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Research Article

The Weighted Fermat Triangle Problem

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We completely solve the *generalized Fermat problem*: given a triangle P_1, P_2, P_3 and three positive numbers λ_1, λ_2, λ_3, find a point P for which the sum $\lambda_1 P_1 P + \lambda_2 P_2 P + \lambda_3 P_3 P$ is minimal. We show that the point always exists and is unique, and indicate necessary and sufficient conditions for the point to lie inside the triangle. We provide geometric interpretations of the conditions and briefly indicate a connection with dynamical systems.

1. Introduction

Pierre Fermat (1601–1665) formulated the following problem.

Given a triangle ABC, find a point P such that the sum of the three distances from P to the vertices A, B, C is minimal.

In the literature, one can find various beautiful ways to solve the problem (see, e.g., [1–4]). In short, the answer is as follows. If every angle of ABC measures less than 120°, then the point P in the interior of the triangle such that $\angle APB = \angle BPC = \angle CPA = 120^\circ$ minimizes the sum of the three distances. If one of the angles of ABC measures 120° or more, then the vertex corresponding to this angle minimizes the sum of the three distances to the vertices.

An important application is the *shortest network problem*, used in the construction of telephone, pipeline, and roadway networks; see, for example, [5].

In this paper, we consider a weighted Fermat triangle problem.

Given a triangle $P_1P_2P_3$ and given three positive numbers λ_1, λ_2, λ_3, find a point P on the triangle such that the weighted sum of the distances to the three vertices $\lambda_1 P_1 P + \lambda_2 P_2 P + \lambda_3 P_3 P$ is the least possible.

A possible application is the problem of constructing a consumer center servicing three given cities in such a way as to minimize the total distance to all three, but also making the distance to a given city inversely proportional to the population of that city. As we found out after working out a solution, this problem had been previously formulated by Greenberg and

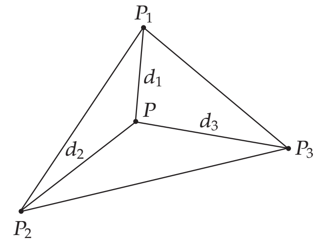

Figure 1

Robertello in 1965 [6] as the *three-factory problem* and solved using trigonometry; two subsequent papers by van de Lindt [7] and Tong and Chua [8] offered geometric solutions. There is also a higher-dimensional generalization in [9]. However, we think our approach is still of interest, firstly, because of the geometric connections explored throughout the paper, and secondly because of its accessibility. Except possibly for the last section, the paper can be understood by students who have completed the calculus sequence.

Using calculus, we obtain necessary and sufficient conditions for $\lambda_1 P_1P + \lambda_2 P_2P + \lambda_3 P_3P$ to attain its absolute minimum in the interior of the triangle $P_1P_2P_3$. We show uniqueness of such minimizing point, and present an elegant geometric construction of this point.

In the event that the absolute minimum of $\lambda_1 P_1P + \lambda_2 P_2P + \lambda_3 P_3P$ does not occur in the interior of the triangle, we show that one and only one vertex of the triangle minimizes this sum, and we locate that vertex.

In the last section, we briefly show a connection between the Fermat problem and a gradient dynamical system.

2. Existence of a minimum inside the triangle

Assuming our plane has Cartesian coordinates (x, y), let P_i have coordinates (x_i, y_i), $i = 1, 2, 3$, and let P have coordinates (x, y). Call d_i the distance between P_i and P, $i = 1, 2, 3$ (see Figure 1).

Then, the problem is to minimize the function

$$f(P) = f(x, y) = \lambda_1 d_1 + \lambda_2 d_2 + \lambda_3 d_3. \tag{2.1}$$

This function is continuous on the whole plane $\mathbb{R}^2$, so it must attain an absolute minimum on the closed triangle $P_1P_2P_3$.

Let us find the gradient of $f(P)$. First of all, we have

$$d_i^2 = (x - x_i)^2 + (y - y_i)^2, \quad i = 1, 2, 3, \tag{2.2}$$

and therefore,

$$\frac{\partial}{\partial x}(d_i)^2 = 2(x - x_i). \tag{2.3}$$

If $P \neq P_i$, then d_i itself is differentiable, in which case we have

$$2d_i \frac{\partial}{\partial x} d_i = 2(x - x_i). \tag{2.4}$$

It follows that

$$\frac{\partial}{\partial x} d_i = \frac{x - x_i}{d_i}, \quad \text{if } P \neq P_i. \tag{2.5}$$

Similarly, we get

$$\frac{\partial}{\partial y} d_i = \frac{y - y_i}{d_i}, \quad \text{if } P \neq P_i. \tag{2.6}$$

Therefore, the gradient ∇d_i of d_i is equal to

$$\nabla d_i = \frac{1}{d_i}(x - x_i, y - y_i), \quad \text{if } P \neq P_i, \text{ where } i = 1, 2, 3. \tag{2.7}$$

Let us call

$$\mathbf{u}_i = \nabla d_i = \frac{1}{d_i}(x - x_i, y - y_i) = \frac{1}{d_i}\overrightarrow{P_i P}. \tag{2.8}$$

This is a unit vector, defined for every $P \neq P_i$, $i = 1, 2, 3$.

