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Research Article

Farthest Points and Subdifferential in p-Normed Spaces

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We study the farthest point mapping in a p-normed space X in virtue of subdifferential of $r(x) = \sup\{\|x-z\|^p : z \in M\}$, where M is a weakly sequentially compact subset of X. We show that the set of all points in X which have farthest point in M contains a dense G_δ subset of X.

1. Introduction

Let X be a real linear space. A quasinorm is a real-valued function on X satisfying the following conditions.

(i) $\|x\| \geq 0$ for all $x \in X$, and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in R$ and all $x \in X$.

(iii) There is a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasinormed space if $\|\cdot\|$ is a quasinorm on X. The smallest possible K is called the modules of concavity of $\|\cdot\|$. By a quasi-Banach space we mean a complete quasinormed space, that is, a quasinormed space in which every Cauchy sequence converges in X.

This class includes Banach spaces. The most significant class of quasi-Banach spaces, which are not Banach spaces, is L_p-spaces for $0 < p < 1$ equipped with the L_p-norms $\|\cdot\|_p$. A quasinorm $\|\cdot\|$ is called a p-norm $(0 < p < 1)$ if $\|x+y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$. In this case, a quasinormed (quasi-Banach) space is called a p-normed (p-Banach) space. By the Aoki-Rolewicz theorem [1], each quasinorm is equivalent to some p-norm. Since it is much easier to work with p-norms than with quasinorms, henceforth we restrict our attention mainly to p-norms. See [2–4] for more information.

If x^* is in X^*, the dual of X, and $x \in X$ we write $x^*(x)$ as $\langle x^*, x\rangle$. We also consider quasinorms with $K > 1$. The case where $K = 1$ turns out to be the classical normed spaces, so we will not discuss it and refer the interested reader to [5–7] for analogue results concerning normed spaces.

In this paper, using some strategies from [5–7], we study the farthest point mapping in a p-normed space X in virtue of subdifferential of $r(x) = \sup\{\|x - z\|^p : z \in M\}$, where M is a weakly sequentially compact subset of X. We show that the set of all points in X which have farthest point in M contains a dense G_δ subset of X.

Let X be a p-normed space and let M be a nonempty bounded subset of X. The mapping $Q_M : X \to 2^M$ defined by $Q_M(x) = \{z \in M : \|x - z\|^p = \sup_{t\in M}\|x - t\|^p\}$ is called the farthest point map of M. We call M a remotal (uniquely remotal, resp.) set if for each $x \in X$ the set $Q_M(x)$ is not empty (is singleton, resp.) [8–10].

2. Main results

Let X be a p-normed space and let M be a bounded subset of X. For each $x \in X$, we define the subdifferential of a function f at x by

$$\partial f(x) = \left\{x^* \in X^* : \operatorname{sgn}\langle x^*, y - x\rangle \left|\langle x^*, y - x\rangle\right|^p + f(x) \le f(y)\ \forall y \in X\right\}. \tag{2.1}$$

This set may be empty even if we consider X to be a Banach space [7, Example 3.8]. In a p-normed space, it may happen that $\partial r(x) \neq \varnothing$, although we should note that a p-normed space may have a trivial dual as well as it might have a nontrivial dual, see [11, Chapter 3], for some examples. To see the nonemptiness, suppose that X is a p-normed space, $x \in X$, and $M = \{x\}$. Thus, $r(x) = 0$ and obviously $0 \in \partial r(x)$. Also for each $x^* \in X^*$ with $\|x^*\| \le 1$, we have

$$\left|\langle x^*, y - x\rangle\right|^p \le \left\|x^*\right\|^p \|y - x\|^p \le \|y - x\|^p \le r(y) = r(y) - r(x) \quad (y \in X). \tag{2.2}$$

It follows that $\operatorname{sgn}\langle x^*, y - x\rangle|\langle x^*, y - x\rangle|^p + r(x) \le r(y)$ $(y \in X)$, and so $\partial r(x) \neq \varnothing$. Throughout the rest, we assume $\partial r(x) \neq \varnothing$ when we deal with this set. For an arbitrary nonempty bounded subset M of X, finding the set of all x for which $\partial r(x) \neq \varnothing$ remains an open question.

