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Research Article

The Order of Generalized Hypersubstitutions of Type $\tau = (2)$

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The order of hypersubstitutions, all idempotent elements on the monoid of all hypersubstitutions of type $\tau = (2)$ were studied by K. Denecke and Sh. L. Wismath and all idempotent elements on the monoid of all hypersubstitutions of type $\tau = (2,2)$ were studied by Th. Changpas and K. Denecke. We want to study similar problems for the monoid of all generalized hypersubstitutions of type $\tau = (2)$. In this paper, we use similar methods to characterize idempotent generalized hypersubstitutions of type $\tau = (2)$ and determine the order of each generalized hypersubstitution of this type. The main result is that the order is 1,2 or infinite.

1. Introduction

The concept of generalized hypersubstitutions was introduced by Leeratanavalee and Denecke [1]. We use it as a tool to study strong hyperidentities and use strong hyperidentities to classify varieties into collections called *strong hypervarieties*. Varieties which are closed under arbitrary application of generalized hypersubstitutions are called *strongly solid*.

A generalized hypersubstitution of type $\tau = (n_i)_{i\in I}$, for short, a generalized hypersubstitution is a mapping σ which maps each n_i-ary operation symbol of type τ to the set $W_\tau(X)$ of all terms of type τ built up by operation symbols from $\{f_i \mid i \in I\}$ where f_i is n_i-ary and variables from a countably infinite alphabet of variables $X := \{x_1, x_2, x_3, \ldots\}$ which does not necessarily preserve the arity. We denote the set of all generalized hypersubstitutions of type τ by $\mathrm{Hyp}_G(\tau)$. First, we define inductively the concept of *generalized superposition of terms* $S^m : W_\tau(X)^{m+1} \to W_\tau(X)$ by the following steps:

(i) if $t = x_j,\ 1 \le j \le m$, then $S^m(x_j, t_1, \ldots, t_m) := t_j$;

(ii) if $t = x_j,\ m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \ldots, t_m) := x_j$;

(iii) if $t = f_i(s_1, \dots, s_{n_i})$, then

$$S^m(t, t_1, \dots, t_m) := f_i(S^m(s_1, t_1, \dots, t_m), \dots, S^m(s_{n_i}, t_1, \dots, t_m)). \tag{1.1}$$

We extend a generalized hypersubstitution σ to a mapping $\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$ inductively defined as follows:

(i) $\widehat{\sigma}[x] := x \in X$;

(ii) $\widehat{\sigma}[f_i(t_1, \dots, t_{n_i})] := S^{n_i}(\sigma(f_i), \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i supposed that $\widehat{\sigma}[t_j]$, $1 \le j \le n_i$ are already defined.

Then we define a binary operation $\circ_G$ on $\text{Hyp}_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in \text{Hyp}_G(\tau)$. Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \dots, x_{n_i})$. We proved the following propositions.

Proposition 1.1 (see [1]). *For arbitrary terms $t, t_1, \dots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ one has*

(i) $S^n(\widehat{\sigma}[t], \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n]) = \widehat{\sigma}[S^n(t, t_1, \dots, t_n)]$;

(ii) $(\widehat{\sigma}_1 \circ \sigma_2)^\wedge = \widehat{\sigma}_1 \circ \widehat{\sigma}_2$.

Proposition 1.2 (see [1]). *$\underline{\text{Hyp}_G(\tau)} = (\text{Hyp}_G(\tau); \circ_G, \sigma_{id})$ is a monoid and the set of all hypersubstitutions of type τ forms a submonoid of $\underline{\text{Hyp}_G(\tau)}$.*

The order of the element a is defined as the order of the cyclic subsemigroup $\langle a \rangle$. The order of any hypersubstitution of type $\tau = (2)$ was determined in [2].

Theorem 1.3 ([2]). *Let $\tau = (2)$ be a type. The order of any hypersubstitution of type τ is 1,2 or infinite.*

In Section 4, we characterize the order of generalized hypersubstitutions of type $\tau = (2)$.

2. Idempotent elements in $\text{Hyp}_G(2)$

In this section, we consider especially the idempotent elements of $\text{Hyp}_G(2)$. We have only one binary operation symbol, say f. The generalized hypersubstitution σ which maps f to the term t is denoted by σ_t. For any term $t \in W_{(2)}(X)$, the set of all variables occurring in t is denoted by $\text{var}(t)$. First, we will recall the definition of an idempotent element.