Getting back to our function $f(P)$, we conclude that f is differentiable on the open domain

$$\Omega = \mathbb{R}^2 \setminus \{P_1, P_2, P_3\}, \tag{2.9}$$

and its gradient is equal to

$$\nabla f = \lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \lambda_3 \mathbf{u}_3 \quad \text{on } \Omega. \tag{2.10}$$

Let us see when f can have stationary points.

Lemma 2.1. *A necessary condition for ∇f to be zero (at some point in Ω) is that*

$$\begin{aligned} \lambda_1 &< \lambda_2 + \lambda_3, \\ \lambda_2 &< \lambda_3 + \lambda_1, \\ \lambda_3 &< \lambda_1 + \lambda_2. \end{aligned} \tag{2.11}$$

(Geometrically, this means that we can construct a nondegenerate triangle with sides λ_1, λ_2, λ_3.)

Indeed, $\nabla f = 0$ is equivalent to $\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \lambda_3 \mathbf{u}_3 = 0$. This, in turn, means that the polygonal curve with sides $\lambda_1 \mathbf{u}_1$, $\lambda_2 \mathbf{u}_2$, $\lambda_3 \mathbf{u}_3$ must be a triangle; see Figures 2 and 3.

Moreover, in our case the triangle cannot be degenerate, since this would imply that the three vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are parallel, which is impossible, for the points P_1, P_2, P_3 do not lie on one line.

Since $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are unit vectors, this triangle has sides λ_1, λ_2, λ_3, so these three numbers must satisfy the triangle inequalities (2.11).

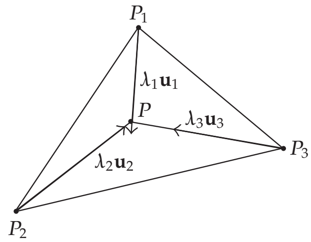

Figure 2

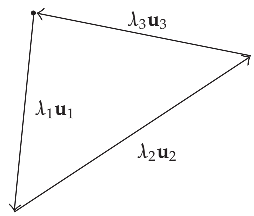

Figure 3

Lemma 2.2. *If $\nabla f(P) = 0$ at some $P \in \Omega$, then at this point one has*

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{u}_2 &= \frac{\lambda_3^2 - \lambda_1^2 - \lambda_2^2}{\lambda_1\lambda_2}, \\ \mathbf{u}_2 \cdot \mathbf{u}_3 &= \frac{\lambda_1^2 - \lambda_2^2 - \lambda_3^2}{2\lambda_2\lambda_3}, \\ \mathbf{u}_3 \cdot \mathbf{u}_1 &= \frac{\lambda_2^2 - \lambda_1^2 - \lambda_3^2}{\lambda_1\lambda_3}. \end{aligned} \tag{2.12}$$

Indeed, if $\nabla f(P) = 0$, then at P we have

$$\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 + \lambda_3\mathbf{u}_3 = 0. \tag{2.13}$$

Let us dot multiply this equality successively by $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$. Recalling that $\mathbf{u}_i \cdot \mathbf{u}_i = \|\mathbf{u}_i\|^2 = 1$, we get

$$\begin{aligned} \lambda_1 + \lambda_2\mathbf{u}_1 \cdot \mathbf{u}_2 + \lambda_3\mathbf{u}_1 \cdot \mathbf{u}_3 &= 0, \\ \lambda_1\mathbf{u}_2 \cdot \mathbf{u}_1 + \lambda_2 + \lambda_3\mathbf{u}_2 \cdot \mathbf{u}_3 &= 0, \\ \lambda_1\mathbf{u}_3 \cdot \mathbf{u}_1 + \lambda_2\mathbf{u}_3 \cdot \mathbf{u}_2 + \lambda_3 &= 0. \end{aligned} \tag{2.14}$$

To simplify matters, let us call for a moment

$$v_3 = \mathbf{u}_1 \cdot \mathbf{u}_2, \qquad v_2 = \mathbf{u}_1 \cdot \mathbf{u}_3, \qquad v_1 = \mathbf{u}_2 \cdot \mathbf{u}_3. \tag{2.15}$$

Then, the previous system looks like

$$\begin{aligned} \lambda_1 + \lambda_2 v_3 + \lambda_3 v_2 &= 0, \\ \lambda_1 v_3 + \lambda_2 + \lambda_3 v_1 &= 0, \\ \lambda_1 v_2 + \lambda_2 v_1 + \lambda_3 &= 0. \end{aligned} \tag{2.16}$$

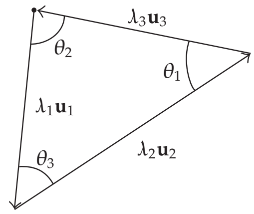

Figure 4

Let us multiply the first equation by λ_1, the second by λ_2, and subtract the second from the first:

$$\lambda_1^2 + \lambda_1\lambda_3 v_2 - \lambda_2^2 - \lambda_2\lambda_3 v_1 = 0. \tag{2.17}$$