Lemma 2.1. *Let X be a p-Banach space and let M be a bounded subset in X. Then for each $x \in X$, each element of $\partial r(x)$ has norm less than or equal to 1 and hence $\partial r(x)$ is w^*-compact.*

Proof. Let $x \in X$ and $x^* \in \partial r(x)$. We have $\operatorname{sgn}\langle x^*, y - x\rangle|\langle x^*, y - x\rangle|^p + r(x) \le r(y)$ $(y \in X)$. By definition of $r(x)$ we have $|r(y) - r(x)| \le \|x - y\|^p$ for all $y \in X$ [10, 12].

Hence $\operatorname{sgn}\langle x^*, y - x\rangle|\langle x^*, y - x\rangle|^p \le \|x - y\|^p$ and therefore $\|x^*\| \le 1$. □

Note that $-r(x) \le -\|x - y\|^p \le -\operatorname{sgn}\langle x^*, x - y\rangle|\langle x^*, x - y\rangle|^p = \operatorname{sgn}\langle x^*, y - x\rangle|\langle x^*, y - x\rangle|^p$, thus $\inf_{z\in M}\operatorname{sgn}\langle x^*, z - x\rangle|\langle x^*, z - x\rangle|^p \ge -r(x)$.

Now we have the following proposition which is interesting on its own right.

Proposition 2.2. *Let X be a p-Banach space and let M be a bounded subset of X. Then the set $F = \{x \in X : \inf_{z\in M} \operatorname{sgn}\langle x^*, z - x\rangle|\langle x^*, z - x\rangle|^p > -r(x)$ for some $x^* \in \partial r(x)\}$ is of the first category in X.*

Proof. Let

$$F_n := \left\{ x \in X : \inf_{z \in M} \operatorname{sgn}\langle x^*, z - x\rangle \left|\langle x^*, z - x\rangle\right|^p \geq -r(x) + \frac{1}{n} \text{ for some } x^* \in \partial r(x) \right\}. \tag{2.3}$$

Then $F = \bigcup_{n=1}^{\infty} F_n$. We will show that for each n,

(i) F_n is a norm closed subset of X;

(ii) F_n has empty interior.

To see (i), let $\{x_m\}$ be a sequence in F_n which converges to an element x in X. For each m, choose $x_m^* \in \partial r(x_m)$ such that

$$\inf_{z \in M} \operatorname{sgn}\langle x_m^*, z - x_m\rangle \left|\langle x_m^*, z - x_m\rangle\right|^p \geq -r(x_m) + \frac{1}{n}. \tag{2.4}$$

By Lemma 2.1 $\|x_m^*\| \leq 1$ for all m. Without loss of generality, we assume that $\{x_m^*\}$ converges weak* to x^*. For every $y \in X$, we have

$$\begin{aligned} \left|\langle x_m^*, y - x_m\rangle - \langle x^*, y - x\rangle\right| &\leq \left|\langle x_m^*, y - x_m\rangle - \langle x_m^*, y - x\rangle\right| + \left|\langle x_m^*, y - x\rangle - \langle x^*, y - x\rangle\right| \\ &\leq \left\|x_m - x\right\| + \left|\langle x_m^* - x^*, y - x\rangle\right|. \end{aligned} \tag{2.5}$$

This shows that $\{\langle x_m^*, y - x_m\rangle\}_{m=1}^{\infty}$ converges to $\langle x^*, y - x\rangle$. Since $x_m^* \in \partial r(x_m)$,

$$\operatorname{sgn}\langle x_m^*, y - x_m\rangle \left|\langle x_m^*, y - x_m\rangle\right|^p + r(x_m) \leq r(y) \quad (y \in X), \tag{2.6}$$

or equivalently

$$\langle x_m^*, y - x_m\rangle \left|\langle x_m^*, y - x_m\rangle\right|^{p-1} + r(x_m) \leq r(y) \quad (y \in X). \tag{2.7}$$

It follows that

$$\langle x^*, y - x\rangle \left|\langle x^*, y - x\rangle\right|^{p-1} + r(x) \leq r(y) \quad (y \in X) \tag{2.8}$$

and hence

$$\operatorname{sgn}\langle x^*, y - x\rangle \left|\langle x^*, y - x\rangle\right|^p + r(x) \leq r(y) \quad (y \in X). \tag{2.9}$$