Definition 2.1. For any semigroup S, an element $e \in S$ is called idempotent if $ee = e$. In general, by $E(S)$ we denote the set of all idempotent elements of S.

Proposition 2.2. *An element $\sigma_t \in \mathrm{Hyp}_G(2)$ is idempotent if and only if $\widehat{\sigma}_t[t] = t$.*

Proof. Assume that σ_t is idempotent, that is, $\sigma_t^2 = \sigma_t$. Then

$$\widehat{\sigma}_t[t] = \widehat{\sigma}_t[\sigma_t(f)] = \sigma_t^2(f) = \sigma_t(f) = t. \tag{2.1}$$

Conversely, let $\widehat{\sigma}_t[t] = t$. We have $(\sigma_t \circ_G \sigma_t)(f) = \widehat{\sigma}_t[\sigma_t(f)] = \widehat{\sigma}_t[t] = t = \sigma_t(f)$. Thus $\sigma_t^2 = \sigma_t$. □

Proposition 2.3. *For every $x_i \in X$, σ_{x_i} and σ_{id} are idempotent.*

Proof. Since for every $x_i \in X$, $\widehat{\sigma}_{x_i}[x_i] = x_i$. By Proposition 2.2 we have σ_{x_i} is idempotent. σ_{id} is idempotent because it is a neutral element. □

Proposition 2.4. *For every $i, j \in \mathbb{N}$, the generalized hypersubstitutions $\sigma_{f(x_1,x_j)}$ and $\sigma_{f(x_i,x_2)}$ are idempotent.*

Proof. Let $i, j \in \mathbb{N}$. Then we have

$$\begin{aligned}\widehat{\sigma}_{f(x_1,x_j)}[f(x_1,x_j)] &= S^2(\sigma_{f(x_1,x_j)}(f), x_1, x_j) = S^2(f(x_1,x_j), x_1, x_j) = f(x_1,x_j),\\ \widehat{\sigma}_{f(x_i,x_2)}[f(x_i,x_2)] &= S^2(\sigma_{f(x_i,x_2)}(f), x_i, x_2) = S^2(f(x_i,x_2), x_i, x_2) = f(x_i,x_2).\end{aligned} \tag{2.2}$$

Note that for any $t \in W_{(2)}(X) \setminus X$ and $x_1, x_2 \notin \mathrm{var}(t)$, σ_t is idempotent. Since there is nothing to substitute in the term $\widehat{\sigma}_t[t]$, thus $\widehat{\sigma}_t[t] = t$. □

Proposition 2.5. *Let $t \in W_{(2)}(X) \setminus X$. Then the following propositions hold:*

(i) *if $x_2 \notin \mathrm{var}(t)$, then $\sigma_{f(x_1,t)}$ is idempotent;*

(ii) *if $x_1 \notin \mathrm{var}(t)$, then $\sigma_{f(t,x_2)}$ is idempotent.*

Proof. (i) Let $x_2 \notin \mathrm{var}(t)$. Then

$$\widehat{\sigma}_{f(x_1,t)}[f(x_1,t)] = S^2(\sigma_{f(x_1,t)}(f), x_1, \widehat{\sigma}_{f(x_1,t)}[t]) = S^2(f(x_1,t), x_1, \widehat{\sigma}_{f(x_1,t)}[t]) = f(x_1,t). \tag{2.3}$$

(ii) Let $x_1 \notin \mathrm{var}(t)$. Then

$$\widehat{\sigma}_{f(t,x_2)}[f(t,x_2)] = S^2(\sigma_{f(t,x_2)}(f), \widehat{\sigma}_{f(t,x_2)}[t], x_2) = S^2(f(t,x_2), \widehat{\sigma}_{f(t,x_2)}[t], x_2) = f(t,x_2). \tag{2.4}$$

□

3. Nonidempotent elements of $\mathrm{Hyp}_G(2)$

In this section, we characterize all elements of $\mathrm{Hyp}_G(2)$ which are not idempotent.