This can be rewritten as

$$\lambda_1\lambda_3 v_2 - \lambda_2\lambda_3 v_1 = \lambda_2^2 - \lambda_1^2. \tag{2.18}$$

Let us adjoin to this equation the last equation in (2.16), previously multiplied by λ_3:

$$\begin{aligned} \lambda_1\lambda_3 v_2 - \lambda_2\lambda_3 v_1 &= \lambda_2^2 - \lambda_1^2, \\ \lambda_1\lambda_3 v_2 + \lambda_2\lambda_3 v_1 + \lambda_3^2 &= 0. \end{aligned} \tag{2.19}$$

Adding both equations in this system, and solving for v_2, we get

$$v_2 = \frac{\lambda_2^2 - \lambda_1^2 - \lambda_3^2}{\lambda_1\lambda_3}. \tag{2.20}$$

Similarly one gets

$$\begin{aligned} v_1 &= \frac{\lambda_1^2 - \lambda_2^2 - \lambda_3^2}{2\lambda_2\lambda_3}, \\ v_3 &= \frac{\lambda_3^2 - \lambda_1^2 - \lambda_2^2}{2\lambda_1\lambda_2}. \end{aligned} \tag{2.21}$$

Our result follows if we recall the notation (2.15).

Geometric interpretation of equalities (2.12)

Assume that conditions (2.11) hold, so that we can construct the triangle in Figure 3. Call θ_i the angle opposite to the side λ_i in this triangle (see Figure 4).

Then, for example,

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = \cos\left(\pi - \theta_3\right) = -\cos\ \theta_3 \tag{2.22}$$

(recall that the $\mathbf{u}_i$ are unit vectors), and the first equality in (2.12) follows from the law of the cosine. The other two equalities in (2.12) have analogous geometric interpretations.

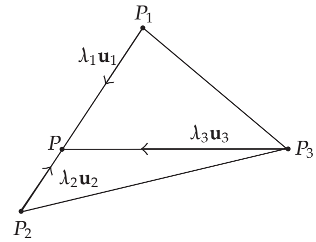

Figure 5

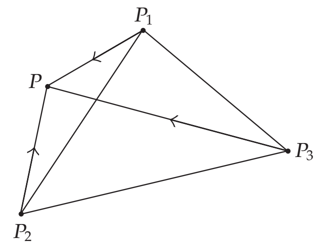

Figure 6

Lemma 2.3. *If P lies on one of the sides of the triangle $P_1P_2P_3$, but does not coincide with one of the vertices, then $\nabla f(P) \neq 0$.*

Indeed, assume, for example, that P lies on the side P_1P_2 (see Figure 5).

Then $\mathbf{u}_1$ and $\mathbf{u}_2$ are parallel. Moreover, the vectors $\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2$ and $\lambda_3\mathbf{u}_3$ are linearly independent, and at least the second one is nonzero. Therefore,

$$\nabla f(P) = \left(\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2\right) + \lambda_3\mathbf{u}_3 \neq 0. \tag{2.23}$$

Lemma 2.4. *If P lies outside the triangle $P_1P_2P_3$, then $\nabla f(P) \neq 0$.*

Indeed, if P lies outside this triangle, then it must lie on one of the half-planes whose boundary is the line joining two vertices, which does not contain the third vertex. Assume, for example, that P lies on the half-plane with the boundary through P_1P_2 which does not contain P_3 (see Figures 6 and 7).

Then, if we draw the vectors $\lambda_1\mathbf{u}_1$, $\lambda_2\mathbf{u}_2$, and $\lambda_3\mathbf{u}_3$ starting at a common origin P, all three will lie on the same half-plane with boundary being the line parallel to P_1P_2 passing through P. Since all three vectors are nonzero, so is their sum

$$\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 + \lambda_3\mathbf{u}_3 = \nabla f(P). \tag{2.24}$$

The gradient ∇f does not exist at the vertices of the triangle. However, we can compute one-sided directional derivatives at these points.

Let us start by analyzing the behavior of $d_1(P) = d_1(x, y)$ near the singular point P_1. Let us fix an arbitrary unit vector $\mathbf{n}$ and a nonzero number h. Then,

$$\frac{d_1(P_1 + h\mathbf{n}) - d_1(P_1)}{h} = \frac{d_1(P_1 + h\mathbf{n})}{h} = \frac{\|h\mathbf{n}\|}{h} = \frac{|h|}{h}. \tag{2.25}$$