This shows that $x^* \in \partial r(x)$. It follows from (2.4) that

$$\operatorname{sgn}\langle x_m^*, z - x_m\rangle \left|\langle x_m^*, z - x_m\rangle\right|^p \geq -r(x_m) + \frac{1}{n} \quad (z \in M). \tag{2.10}$$

We use the fact that $\operatorname{sgn}\langle x_m^*, y - x_m\rangle |\langle x_m^*, y - x_m\rangle|^p = \langle x_m^*, y - x_m\rangle |\langle x_m^*, y - x_m\rangle|^{p-1}$ once more to obtain the inequality

$$\operatorname{sgn}\langle x^*, z - x\rangle \left|\langle x^*, z - x\rangle\right|^p \geq -r(x) + \frac{1}{n} \quad (z \in M). \tag{2.11}$$

Therefore $x \in F_n$. So F_n is a closed subset of X.

To see (ii), suppose that some F_k has nonempty interior. Then, there exists an open ball U in X of radius $\lambda(2r(y_0))^{1/p}$ for some $\lambda > 0$ and center at y_0 such that $U \subseteq F_k$. Let $\alpha = 1/\lambda^p$, $\beta = 1/(\lambda+1)^p$, and $\epsilon \leq ((1+\lambda)^p - 1)/k(\alpha+\beta+1)((1+\lambda)^p)\min\{r^{1/p}(y_0), 1\}$ and choose $z_0 \in M$ such that

$$r(y_0) - \epsilon < \|y_0 - z_0\|^p \leq r(y_0). \tag{2.12}$$

Let $x_0 = y_0 + \lambda(y_0 - z_0)$ then

$$x_0 - z_0 = (1+\lambda)(y_0 - z_0). \tag{2.13}$$

Choose $x_1 \in U \subseteq F_k$ such that

$$\|x_1 - x_0\|^p = \epsilon. \tag{2.14}$$

Then there exists $x_1^* \in \partial r(x_1)$ such that

$$\inf_{z\in M} \operatorname{sgn}\langle x_1^*, z - x_1\rangle \left|\langle x_1^*, z - x_1\rangle\right|^p \geq -r(x_1) + \frac{1}{k}. \tag{2.15}$$

We will show that

$$\operatorname{sgn}\langle x_1^*, y_0 - x_1\rangle \left|\langle x_1^*, y_0 - x_1\rangle\right|^p + r(x_1) > r(y_0). \tag{2.16}$$

This will contradict the fact that x_1^* is a subdifferential of r at x_1 and the proof would be completed. To achieve a contradiction, we will consider four cases as follows.

(i) $\operatorname{sgn}\langle x_1^*, z_0 - x_1\rangle < 0$ and $\operatorname{sgn}\langle x_1^*, y_0 - x_1\rangle > 0$.

(ii) $\operatorname{sgn}\langle x_1^*, z_0 - x_1\rangle > 0$ and $\operatorname{sgn}\langle x_1^*, y_0 - x_1\rangle > 0$.

(iii) $\operatorname{sgn}\langle x_1^*, z_0 - x_1\rangle > 0$ and $\operatorname{sgn}\langle x_1^*, y_0 - x_1\rangle < 0$.

(iv) $\operatorname{sgn}\langle x_1^*, z_0 - x_1\rangle < 0$ and $\operatorname{sgn}\langle x_1^*, y_0 - x_1\rangle < 0$.

We investigate case (i) in detail. The other cases can be studied similarly. First of all note that

$$z_0 - x_1 = z_0 - y_0 + y_0 - x_1 = \frac{y_0 - x_0}{\lambda} + y_0 - x_1 = \left(1 + \frac{1}{\lambda}\right)(y_0 - x_1) + \frac{1}{\lambda}(x_1 - x_0). \tag{2.17}$$