Proposition 3.1. *If $i, j \in \mathbb{N}$ with $i \neq 1$ and $j \neq 2$, then $\sigma_{f(x_2,x_j)}$ and $\sigma_{f(x_i,x_1)}$ are not idempotent.*

Proof. Let $i, j \in \mathbb{N}$ with $i \neq 1$ and $j \neq 2$. Since $j \neq 2$, $\widehat{\sigma}_{f(x_2,x_j)}[f(x_2,x_j)] = S^2(f(x_2,x_j), x_2, x_j) \neq f(x_2,x_j)$. Since $i \neq 1$, $\widehat{\sigma}_{f(x_i,x_1)}[f(x_i,x_1)] = S^2(f(x_i,x_1), x_i, x_1) \neq f(x_i,x_1)$. □

Proposition 3.2. *Let $t \in W_{(2)}(X) \setminus X$. Then the following propositions hold:*

(i) *if $x_2 \in \text{var}(t)$, then $\sigma_{f(x_1,t)}$ is not idempotent;*

(ii) *if $x_1 \in \text{var}(t)$, then $\sigma_{f(t,x_2)}$ is not idempotent;*

(ii) *$\sigma_{f(t,x_1)}$ and $\sigma_{f(x_2,t)}$ are not idempotent;*

(iv) *if $x_1 \in \text{var}(t)$ or $x_2 \in \text{var}(t)$, then $\sigma_{f(x_i,t)}$ and $\sigma_{f(t,x_i)}$ are not idempotent for any $i \in \mathbb{N}$ with $i > 2$.*

Proof. (i) Let $x_2 \in \text{var}(t)$. Then we have $\widehat{\sigma}_{f(x_1,t)}[f(x_1,t)] = S^2(f(x_1,t), x_1, \widehat{\sigma}_{f(x_1,t)}[t])$. Since $x_2 \in \text{var}(t)$, then we have to substitute x_2 in the term t by $\widehat{\sigma}_{f(x_1,t)}[t]$. $S^2(f(x_1,t), x_1, \widehat{\sigma}_{f(x_1,t)}[t]) \neq f(x_1,t)$.

The proofs of (ii), (iii), and (iv) are similar to (i). □

Proposition 3.3. *Let $t_1, t_2 \in W_{(2)}(X) \setminus X$. If $x_1 \in \text{var}(t_1) \cup \text{var}(t_2)$ or $x_2 \in \text{var}(t_1) \cup \text{var}(t_2)$, then $\sigma_{f(t_1,t_2)}$ is not idempotent.*

Proof. The proof is similar to that of Proposition 3.2. □

By Sections 2 and 3, we get $P_G(2) \cup E^G_{x_1} \cup E^G_{x_2} \cup G \cup \{\sigma_{\text{id}}\}$ is the set of all idempotent elements in $\text{Hyp}_G(2)$ where $P_G(2) := \{\sigma_{x_i} \in \text{Hyp}_G(2) \mid \text{i} \in \mathbb{N},\ \text{x}_\text{i} \in \text{X}\}$, $\text{E}^{\text{G}}_{\text{x}_1} := \{\text{œ}_{\text{f}(\text{x}_1,\text{s})} \in \text{Hyp}_\text{G}(2) \mid s \in W_{(2)}(X),\ x_2 \notin \text{var}(s)\}$, $E^G_{x_2} := \{\sigma_{f(s,x_2)} \in \text{Hyp}_G(2) \mid s \in W_{(2)}(X),\ x_1 \notin \text{var}(s)\}$, and $G := \{\sigma_s \in \text{Hyp}_G(2) \mid s \in W_{(2)}(X) \setminus X,\ x_1, x_2 \notin \text{var}(s)\}$.

4. The order of generalized hypersubstitutions of type $\tau = (2)$

In this section, we characterize the order of generalized hypersubstitutions of type $\tau = (2)$. First, we introduce some notations. For $s, f(c,d) \in W_{(2)}(X)$, $x_i, x_j \in X$, $i, j \in \mathbb{N}$ we denote

$vb(s)$:= the total number of variables occurring in the term s;

leftmost(s) := the first variable (from the left) that occurs in s;

rightmost(s) := the last variable that occurs in s;

$W^G_{(2)}(\{x_1\}) := \{s \in W_{(2)}(X) \mid x_1 \in \text{var}(s),\ x_2 \notin \text{var}(s)\}$,

$W^G_{(2)}(\{x_2\}) := \{s \in W_{(2)}(X) \mid x_2 \in \text{var}(s),\ x_1 \notin \text{var}(s)\}$,

$\overline{f(c,d)}$:= the term obtained from $f(c,d)$ by interchanging all occurrences of the letters x_1 and x_2, that is, $\overline{f(c,d)} = S^2(f(c,d), x_2, x_1)$ and $f(c,d) = S^2(\overline{f(c,d)}, x_2, x_1)$;

$f(c,d)'$:= the term defined inductively by $x_i' = x_i$ and $f(c,d)' = f(d', c')$;

