \mathcal{C}_2(\mathscr{K}), \\ \left\| f(N_A)\widetilde{X} - \widetilde{X}f(N_{B^*}^*) \right\|_2 \leq L\left\| N_A\widetilde{X} - \widetilde{X}N_{B^*}^* \right\|_2. \end{gathered} \tag{2.6}$$

Since $f(N_A)\widetilde{X} - \widetilde{X}f(N_{B^*}^*) = (f(A)X - Xf(B)) \oplus 0$, the proof is finished. □

Corollary 2.3. *Let $A, B^* \in \mathcal{L}(\mathscr{H})$ be subnormal operators and $X \in \mathcal{L}(\mathscr{H})$ such that $AX - XB \in \mathcal{C}_2(\mathscr{H})$, and let $\Sigma = \sigma(A) \cup \sigma(B)$ and $f \in \mathrm{Lip}(\Sigma) \cap R(\Sigma)$. Then*

$$\left\| f(A)^*X - Xf(B)^* \right\|_2 \leq \left\| f(A)X - Xf(B) \right\|_2, \tag{2.7}$$

and thus

$$||f(A)^*X - Xf(B)^*||_2 \leq L||AX - XB||_2, \tag{2.8}$$

where L is the Lipschitz constant of the function f.

Proof. The first inequality is a consequence of Theorem 1.2 after observing that $f(A)$ and $f(B)^*$ are subnormal operators. The second inequality follows from Theorem 2.2. □

3. Hyponormal derivations

In this section, we approach the same problem, but in the case in which $A = S$, $B = T$ are hyponormal operators and the Hilbert-Schmidt class is replaced with an arbitrary norm ideal.

For a hyponormal operator $T \in \mathcal{L}(\mathcal{H})$, the analytic functional calculus can be extended to a class $A^{\alpha}(\sigma(T))$ of "pseudo-analytic" functions on $\sigma(T)$ that satisfy a certain growth condition at the boundary.

The extension of the analytic functional calculus for a hyponormal operator was introduced by Dyn'kin (cf. [7, 8]) and it also can be found in [9].

We briefly review the definition and the main tools that are necessary. Let Σ be a perfect compact set of the complex plane and let α be a positive noninteger with k its integer part, $[\alpha]$. The class $A^{\alpha}(\Sigma)$ is defined as the set of $(k+1)$ tuples of continuous functions on Σ, $(f_0, \dots, f_k) : \Sigma \to \mathbb{C}^{k+1}$ that are related by

$$f_j(z) = f_j(z_0) + \frac{f_{j+1}(z_0)}{1!}(z - z_0) + \cdots + \frac{f_k(z_0)}{(k-j)!}(z - z_0)^{k-j} + R_j(z_0, z),$$
$$\left|R_j(z_0, z)\right| \le C_j \left|z - z_0\right|^{\alpha - j} \tag{3.1}$$

for $j = 0, \dots, k$ and $z, z_0 \in \Sigma$. Since Σ is a perfect set,

$$f_j(z_0) = \lim_{z \to z_0} \frac{f_{j-1}(z) - f_{j-1}(z_0)}{z - z_0}, \quad j = 0, \dots, k-1, \tag{3.2}$$

and thus the $(k+1)$ tuple depends only on f_0. The space $A^{\alpha}(\Sigma)$, endowed with the maximum of the smallest constants that satisfy (3.1) plus the supremum norm on Σ of f_0, becomes a unital Banach algebra and is a closed subalgebra of $\operatorname{Lip}(\alpha, \Sigma)$, the algebra of Lipschitz functions of order α.