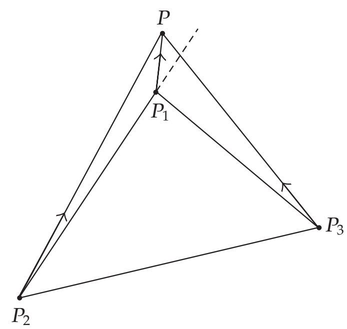

Figure 7

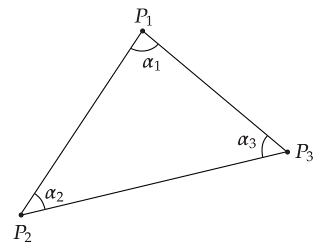

Figure 8

Therefore,

$$\begin{aligned} D_{\mathbf{n}}^{+}d_1(P_1) &\overset{\text{def}}{=} \lim_{h\to 0^{+}} \frac{d_1(P_1 + h\mathbf{n}) - d_1(P_1)}{h} = 1, \\ D_{\mathbf{n}}^{-}d_1(P_1) &\overset{\text{def}}{=} \lim_{h\to 0^{-}} \frac{d_1(P_1 + h\mathbf{n}) - d_1(P_1)}{h} = -1. \end{aligned} \tag{2.26}$$

Let us denote by α_i the angle of the triangle $P_1P_2P_3$ at P_i, where $i = 1, 2, 3$ (see Figure 8).

Lemma 2.5. *If*

$$\cos\ \alpha_1 > \frac{\lambda_1^2 - \lambda_2^2 - \lambda_3^2}{2\lambda_2\lambda_3}, \tag{2.27}$$

then the absolute minimum of the function $f(P)$ *given by* (2.1) *on the triangle* $P_1P_2P_3$ *is not attained at* P_1.

To show this, let us compute the one-sided directional derivative $D_{\mathbf{n}}^{+}f(P_1)$. Now, only the first term of $f(P)$, that is, $\lambda_1 d_1$, is not differentiable at P_1; for the other two terms, we can compute the directional derivative in the usual way. Therefore, we have

$$\begin{aligned} D_{\mathbf{n}}^{+}f(P_1) &= \lambda_1 D_{\mathbf{n}}^{+}d_1(P_1) + \lambda_2 \nabla d_2(P_1)\cdot\mathbf{n} + \lambda_3 \nabla d_3(P_1)\cdot\mathbf{n} \\ &= \lambda_1 + \left[\lambda_2\mathbf{u}_2(P_1) + \lambda_3\mathbf{u}_3(P_1)\right]\cdot\mathbf{n}. \end{aligned} \tag{2.28}$$

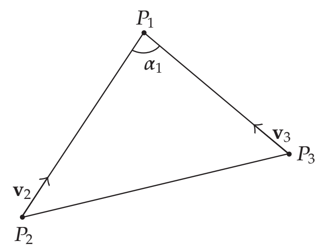

Figure 9

The smallest value of this derivative will happen when $\mathbf{n}$ is parallel to the vector $\lambda_2\mathbf{u}_2(P_1) + \lambda_3\mathbf{u}_3(P_1)$ and has the opposite direction. For such $\mathbf{n}$ we get

$$D_{\mathbf{n}}^{+} f(P_1) = \lambda_1 - \left\| \lambda_2\mathbf{u}_2(P_1) + \lambda_3\mathbf{u}_3(P_1) \right\|. \tag{2.29}$$

Let us denote

$$\mathbf{v}_2 = \mathbf{u}_2(P_1), \qquad \mathbf{v}_3 = \mathbf{u}_3(P_1); \tag{2.30}$$

these are unit vectors directed along the sides P_2P_1 and P_3P_1, respectively, as shown in Figure 9. With this notation, we have

$$D_{\mathbf{n}}^{+} f(P_1) = \lambda_1 - \left\| \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 \right\|. \tag{2.31}$$

Notice that the vector $\mathbf{n}$ we have chosen, directed opposite to $\lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3$, points towards *the interior* of the triangle $P_1P_2P_3$.

If this derivative is negative, this means that when we move from P_1 in the direction of $\mathbf{n}$, the function $f(P)$ will decrease, so that P_1 cannot be a minimum. For this derivative to be negative, we must have

$$\left\| \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 \right\|^2 > \lambda_1^2 \tag{2.32}$$

or

$$(\lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3) \cdot (\lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3) > \lambda_1^2, \tag{2.33}$$

or still

$$\lambda_2^2 + 2\lambda_2\lambda_3\mathbf{v}_2 \cdot \mathbf{v}_3 + \lambda_3^2 > \lambda_1^2. \tag{2.34}$$

This implies that

$$\mathbf{v}_2 \cdot \mathbf{v}_3 > \frac{\lambda_1^2 - \lambda_2^2 - \lambda_3^2}{2\lambda_2\lambda_3}. \tag{2.35}$$

Finally, notice that $\mathbf{v}_2 \cdot \mathbf{v}_3 = \cos\ \alpha_1$ (see Figure 9).

In a totally similar way, one can prove that if

$$\cos\ \alpha_2 > \frac{\lambda_2^2 - \lambda_3^2 - \lambda_1^2}{2\lambda_3\lambda_1}, \tag{2.36}$$

then the absolute minimum $f(P)$ on $P_1P_2P_3$ cannot be attained at P_2, and if

$$\cos\ \alpha_3 > \frac{\lambda_3^2 - \lambda_1^2 - \lambda_2^2}{2\lambda_1\lambda_2}, \tag{2.37}$$

then the absolute minimum $f(P)$ on $P_1P_2P_3$ cannot be attained at P_3.