${}_{xi}C[f(c,d)]$:= the term obtained from $f(c,d)$ by replacing each of the occurrences of the letter x_1 by x_i, that is, ${}_{xi}C[f(c,d)] = S^2(f(c,d), x_i, x_2)$;

$C_{x_i}[f(c,d)]$:= the term obtained from $f(c,d)$ by replacing each of the occurrences of the letter x_2 by x_i, that is, $C_{x_i}[f(c,d)] = S^2(f(c,d), x_1, x_i)$;

${}_{xi}C_{xj}[f(c,d)]$:= the term obtained from $f(c,d)$ by replacing each of the occurrences of the letter x_1 by x_i and the letter x_2 by x_j, that is, ${}_{xi}C_{xj}[f(c,d)] = S^2(f(c,d), x_i, x_j)$.

An element a in a semigroup S is idempotent if and only if the order of a is 1. Then we consider only the order of generalized hypersubstitutions of type $\tau = (2)$ which are not idempotent. We have the following lemmas and propositions.

Lemma 4.1. *Let $f(c,d), f(u,v) \in W_{(2)}(X)$ and $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_w$. Then $vb(w) > vb(f(c,d))$ unless $f(c,d)$ and $f(u,v)$ match one of the following 16 possibilities:*

E(1) $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_{f(c,d)}$ *where* $\sigma_{f(c,d)} \in G$;

E(2) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_1,x_1)} = \sigma_{C_{x_1}[f(c,d)]}$;

E(3) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2,x_2)} = \sigma_{{}_{x_2}C[f(c,d)]}$;

E(4) $\sigma_{f(c,d)} \circ_G \sigma_{id} = \sigma_{f(c,d)}$;

E(5) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_1,x_i)} = \sigma_{C_{x_i}[f(c,d)]}$ *where* $x_i \in X$, $i > 2$;

E(6) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2,x_1)} = \sigma_{\overline{f(c,d)}}$;

E(7) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2,x_i)} = \sigma_{{}_{x_2}C_{x_i}[f(c,d)]}$ *where* $x_i \in X$, $i > 2$;

E(8) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i,x_1)} = \sigma_{{}_{x_i}C_{x_1}[f(c,d)]}$ *where* $x_i \in X$, $i > 2$;

E(9) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i,x_2)} = \sigma_{{}_{x_i}C[f(c,d)]}$ *where* $x_i \in X$, $i > 2$;

E(10) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i,x_j)} = \sigma_{{}_{x_i}C_{x_j}[f(c,d)]}$ *where* $x_i, x_j \in X$, $i, j > 2$;

E(11) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_1,v)} = \sigma_{f(c,d)}$ *where* $v \notin X$, $f(c,d) \in W^G_{(2)}(\{x_1\})$;

E(12) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2,v)} = \sigma_{\overline{f(c,d)}}$ *where* $v \notin X$, $f(c,d) \in W^G_{(2)}(\{x_1\})$;

E(13) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i,v)} = \sigma_{{}_{x_i}C[f(c,d)]}$ *where* $x_i \in X$, $i > 2$, $v \notin X$, $f(c,d) \in W^G_{(2)}(\{x_1\})$;

E(14) $\sigma_{f(c,d)} \circ_G \sigma_{f(u,x_1)} = \sigma_{\overline{f(c,d)}}$ *where* $u \notin X$, $f(c,d) \in W^G_{(2)}(\{x_2\})$;

E(15) $\sigma_{f(c,d)} \circ_G \sigma_{f(u,x_2)} = \sigma_{f(c,d)}$ *where* $u \notin X$, $f(c,d) \in W^G_{(2)}(\{x_2\})$;

E(16) $\sigma_{f(c,d)} \circ_G \sigma_{f(u,x_i)} = \sigma_{C_{x_i}[f(c,d)]}$ *where* $x_i \in X$, $i > 2$, $u \notin X$, $f(c,d) \in W^G_{(2)}(\{x_2\})$.

Proof. Assume that $f(c,d), f(u,v) \in W_{(2)}(X)$ and $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_w$. We want to compare $vb(w)$ with $vb(f(c,d))$. From $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_w$, it follows that $w = S^2(f(c,d), \widehat{\sigma}_{f(c,d)}[u], \widehat{\sigma}_{f(c,d)}[v])$. If $\sigma_{f(c,d)} \in G$, then $w = f(c,d)$ and we have E(1). Assume that $\sigma_{f(c,d)} \notin G$. Then $x_1 \in \operatorname{var}(f(c,d))$ or $x_2 \in \operatorname{var}(f(c,d))$. We will consider the following cases.