Now, we have

$$\begin{aligned}
r(y_0) - r(x_1) &< \|y_0 - z_0\|^p + \epsilon - r(x_1) \\
&= \frac{1}{|1+\lambda|^p}\|x_0 - z_0\|^p + \epsilon - r(x_1) \quad (\text{by } (2.13)) \\
&\leq \frac{1}{(1+\lambda)^p}\left[\|x_0 - x_1\|^p + \|x_1 - z_0\|^p\right] + \epsilon - r(x_1) \\
&= \left(1 - \frac{1}{(1+\lambda)^p}\right)(-r(x_1)) + \left(\frac{1}{(1+\lambda)^p} + 1\right)\epsilon \quad (\text{by the defination of } r(\cdot)) \\
&\leq \left(1 - \frac{1}{(1+\lambda)^p}\right)\left[\operatorname{sgn}\langle x_1^*, z_0 - x_1\rangle\left|\langle x_1^*, z_0 - x_1\rangle\right|^p - \frac{1}{k}\right] + \left(\frac{1}{(1+\lambda)^p} + 1\right)\epsilon \\
&\leq \left(1 - \frac{1}{(1+\lambda)^p}\right)\left[-\left|\langle x_1^*, z_0 - x_1\rangle\right|^p - \frac{1}{k}\right] + \left(\frac{1}{(1+\lambda)^p} + 1\right)\epsilon
\end{aligned}$$

$$\leq \left(1-\frac{1}{(1+\lambda)^p}\right)\left[-\left|\left\langle x_1^*, \frac{\lambda+1}{\lambda}(y_0-x_1)+\frac{x_1-x_0}{\lambda}\right\rangle\right|^p-\frac{1}{k}\right]$$

$$+\left(\frac{1}{(1+\lambda)^p}+1\right)\epsilon \quad (\text{by } (2.17))$$

$$\leq \left(1-\frac{1}{(1+\lambda)^p}\right)\left[-\left|\left\langle x_1^*, \frac{\lambda+1}{\lambda}(y_0-x_1)\right\rangle\right|^p+\left|\left\langle x_1^*, \frac{x_1-x_0}{\lambda}\right\rangle\right|^p-\frac{1}{k}\right]$$

$$+\left(\frac{1}{(1+\lambda)^p}+1\right)\epsilon \quad (\text{by the fact that } -|a+b|^p \leq -|a|^p+|b|^p)$$

$$\leq \left(1-\frac{1}{(1+\lambda)^p}\right)\left(\frac{\lambda+1}{\lambda}\right)^p\left[-\langle x_1^*, y_0-x_1\rangle|^p\right]+\frac{\epsilon}{\lambda^p}\left(1-\frac{1}{(1+\lambda)^p}\right)$$

$$+\left(1-\frac{1}{(1+\lambda)^p}\right)\left(-\frac{1}{k}\right)+\left(\frac{1}{(1+\lambda)^p}+1\right)\epsilon$$

$$\leq \left(1-\frac{1}{(1+\lambda)^p}\right)\left[-|\langle x_1^*, y_0-x_1\rangle|^p\right] \leq \operatorname{sgn}\langle x_1^*, y_0-x_1\rangle\left|\langle x_1^*, y_0-x_1\rangle\right|^p. \tag{2.18}$$

□

We recall that a set $G \subset X$ is said to be weakly sequentially compact if each sequence of elements of G contains a subsequence converging weakly to some element $x \in G$.

Theorem 2.3. *Let M be a weakly sequentially compact subset in a p-Banach space X. Then the set $\{x \in X : \|x-z\|^p = r(x)$ for some $z \in M\}$ contains a dense G_δ-set in X. In particular, the set of farthest points of S is nonempty.*

Proof. Let F and F_n bedefined as in Proposition 2.2 and let $D(M) = X \setminus F$. Then

$$D(M) = X \setminus \bigcup_{n\in N} F_n = \bigcap_{n\in N}(X \setminus F_n), \tag{2.19}$$

where each $X \setminus F_n$ is an open dense subset of X. Hence $D(M)$ is a dense G_δ-set in X. For each $x \in D(M)$ and $x^* \in \partial r(x)$, we have

$$\inf_{z\in M} \operatorname{sgn}\langle x^*, z-x\rangle\left|\langle x^*, z-x\rangle\right|^p = -r(x). \tag{2.20}$$

By the weak compactness of M, there exists a point $z \in M$ with $\operatorname{sgn}\langle x^*, z_0-x\rangle|\langle x^*, z_0-x\rangle|^p = -r(x)$. Hence

$$r(x) \geq \left\|x-z_0\right\|^p \geq \operatorname{sgn}\langle x^*, x-z_0\rangle\left|\langle x^*, x-z_0\rangle\right|^p = -\operatorname{sgn}\langle x^*, z_0-x\rangle\left|\langle x^*, z_0-x\rangle\right|^p = r(x). \tag{2.21}$$

This shows that $D(M) \subseteq \{x : \|x-z\|^p = r(x)$ for some $z \in M\}$. □

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