Each of conditions (2.27), (2.36), (2.37) implies the corresponding one in (2.11) (condition (2.27) implies the first one, etc.). Indeed, assume, for example, that (2.27) holds. Then, since $\cos\ \alpha_1 < 1$, we have

$$\frac{\lambda_1^2 - \lambda_2^2 - \lambda_3^2}{2\lambda_2\lambda_3} < 1. \tag{2.38}$$

This is equivalent to

$$\lambda_1^2 - \lambda_2^2 - \lambda_3^2 < 2\lambda_2\lambda_3 \tag{2.39}$$

or

$$\lambda_1^2 < \lambda_2^2 + \lambda_3^2 + 2\lambda_2\lambda_3 = \left(\lambda_2 + \lambda_3\right)^2. \tag{2.40}$$

Since all λ_i are positive, this in turn is equivalent to

$$\lambda_1 < \lambda_2 + \lambda_3. \tag{2.41}$$

The other two inequalities are proved in a similar way.

Hence, if (2.27), (2.36), (2.37) hold, we can construct a nondegenerate triangle with sides λ_1, λ_2, λ_3, as in Figure 4. Calling, as before, θ_i the angle opposite to λ_i on this triangle, and recalling that the cosine function decreases on $[0, \pi]$, we can rewrite conditions (2.27), (2.36), (2.37) in the following very natural way:

$$\alpha_i < \pi - \theta_i, \quad \text{for } i = 1, 2, 3. \tag{2.42}$$

Lemma 2.6. *Conditions* (2.27), (2.36), (2.37) *are necessary for the existence of the absolute minimum of $f(P)$ in the interior of the triangle $P_1P_2P_3$.*

Indeed, assume, for example, that

$$\cos\ \alpha_1 \leq \frac{\lambda_1^2 - \lambda_2^2 - \lambda_3^2}{2\lambda_2\lambda_3}. \tag{2.43}$$

Pick any point P in the interior of the triangle $P_1P_2P_3$ (see Figure 10).

Then the angle $\angle P_2PP_3$ is strictly bigger than α_1. Therefore,

$$\cos\ \alpha_1 > \cos\angle P_2PP_3 = \mathbf{u}_2 \cdot \mathbf{u}_3, \tag{2.44}$$

and our assumption implies that

$$\frac{\lambda_1^2 - \lambda_2^2 - \lambda_3^2}{2\lambda_2\lambda_3} > \mathbf{u}_2 \cdot \mathbf{u}_3. \tag{2.45}$$

Then, we cannot have $\nabla f(P) = 0$ at this point, for otherwise we would get a contradiction with the second equality in (2.12).

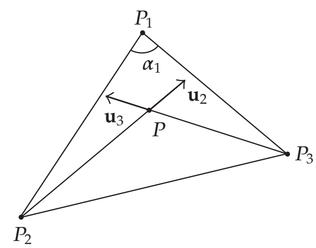

Figure 10

Theorem 2.7. *The function $f(P)$ attains its absolute minimum in the interior of the triangle $P_1P_2P_3$ if, and only if, conditions* (2.27)*,* (2.36)*,* (2.37) *hold, or, equivalently, if conditions* (2.42) *hold.*

Indeed, we know, by Lemma 2.6, that these conditions are necessary. Conversely, assume that the conditions hold.

Consider a circle C_R, with center anywhere on the triangle, and with a radius R so large that the whole triangle lies in its interior and, moreover, on the boundary of C_R the minimum of $f(P)$ is larger than, say, $f(P_1)$. (This can be achieved because $f(P)$ tends to infinity as P tends to infinity in any direction.)

Now, the continuous function $f(P)$ must attain a minimum on the compact set C_R. By our choice of R, this minimum is not on the boundary of C_R. Further, by Lemma 2.5 and the remark following it, this minimum is not attained on either vertex P_1, P_2, P_3. Since these are the only singular points of $\nabla f(P)$, it follows that the minimum must occur at a point P inside C_R at which $\nabla(P) = 0$. (This also proves that the gradient *must* vanish somewhere.) By Lemma 2.4, we conclude that the minimum must lie on the triangle $P_1P_2P_3$ (vertices excepted). Therefore, by Lemma 2.3, the minimum must lie in the interior of the triangle $P_1P_2P_3$, at a point for which $\nabla f(P) = 0$.

3. Uniqueness of the minimum

As in the classical case, the function $f(P)$ attains its absolute minimum value exactly at one point.

Theorem 3.1. *Assume that conditions* (2.27)*,* (2.36)*,* (2.37) *hold. Then $f(P)$ attains its absolute minimum value in the interior of the triangle $P_1P_2P_3$ at exactly one point.*

We already know, by Theorem 2.7, that $f(P)$ attains its minimum at some point P inside the triangle $P_1P_2P_3$.