Case 1. If $u, v \in X$, then $\widehat{\sigma}_{f(c,d)}[u] = u$ and $\widehat{\sigma}_{f(c,d)}[v] = v$. This gives 9 possible subcases:

(1) $u = v = x_1$: then we have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_1,x_1)} = \sigma_{C_{x_1}[f(c,d)]}$, which is E(2);

(2) $u = v = x_2$: then we have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2,x_2)} = \sigma_{{}_{x_2}C[f(c,d)]}$, which is E(3);

(3) $u = x_1$, $v = x_2$: then we have $\sigma_{f(c,d)} \circ_G \sigma_{\mathrm{id}} = \sigma_{f(c,d)}$, which is E(4);

(4) $u = x_1$, $v = x_i$, $i > 2$: then we have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_1,x_i)} = \sigma_{C_{x_i}[f(c,d)]}$, which is E(5);

(5) $u = x_2$, $v = x_1$: then we have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2,x_1)} = \sigma_{\overline{f(c,d)}}$, which is E(6);

(6) $u = x_2$, $v = x_i$, $i > 2$: then we have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2,x_i)} = \sigma_{{}_{x_2}C_{x_i}[f(c,d)]}$, which is E(7);

(7) $u = x_i$, $v = x_1$, $i > 2$: then we have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i,x_1)} = \sigma_{{}_{x_i}C_{x_1}[f(c,d)]}$, which is E(8);

(8) $u = x_i$, $v = x_2$, $i > 2$: then we have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i,x_2)} = \sigma_{{}_{x_i}C[f(c,d)]}$, which is E(9);

(9) $u = x_i$, $v = x_j$, $i, j > 2$: then we have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i,x_j)} = \sigma_{{}_{x_i}C_{x_j}[f(c,d)]}$, which is E(10).

Case 2. If $u = x_1$ and $v \notin X$, then $w = S^2(f(c,d), x_1, \widehat{\sigma}_{f(c,d)}[v])$. If $f(c,d) \in W^G_{(2)}(\{x_1\})$, then $w = f(c,d)$, as in E(11). Assume that $x_2 \in \operatorname{var}(f(c,d))$. Since $vb(\widehat{\sigma}_{f(c,d)}[v]) > 1$ and we have to substitute x_2 in $f(c,d)$ by $\widehat{\sigma}_{f(c,d)}[v]$, we get $vb(w) > vb(f(c,d))$.

Case 3. $u = x_2$ and $v \notin X$. In this case we get E(12) or $vb(w) > vb(f(c,d))$.

Case 4. $u = x_i, i > 2$ and $v \notin X$. In this case we get E(13) or $vb(w) > vb(f(c,d))$.

Case 5. $u \notin X$ and $v = x_1$. In this case we get E(14) or $vb(w) > vb(f(c,d))$.

Case 6. $u \notin X$ and $v = x_2$. In this case we get E(15) or $vb(w) > vb(f(c,d))$.

Case 7. $u \notin X$ and $v = x_i,\ i > 2$. In this case we get E(16) or $vb(w) > vb(f(c,d))$.

Case 8. If $u, v \notin X$, then $vb(\widehat{\sigma}_{f(c,d)}[u]) > 1$ and $vb(\widehat{\sigma}_{f(c,d)}[v]) > 1$. Since $vb(\widehat{\sigma}_{f(c,d)}[u]) > 1$, $vb(\widehat{\sigma}_{f(c,d)}[v]) > 1$ and we have to substitute x_1 in $f(c,d)$ by $\widehat{\sigma}_{f(c,d)}[u]$ or x_2 in $f(c,d)$ by $\widehat{\sigma}_{f(c,d)}[v]$, we get $vb(w) > vb(f(c,d))$. □

Lemma 4.2. *Let $s \in W_{(2)}(X) \setminus X$, $x_1, x_2 \in \text{var}(s)$, $t \in W_{(2)}(X)$ and $x_i \in X$ where $i \in \mathbb{N}$. If $x_i \in \text{var}(t)$, then $x_i \in \text{var}(\widehat{\sigma}_s[t])$.*