Theorem 3.1 (see [8]). *Let Σ be a perfect compact set, $f \in C(\Sigma)$, and α a positive noninteger. The following are equivalent:*

(a) $f \in A^{\alpha}(\Sigma)$;

(b) *f has an extension $F \in C^1(\mathbb{C} \setminus \Sigma)$ with $|\bar{\partial} F(z)| \le C \cdot \operatorname{dist}(z, \Sigma)^{\alpha - 1}$, $z \notin \Sigma$;*

(c) *there exists $\phi \in C_0(\mathbb{C})$ such that*

$$f(z) = \int \frac{\phi(w)}{w - z} d\mu(w), \quad z \in \Sigma, \tag{3.3}$$

and $|\phi(w)| \le C_0 \cdot \|f\|_{A^{\alpha}(\Sigma)} \cdot \operatorname{dist}(w, \Sigma)^{\alpha - 1}$, $w \in \mathbb{C}$, where μ is planar Lebesgue measure and C_0 is a constant that does not depend on f.

If $T \in \mathcal{L}(\mathcal{H})$ is a hyponormal operator, then $\|T\| = \|T\|_{\sigma}$, where $\|T\|_{\sigma}$ denotes the spectral radius of T, that is $\sup_{z \in \sigma(T)} |z|$. It is well known that if $z \notin \sigma(T)$, then $(z - T)^{-1}$ is also hyponormal and thus

$$\left\|(z - T)^{-1}\right\| = \frac{1}{\operatorname{dist}(z, \sigma(T))}. \tag{3.4}$$

Thus, for a hyponormal operator T whose spectrum $\sigma(T)$ is a perfect set and for a function $f \in A^{\alpha}(\sigma(T))$ with $\alpha > 2$, one can associate an operator defined by

$$\int \phi(w)(w-T)^{-1} d\mu(w) \tag{3.5}$$

that will be denoted by $f(T)$. The above integral does not depend on ϕ, that is, the definition of $f(T)$ is not ambiguous, and the mapping $\phi \mapsto f(T)$ acting from $A^{\alpha}(\sigma(T))$ into $\mathcal{L}(\mathcal{H})$ is a continuous, unital morphism of Banach algebras, and which extends the Riesz-Dunford calculus.

Let $(J, \|\cdot\|_J)$ be a norm ideal, that is, a proper two-sided ideal J of $\mathcal{L}(\mathcal{H})$ with a norm $\|\cdot\|_J$ that satisfies th following: $(J, \|\cdot\|_J)$ is a Banach space and $\|AXB\|_J \le \|A\|\|B\|\|X\|_J$, for all $X \in J$ and any $A, B \in \mathcal{L}(\mathcal{H})$. In particular, the Shatten-von Neumann p-classes, $\mathcal{C}_p(\mathcal{H})$, for $p \ge 1$, are instances of norm ideals.

Theorem 3.2. *Let $(J, \|\cdot\|_J)$ be a norm ideal, let $S, T \in \mathcal{L}(\mathcal{H})$ be hyponormal operators for which both $\sigma(S)$ and $\sigma(T)$ are perfect sets, let f belong to $A^{\alpha}(\Sigma)$ with $\alpha > 3$ and $\Sigma = \sigma(S) \cup \sigma(T)$, and let $X \in \mathcal{L}(\mathcal{H})$ such that $SX - XT \in J$. Then $f(S)X - Xf(T) \in J$ and*

$$\|f(S)X - Xf(T)\|_J \le C_1 \cdot \|f\|_{A^{\alpha}(\Sigma)} \cdot \|SX - XT\|_J, \tag{3.6}$$

where C_1 is a constant that depends on Σ but it does not depend on f.

Proof. For $f \in A^{\alpha}(\Sigma)$, according to Theorem 3.1, there exists $\phi \in C_0(\mathbb{C})$ such that

$$\begin{gathered} f(z) = \int \frac{\phi(w)}{w-z} d\mu(w), \quad z \in \Sigma, \\ |\phi(w)| \le C_0 \cdot \|f\|_{A^{\alpha}(\Sigma)} \cdot \operatorname{dist}(w, \Sigma)^{\alpha-1}, \quad w \in \mathbb{C}. \end{gathered} \tag{3.7}$$

Therefore,

$$f(S)X - Xf(T) = \int \phi(w)\left[(w-S)^{-1}X - X(w-T)^{-1}\right] d\mu(w). \tag{3.8}$$