Arguing by contradiction, assume that the minimum is also attained at some other point P' inside the triangle. Then P' must lie on one of the triangles PP_1P_2, PP_2P_3, or PP_3P_1 (possibly on one of the sides PP_i). Assume that P lies on PP_2P_3 as in Figure 11.

Since we have both $\nabla f(P) = 0$ and $\nabla f(P') = 0$, by Lemma 2.2 we must have

$$\mathbf{u}_2 \cdot \mathbf{u}_3 = \mathbf{u}_2' \cdot \mathbf{u}_3' = \frac{\lambda_1^2 - \lambda_2^2 - \lambda_3^2}{2\lambda_2\lambda_3}. \tag{3.1}$$

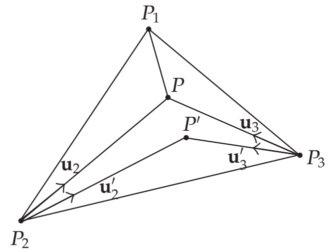

Figure 11

On the other hand, we have

$$\mathbf{u}_2 \cdot \mathbf{u}_3 = \cos \angle P_2 P P_3, \qquad \mathbf{u}_2' \cdot \mathbf{u}_3' = \cos \angle P_2 P' P_3. \tag{3.2}$$

But $\angle P_2 P' P_3$ is strictly bigger than $\angle P_2 P P_3$, so we must have $\mathbf{u}_2' \cdot \mathbf{u}_3' < \mathbf{u}_2 \cdot \mathbf{u}_3$, a contradiction.

4. Construction of the interior minimizing point

Let us assume that conditions (2.27), (2.36), (2.37), or, equivalently, conditions (2.42), are satisfied. Then, by Theorem 2.7, there is a point P at which the function $f(P)$ attains its minimum; moreover, P lies in the interior of the triangle $P_1P_2P_3$. Also, by Theorem 3.1, this point is unique.

To actually find the point, we can use a construction inspired by the one for the classical case (see, e.g., [4]).

Taking P_1P_2 as one of the sides, let us construct a triangle $P_1P_2P_3'$, as in Figure 12, which is similar to the triangle in Figure 4, with sides λ_1, λ_2, λ_3. Moreover, let us choose the angles so that the angle at P_1 is θ_1, the angle at P_2 is θ_2, and the angle at P_3' is θ_3. Further, let us draw the circumcircle to this triangle, and let O be its center.

The arc $P_1P_3'P_2$ of this circle spans the angle θ_3, so the complementary arc will span $\pi - \theta_3$.

Similarly, let us construct $P_1P_2'P_3$, also similar to the triangle with sides λ_1, λ_2, λ_3, as in Figure 12, so that the angle at P_1 is θ_1, the angle at P_2' is θ_2, and the angle at P_3 is θ_3. Let us also draw the circumcircle to $P_1P_2'P_3$, and let O' be its center.

Now, the formula for the radius of the circumscribed circle (see, e.g., [1, page 13]), applied to the triangle $P_1P_3'P_2$, yields

$$\frac{P_1P_2}{\sin\theta_3} = 2P_1O, \quad \text{whence } \frac{P_1P_2}{P_1O} = 2\sin\theta_3. \tag{4.1}$$

Applying the same result to the triangle $P_1P_3P_2'$, we obtain

$$\frac{P_1P_2'}{\sin\theta_3} = 2P_1O', \quad \text{whence } \frac{P_1P_2'}{P_1O'} = 2\sin\theta_3. \tag{4.2}$$

We conclude that

$$\frac{P_1P_2}{P_1O} = \frac{P_1P_2'}{P_1O'}. \tag{4.3}$$

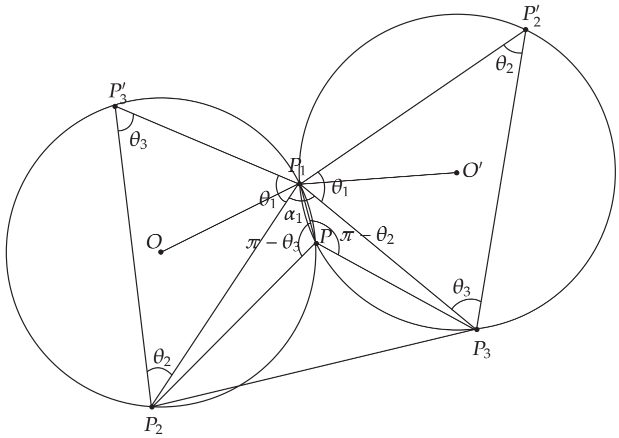

Figure 12

Hence, the isosceles triangles P_1OP_2 and $P_1O'P_2'$ are similar. Therefore, the angle $\angle OP_1P_2$ is equal to the angle $\angle O'P_1P_2'$. Hence,

$$\angle OP_1O' = \angle OP_1P_2 + \angle P_2P_1P_3 + \angle P_3P_1O' = \angle O'P_1P_2' + \alpha_1 + \angle P_3P_1O' = \alpha_1 + \theta_1. \tag{4.4}$$

Now, by our assumption (2.42), $\alpha_1 < \pi - \theta_1$, whence $\angle OP_1O' < \pi$. This guarantees that, firstly, the two circles are not tangent, and secondly, the other point P of intersection of these circles, besides P_1, will occur *inside* the triangle $P_1P_2P_3$. Indeed, from our construction it follows that $\angle P_2PP_1 = \pi - \theta_3$ and $\angle P_1PP_3 = \pi - \theta_2$. Consequently,

$$2\pi - (\pi - \theta_2) - (\pi - \theta_2) = \theta_3 + \theta_2 = \pi - \theta_1, \tag{4.5}$$

which is less than π, so P cannot lie below the line P_2P_3.