Proof. If $t \in X$, then $t = x_i$. So $\widehat{\sigma}_s[t] = x_i$ and thus $x_i \in \text{var}(\widehat{\sigma}_s[t])$. Assume that $t = f(t_1, t_2)$ for some $t_1, t_2 \in W_{(2)}(X)$. Then $x_i \in \text{var}(t_1)$ or $x_i \in \text{var}(t_2)$. Assume that $x_i \in \text{var}(t_1)$ and $x_i \in \text{var}(\widehat{\sigma}_s[t_1])$. Consider $\widehat{\sigma}_s[t] = \widehat{\sigma}_s[f(t_1, t_2)] = S^2(s, \widehat{\sigma}_s[t_1], \widehat{\sigma}_s[t_2])$. Since $x_1 \in \text{var}(s)$ and $x_i \in \text{var}(\widehat{\sigma}_s[t_1])$, we get $x_i \in \text{var}(\widehat{\sigma}_s[t])$. By the same way, we can show that if $x_i \in \text{var}(t_2)$, then $x_i \in \text{var}(\widehat{\sigma}_s[t])$. □

Lemma 4.3. *Let $s \in W_{(2)}(X) \setminus X$. If $x_1, x_2 \in \text{var}(s)$, then $x_1, x_2 \in \text{var}(\sigma_s^n(f))$ for all $n \in \mathbb{N}$.*

Proof. Assume that $s = f(s_1, s_2)$ for some $s_1, s_2 \in W_{(2)}(X)$. For $n = 1$, $\sigma_s^1(f) = \sigma_s(f) = s$. So $x_1, x_2 \in \text{var}(\sigma_s^1(f))$. Assume that $x_1, x_2 \in \text{var}(\sigma_s^n(f))$. Consider $\sigma_s^{n+1}(f) = (\sigma_s^n \circ_G \sigma_s)(f) = \widehat{\sigma}_s^n[\sigma_s(f)] = \widehat{\sigma}_s^n[s] = \widehat{\sigma}_s^n[f(s_1, s_2)] = S^2(\sigma_s^n(f), \widehat{\sigma}_s^n[s_1], \widehat{\sigma}_s^n[s_2])$. If $x_1, x_2 \in \text{var}(s_1)$, then by Lemma 4.2 we get $x_1, x_2 \in \text{var}(\widehat{\sigma}_s^n[s_1])$. Since $x_1 \in \text{var}(\sigma_s^n(f))$ and $x_1, x_2 \in \text{var}(\widehat{\sigma}_s^n[s_1])$, we get $x_1, x_2 \in \text{var}(\sigma_s^{n+1}(f))$. If $s_1 \in W_{(2)}^G(\{x_1\})$, then $x_2 \in \text{var}(s_2)$. By Lemma 4.2, we get $x_1 \in \text{var}(\widehat{\sigma}_s^n[s_1])$ and $x_2 \in \text{var}(\widehat{\sigma}_s^n[s_2])$. Since $x_1, x_2 \in \text{var}(\sigma_s^n(f))$, we get $x_1, x_2 \in \text{var}(\sigma_s^{n+1}(f))$. If $s_1 \in W_{(2)}^G(\{x_2\})$, then by the same proof as for the case $s_1 \in W_{(2)}^G(\{x_1\})$ we get $x_1, x_2 \in \text{var}(\sigma_s^{n+1}(f))$. If $x_1, x_2 \notin \text{var}(s_1)$, then $x_1, x_2 \in \text{var}(s_2)$. By the same proof as for the case $x_1, x_2 \in \text{var}(s_1)$, we get $x_1, x_2 \in \text{var}(\sigma_s^{n+1}(f))$. □

Lemma 4.4. *Let $s \in W_{(2)}(X) \setminus X$ and $s \in W_{(2)}^G(\{x_1\})$. If* leftmost$(s) = x_i$ *where $x_i \in X$, $i > 2$, then $x_1, x_2 \notin \text{var}(\sigma_s^2(f))$.*