The domain of integration is $\operatorname{supp}(\phi)$, which is a compact set that has in common with Σ only possibly boundary points of Σ. For $w \in \operatorname{supp}(\phi) \cap (\mathbb{C} \setminus \Sigma)$,

$$\begin{aligned} (w-S)^{-1}X - X(w-T)^{-1} &= (w-S)^{-1}\left[X(w-T) - (w-S)X\right](w-T)^{-1} \\ &= (w-S)^{-1}[SX - XT](w-T)^{-1} \in J, \end{aligned} \tag{3.9}$$

and, according to (3.4),

$$\begin{aligned} \left\|(w-S)^{-1}X - X(w-T)^{-1}\right\|_J &\le \operatorname{dist}(w, \sigma(S))^{-1} \cdot \operatorname{dist}(w, \sigma(T))^{-1} \cdot \|SX - XT\|_J \\ &\le C' \cdot \operatorname{dist}(w, \Sigma)^{-2} \cdot \|SX - XT\|_J, \end{aligned} \tag{3.10}$$

where C' is a constant that depends on Σ. Therefore, the integrant in (3.8) belongs to the norm ideal J and

$$\left\|\phi(w)(w-S)^{-1}X - X(w-T)^{-1}\right\|_J \le C_0 \cdot C' \cdot \|f\|_{A^{\alpha}(\Sigma)} \cdot \operatorname{dist}(w, \Sigma)^{\alpha-3}\|SX - XT\|_J, \tag{3.11}$$

for $w \in \operatorname{supp}(\phi) \cap (\mathbb{C} \setminus \Sigma)$. After integration one obtains the desired conclusion of the theorem. □

References

[1] G. Weiss, "The Fuglede commutativity theorem modulo the Hilbert-Schmidt class and generating functions for matrix operators. II," *Journal of Operator Theory*, vol. 5, no. 1, pp. 3–16, 1981.
[2] G. Weiss, "The Fuglede commutativity theorem modulo operator ideals," *Proceedings of the American Mathematical Society*, vol. 83, no. 1, pp. 113–118, 1981.
[3] G. Weiss, "An extension of the Fuglede commutativity theorem modulo the Hilbert-Schmidt class to operators of the form $\Sigma M_n X N_n$," *Transactions of the American Mathematical Society*, vol. 278, no. 1, pp. 1–20, 1983.
[4] T. Furuta, "An extension of the Fuglede-Putnam theorem to subnormal operators using a Hilbert-Schmidt norm inequality," *Proceedings of the American Mathematical Society*, vol. 81, no. 2, pp. 240–242, 1981.
[5] F. Kittaneh, "On Lipschitz functions of normal operators," *Proceedings of the American Mathematical Society*, vol. 94, no. 3, pp. 416–418, 1985.
[6] D. Voiculescu, "Some results on norm-ideal perturbations of Hilbert space operators," *Journal of Operator Theory*, vol. 2, no. 1, pp. 3–37, 1979.
[7] E. M. Dyn'kin, "An operator calculus based on the Cauchy-Green formula. Investigations on linear operators and the theory of functions, III," *Zapiski Naučnyh Seminarov Leningradskogo Otdelenija Matematičeskogo Instituta im. V. A. Steklova Akademii Nauk SSSR (LOMI)*, vol. 30, pp. 33–39, 1972, (Russian).
[8] E. M. Dyn'kin, "Pseudoanalytic continuation of smooth functions. Uniform scale," in *Mathematical Programming and Related Questions (Proc. Seventh Winter School, Drogobych, 1974), Theory of Functions and Functional Analysis (Russian)*, pp. 40–73, Central Èkonom.-Mat. Inst. Akad. Nauk SSSR, Moscow, Russia, 1976.
[9] M. Martin and M. Putinar, *Lectures on Hyponormal Operators*, vol. 39 of *Operator Theory: Advances and Applications*, Birkhäuser, Basel, Switzerland, 1989.