We claim that P is the desired minimizing point.

Indeed, geometrically, the fact that $\angle P_2PP_1 = \pi - \theta_3$, $\angle P_1PP_3 = \pi - \theta_2$, and $\angle P_1PP_3 = \pi - \theta_1$ guarantees that at P one can arrange the vectors $\lambda_i \mathbf{u}_i$ as in Figure 4, and therefore,

$$\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 + \lambda_3\mathbf{u}_3 = 0, \tag{4.6}$$

that is, $\nabla f(P) = 0$.

Here is an algebraic proof of the same fact. At P, we have

$$\begin{aligned}
\left\|\nabla f(P)\right\|^2 &= (\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 + \lambda_3\mathbf{u}_3)\cdot(\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 + \lambda_3\mathbf{u}_3) \\
&= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2\mathbf{u}_1\cdot\mathbf{u}_2 + 2\lambda_1\lambda_3\mathbf{u}_1\cdot\mathbf{u}_3 + 2\lambda_2\lambda_3\mathbf{u}_2\cdot\mathbf{u}_3 \\
&= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2\cos(\pi - \theta_3) + 2\lambda_1\lambda_3\cos(\pi - \theta_2) + 2\lambda_2\lambda_3\cos(\pi - \theta_1) \\
&= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2\lambda_1\lambda_2\cos\theta_3 - 2\lambda_1\lambda_3\cos\theta_2 - 2\lambda_2\lambda_3\cos\theta_1.
\end{aligned} \tag{4.7}$$

Now, by the cosine law,

$$\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 \cos\ \theta_3 = \lambda_3^2, \tag{4.8}$$

so the above expression simplifies to

$$\|\nabla f(P)\|^2 = \lambda_3^2 + \lambda_3^2 - 2\lambda_1\lambda_3\ \cos\theta_2 - 2\lambda_2\lambda_3 \cos\ \theta_1. \tag{4.9}$$

Now we add and subtract λ_1^2 and apply the cosine law twice again:

$$\begin{aligned}\|\nabla f(P)\|^2 &= \left(\lambda_3^2 + \lambda_1^2 - 2\lambda_1\lambda_3 \cos\ \theta_2\right) + \lambda_3^2 - 2\lambda_2\lambda_3 \cos\ \theta_1 - \lambda_1^2 \\ &= \left(\lambda_2^2 + \lambda_3^2 - 2\lambda_2\lambda_3 \cos\ \theta_1\right) - \lambda_1^2 \\ &= \lambda_1^2 - \lambda_1^2 = 0.\end{aligned} \tag{4.10}$$

Note. As for the classical case, it is not hard to show that actually the points P, P_3, and P_3' lie on the same line, and so do the points P, P_2, and P_2'; this provides another geometric way of constructing the minimizing point P; see [8]. Moreover, generalizing the situation in the classical case, one can see that $P_3P_3' = d/\lambda_3$ and $P_2P_2' = d/\lambda_2$, where d is the minimum of our function $f(P)$ (attained at the point P we just constructed).

5. Degenerate cases

Case 1. When one of the triangle inequalities (2.11) fails to hold then, by Lemma 2.1, the absolute minimum cannot occur in Ω, so it must happen at one of the vertices of our triangle $P_1P_2P_3$.

Assume, for example, that

$$\lambda_1 \geq \lambda_2 + \lambda_3. \tag{5.1}$$

Then, we claim that the minimum is attained at P_1.

Indeed, we have

$$f(P_2) = \lambda_1P_1P_2 + \lambda_3P_2P_3 \geq (\lambda_2 + \lambda_3)P_1P_2 + \lambda_3P_2P_3 = \lambda_2P_1P_2 + \lambda_3(P_1P_2 + P_2P_3). \tag{5.2}$$

By the triangle inequality, applied to $P_1P_2P_3$, we have $P_1P_2 + P_2P_3 > P_1P_3$. Therefore, the last expression is strictly greater than

$$\lambda_2P_1P_2 + \lambda_3P_1P_3 = f(P_1). \tag{5.3}$$

This shows that $f(P_2) > f(P_1)$. One shows analogously that $f(P_3) > f(P_1)$.

The other two possibilities of failure of (2.11) are discussed analogously; this leads to the following result.