Proof. Assume that $s = f(s_1, s_2)$ for some $s_1, s_2 \in W_{(2)}(X)$. Consider $\sigma_s^2(f) = (\sigma_s \circ_G \sigma_s)(f) = \widehat{\sigma}_s[\sigma_s(f)] = \widehat{\sigma}_s[s] = \widehat{\sigma}_s[f(s_1, s_2)] = S^2(s, \widehat{\sigma}_s[s_1], \widehat{\sigma}_s[s_2])$. If $s_1 \in X$, then s_1 is the leftmost variable of s, so $s_1 = x_i$. Thus $\widehat{\sigma}_s[s_1] = x_i$. Since $s \in W_{(2)}^G(\{x_1\})$, $x_1, x_2 \notin \text{var}(\widehat{\sigma}_s[s_1])$ and $\sigma_s^2(f) = S^2(s, \widehat{\sigma}_s[s_1], \widehat{\sigma}_s[s_2])$, we get $x_1, x_2 \notin \text{var}(\sigma_s^2(f))$. Assume that $s_1 = f(s_3, s_4)$ for some $s_3, s_4 \in W_{(2)}(X)$. Consider $\widehat{\sigma}_s[s_1] = \widehat{\sigma}_s[f(s_3, s_4)] = S^2(s, \widehat{\sigma}_s[s_3], \widehat{\sigma}_s[s_4])$. If $s_3 \in X$, then s_3 is the leftmost variable of s, so $s_3 = x_i$. Thus $\widehat{\sigma}_s[s_3] = x_i$. Since $s \in W_{(2)}^G(\{x_1\})$, $x_1, x_2 \notin \text{var}(\widehat{\sigma}_s[s_3])$ and $\widehat{\sigma}_s[s_1] = S^2(s, \widehat{\sigma}_s[s_3], \widehat{\sigma}_s[s_4])$, we get $x_1, x_2 \notin \text{var}(\widehat{\sigma}_s[s_1])$. This implies $x_1, x_2 \notin \text{var}(\sigma_s^2(f))$. This procedure stops after finitely many steps at leftmost$(s) = x_i$. □

Lemma 4.5. *Let $s \in W_{(2)}(X) \setminus X$. If* leftmost$(s) = x_1$, *then* leftmost$(\sigma_s^n(f)) = x_1$ *for all $n \in \mathbb{N}$.*

Proof. Assume that $s = f(s_1, s_2)$ for some $s_1, s_2 \in W_{(2)}(X)$. For $n = 1$, $\sigma_s^1(f) = \sigma_s(f) = s$. So $\text{leftmost}(\sigma_s^1(f)) = x_1$. Assume that $\text{leftmost}(\sigma_s^n(f)) = x_1$. Consider $\sigma_s^{n+1}(f) = (\sigma_s^n \circ_G \sigma_s)(f) = \widehat{\sigma}_s^n[\sigma_s(f)] = \widehat{\sigma}_s^n[s] = \widehat{\sigma}_s^n[f(s_1, s_2)] = S^2(\sigma_s^n(f), \widehat{\sigma}_s^n[s_1], \widehat{\sigma}_s^n[s_2])$. If $s_1 \in X$, then s_1 is the leftmost variable of s, so $s_1 = x_1$. Thus $\widehat{\sigma}_s^n[s_1] = x_1$. Since $\sigma_s^{n+1}(f) = S^2(\sigma_s^n(f), \widehat{\sigma}_s^n[s_1], \widehat{\sigma}_s^n[s_2])$, $\text{leftmost}(\sigma_s^n(f)) = x_1$ and $\widehat{\sigma}_s^n[s_1] = x_1$, we get $\text{leftmost}(\sigma_s^{n+1}(f)) = x_1$. Assume that $s_1 = f(s_3, s_4)$ for some $s_3, s_4 \in W_{(2)}(X)$. Consider $\widehat{\sigma}_s^n[s_1] = \widehat{\sigma}_s^n[f(s_3, s_4)] = S^2(\sigma_s^n(f), \widehat{\sigma}_s^n[s_3], \widehat{\sigma}_s^n[s_4])$. If $s_3 \in X$, then s_3 is the leftmost variable of s, so $s_3 = x_1$. Thus $\widehat{\sigma}_s^n[s_3] = x_1$. Since $\widehat{\sigma}_s^n[s_1] = S^2(\sigma_s^n(f), \widehat{\sigma}_s^n[s_3], \widehat{\sigma}_s^n[s_4])$, $\text{leftmost}(\sigma_s^n(f)) = x_1$ and $\widehat{\sigma}_s^n[s_3] = x_1$, we get $\text{leftmost}(\widehat{\sigma}_s^n[s_1]) = x_1$. This implies $\text{leftmost}(\sigma_s^{n+1}(f)) = x_1$. This procedure stops after finitely many steps at $\text{leftmost}(s) = x_1$. □

Lemma 4.6. *Let $s \in W_{(2)}(X) \setminus X$ and $s \in W_{(2)}^G(\{x_2\})$. If* $\text{rightmost}(s) = x_i$ *where $x_i \in X$, $i > 2$, then $x_1, x_2 \notin \text{var}(\sigma_s^2(f))$.*