Theorem 5.1. *If $\lambda_1 \geq \lambda_2 + \lambda_3$, then the absolute minimum of $f(P)$ is attained at P_1 and only at P_1. Similarly, if $\lambda_2 \geq \lambda_1 + \lambda_3$, the minimum is attained at P_2, and if $\lambda_3 \geq \lambda_1 + \lambda_2$, the minimum is attained at P_3.*

Case 2. Assume now that the triangle inequalities (2.11) hold, but one of the conditions (2.42) fails to hold. We claim that only one of these conditions can fail.

Indeed, if we had, say, both

$$\alpha_1 \geq \pi - \theta_1,$$
$$\alpha_2 \geq \pi - \theta_2, \tag{5.4}$$

then we would have both

$$\alpha_1 + \theta_1 \geq \pi,$$
$$\alpha_2 + \theta_2 \geq \pi. \tag{5.5}$$

Adding up, we would get

$$(\alpha_1 + \alpha_2) + (\theta_1 + \theta_2) \geq 2\pi, \tag{5.6}$$

which is impossible, since $\alpha_1 + \alpha_2 + \alpha_3 < \pi$ and $\theta_1 + \theta_2 + \theta_3 < \pi$. So, only one of the inequalities in (2.42) can fail to hold.

If we have, for example,

$$\alpha_1 \geq \pi - \theta_1, \tag{5.7}$$

then we will have both $\alpha_2 < \pi - \theta_2$ and $\alpha_3 < \pi - \theta_3$. This implies, by Lemma 2.5, the remark following it, and Theorem 2.7, that the minimum must be attained at P_1.

The other two possibilities of failure of (2.11) are discussed similarly. The following result summarizes our discussion.

Theorem 5.2. *If conditions* (2.11) *hold and* $\alpha_1 \geq \pi - \theta_1$, *then the absolute minimum of* $f(P)$ *is attained at* P_1. *Similarly, if* $\alpha_2 \geq \pi - \theta_2$, *then the minimum is attained at* P_2, *and if* $\alpha_3 \geq \pi - \theta_3$, *the minimum is attained at* P_3.

6. The classical case

The classical Fermat triangle problem happens when

$$\lambda_1 = \lambda_2 = \lambda_3. \tag{6.1}$$

Then, the triangles in Figures 3 and 4 are equilateral, and therefore

$$\theta_1 = \theta_2 = \theta_3 = 60^\circ. \tag{6.2}$$

Also, all the right-hand sides in (2.12) are equal to $-1/2$. Conditions (2.42) become

$$\alpha_i < 120^\circ \quad \text{for } i = 1, 2, 3. \tag{6.3}$$

From here one can easily deduce the classical result stated in the introduction, especially as discussed in [3].

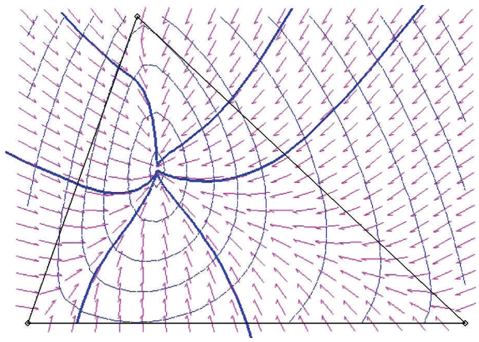

Figure 13

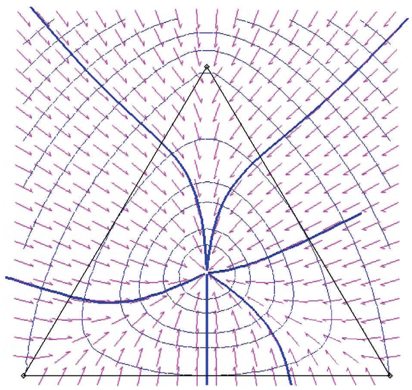

Figure 14

7. The Fermat gradient system

Assume conditions (2.42) hold. As we observed before, the gradient (2.10) of the weighted distance sum $f(x, y)$ given by (2.1) is defined in all of $\Omega = \mathbb{R}^2 \setminus \{P_1, P_2, P_3\}$. Since the function $f(x, y)$ has a global minimum at the optimal point P, the trajectories in Ω of the gradient system

$$(\dot{x}, \dot{y}) = -\nabla f(x, y) \tag{7.1}$$

will converge to the asymptotically stable equilibrium P. This follows immediately from the fact that $V(x, y) = -\|\nabla f(x, y)\|^2$ is a global Lyapunov function for the system on Ω (see, e.g., [10, Section 9.3]). Moreover, the trajectories of (7.1) are orthogonal to the level curves of the weighted sum $f(x, y)$. Figure 13 illustrates the situation for a more or less randomly chosen triangle, for the *classical* Fermat problem, when all the weights λ_i coincide. We have depicted the direction field of the gradient system plus several trajectories. The closed lines are the level curves of f.

Several intriguing questions arise. For example, when the triangle is equilateral, a single level curve will be tangent to all three sides as shown in Figure 14.

In general, under which condition(s) is there a level curve simultaneously tangent to two sides? To all three sides? For a generic triangle, and for the *generalized* Fermat problem, is it always possible to pick the weights λ_i so that a single level curve of f will be tangent to all three sides?

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