Proof. The proof is similar to the proof of Lemma 4.4. □

Lemma 4.7. *Let $s \in W_{(2)}(X) \setminus X$. If* $\text{rightmost}(s) = x_2$, *then* $\text{rightmost}(\sigma_s^n(f)) = x_2$ *for all $n \in \mathbb{N}$.*

Proof. The proof is similar to the proof of Lemma 4.5. □

Note that $\{\sigma_{f(x_2,x_1)}^n \mid n \in \mathbb{N}\} = \{\sigma_{\text{id}}, \sigma_{f(x_2,x_1)}\}$, and that the order of $\sigma_{f(x_2,x_1)}$ is 2.

Proposition 4.8. *Let $s \in W_{(2)}(X)$, $x_1, x_2 \in \text{var}(s)$, σ_s be not idempotent and not equal to $\sigma_{f(x_2,x_1)}$. Then the order of σ_s is infinite.*

Proof. Let $n \in \mathbb{N}$. Let $\sigma_s^n(f) = w$. By Lemma 4.3, we get $x_1, x_2 \in \text{var}(w)$. Then the equation $\sigma_s^{n+1} = \sigma_s^n \circ_G \sigma_s$ does not fit any of E(1) to E(16), so by Lemma 4.1, we have that the term for σ_s^{n+1} is longer than w. This implies the order of σ_s is infinite. □

Proposition 4.9. *Let $s \in W_{(2)}^G(\{x_1\})$, and σ_s be not idempotent. If* $\text{leftmost}(s) = x_1$, *then the order of σ_s is infinite.*

Proof. Let $n \in \mathbb{N}$. Let $\sigma_s^n(f) = w$. By Lemma 4.5, we get $\text{leftmost}(w) = x_1$. Then the equation $\sigma_s^{n+1} = \sigma_s^n \circ_G \sigma_s$ does not fit any of E(1) to E(16), so by Lemma 4.1 we have that the term for σ_s^{n+1} is longer than w. This implies the order of σ_s is infinite. □

Proposition 4.10. *Let $s \in W_{(2)}^G(\{x_1\})$ and σ_s be not idempotent. If* $\text{leftmost}(s) = x_i$ *where $x_i \in X$, $i > 2$, then the order of σ_s is 2.*

Proof. Let $\sigma_s^2(f) = w$. By Lemma 4.4, we get $x_1, x_2 \notin \text{var}(w)$. This implies $\sigma_s^n = \sigma_s^2$ for all $n \in \mathbb{N}$ where $n \geq 2$. So the order of σ_s is 2. □

Proposition 4.11. *Let $s \in W_{(2)}^G(\{x_2\})$ and σ_s be not idempotent. If* $\text{rightmost}(s) = x_2$, *then the order of σ_s is infinite.*

Proof. The proof is similar to the proof of Proposition 4.9. □

Proposition 4.12. *Let $s \in W_{(2)}^G(\{x_2\})$ and σ_s be not idempotent. If* $\text{rightmost}(s) = x_i$ *where $x_i \in X$, $i > 2$, the order of σ_s is 2.*

Proof. The proof is similar to the proof of Proposition 4.10. □

Now we have the main result.

Theorem 4.13. *The order of any generalized hypersubstitution of type* $\tau = (2)$ *is 1,2 or infinite.*

Proof. Let $\sigma_t \in \text{Hyp}_G(2)$. If σ_t is idempotent, then the order of σ_t is 1. If σ_t is not idempotent, then $x_1 \in \text{var}(t)$ or $x_2 \in \text{var}(t)$. Assume that $x_1, x_2 \in \text{var}(t)$. If $\sigma_t = \sigma_{f(x_2,x_1)}$, then the order of σ_t is 2. If $\sigma_t \neq \sigma_{f(x_2,x_1)}$, then by Proposition 4.8 we get the order of σ_t is infinite. Assume that $x_1 \in \text{var}(t)$ and $x_2 \notin \text{var}(t)$. If $\text{leftmost}(t) = x_1$, then by Proposition 4.9 we get the order of σ_t is infinite. If $\text{leftmost}(t) = x_i$ where $i > 2$, then by Proposition 4.10 we get the order of σ_t is 2. By the same way we can show that if $x_2 \in \text{var}(t)$ and $x_1 \notin \text{var}(t)$, then the order of σ_t is 2 or infinite. □

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