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Research Article

Second Hankel Determinant for a Class of Analytic Functions Defined by Fractional Derivative

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By making use of the fractional differential operator Ω_z^λ due to Owa and Srivastava, a class of analytic functions $\mathcal{R}_\lambda(\alpha,\rho)$ $(0 \le \rho \le 1,\ 0 \le \lambda < 1,\ |\alpha| < \pi/2)$ is introduced. The sharp bound for the nonlinear functional $|a_2a_4 - a_3^2|$ is found. Several basic properties such as inclusion, subordination, integral transform, Hadamard product are also studied.

1. Introduction

Let $\mathcal{A}$ denote the class of functions analytic in the *open* unit disc

$$\mathcal{U} := \{z : z \in \mathbb{C},\ |z| < 1\} \tag{1.1}$$

and let $\mathcal{A}_0$ be the class of functions f in $\mathcal{A}$ given by the *normalized* power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathcal{U}). \tag{1.2}$$

Also let $\mathcal{S}$, $\mathcal{S}^*(\beta)$, $\mathcal{CV}(\beta)$, and $\mathcal{K}$ denote, respectively, the subclasses of $\mathcal{A}_0$ consisting of functions which are *univalent*, *starlike* of order β, *convex* of order β (cf. [1]), and *close-to-convex* (cf. [2]) in $\mathcal{U}$. In particular, $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{CV}(0) = \mathcal{CV}$ are the familiar classes of starlike and convex functions in $\mathcal{U}$ (cf. [2]).

Given f and g in $\mathcal{A}$, the function f is said to be *subordinate* to g in $\mathcal{U}$ if there exits a function $\omega \in \mathcal{A}$ satisfying the conditions of the Schwarz Lemma such that $f(z) = g(\omega(z))$, $(z \in \mathcal{U})$. We denote the subordination by

$$f(z) \prec g(z) \quad (z \in \mathcal{U}) \text{ or } f \prec g \text{ in } \mathcal{U}. \tag{1.3}$$

It is well known [2] that if g is univalent in $\mathcal{U}$, then $f \prec g$ in $\mathcal{U}$ is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

For the functions f and g given by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n \quad (z \in \mathcal{U}), \tag{1.4}$$

their Hadamard product (or *convolution*), denoted by $f*g$, is defined by

$$(f*g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (g*f)(z) \quad (z \in \mathcal{U}). \tag{1.5}$$

Note that $f*g \in \mathcal{A}$.

By making use of the Hadamard product, Carlson-Shaffer [3] defined the linear operator $\mathcal{L}(a,c) : \mathcal{A} \to \mathcal{A}$ by

$$\left(\mathcal{L}(a,c)f\right)(z) := \Phi(a,c;z)*f(z) \quad (z \in \mathcal{U},\ f \in \mathcal{A}), \tag{1.6}$$

where

$$\Phi(a,c;z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1} \quad \left(z \in \mathcal{U},\ c \notin \mathbb{Z}_0^- = \{0\} \cup \{-1,-2,-3,\ldots\}\right) \tag{1.7}$$

and $(\lambda)_k$ is the Pochhammer symbol (or *shifted factorial*) defined in terms of the gamma function by

$$(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k=0), \\ \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+k-1) & (k \in \mathbb{N} := \{1,2,\ldots\}). \end{cases} \tag{1.8}$$

It can be readily verified that $\mathcal{L}(a,a)\ (a \notin \mathbb{Z}_0^-)$ is the identity operator; the operators $\mathcal{L}(a,b)$, $\mathcal{L}(c,d)$ commute, where $b,\ d \notin \mathbb{Z}_0^-$, that is,

$$\mathcal{L}(a,b)\mathcal{L}(c,d)f = \mathcal{L}(c,d)\mathcal{L}(a,b)f \quad (f \in \mathcal{A}), \tag{1.9}$$

and the transitive property, that is,

$$\mathcal{L}(a,b)\mathcal{L}(b,c)f = \mathcal{L}(a,c)f \quad \left(b,c \notin \mathbb{Z}_0^-,\ f \in \mathcal{A}\right), \tag{1.10}$$

holds. Each of the following definitions will also be required in our present investigation.

Definition 1.1 (cf. [4, 5], see also [6]). Let the function f be analytic in a simply connected region of the z-plane containing the origin. The *fractional derivative of f of order λ* is defined by

$$\left(D_z^{\lambda} f\right)(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \le \lambda < 1), \tag{1.11}$$

where the multiplicity of $(z-\zeta)^{\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Using Definition 1.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [5] introduced the *fractional differintegral operator* $\Omega_z^\lambda : \mathcal{A}_0 \to \mathcal{A}_0$ defined by

$$\left(\Omega_z^\lambda f\right)(z) = \Gamma(2-\lambda) z^\lambda \left(D_z^\lambda f\right)(z) \quad (\lambda \neq 2,3,\ldots,\ z \in \mathcal{U}). \tag{1.12}$$

Note that $\Omega_z^0 f(z) = f(z)$, $\Omega_z^1 f(z) = zf'(z)$, and

$$\left(\Omega_z^\lambda f\right)(z) = \left(\mathcal{L}(2,2-\lambda) f\right)(z) \quad (0 \leq \lambda < 1,\ z \in \mathcal{U}). \tag{1.13}$$

Definition 1.2 (cf. [7]). For the function f given by (1.2) and $q \in \mathbb{N} := \{1,2,3,\ldots\}$, the qth Hankel determinant of f is defined by

$$\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.14}$$

We now introduce the following class of functions.

Definition 1.3. The function $f \in \mathcal{A}_0$ is said to be in the class $\mathcal{R}_\lambda(\alpha,\rho)$ $(0 \leq \lambda < 1,\ |\alpha| < \pi/2,\ 0 \leq \rho \leq 1)$ if it satisfies the inequality

$$\Re\left\{ e^{i\alpha} \frac{\Omega_z^\lambda f(z)}{z} \right\} > \rho \cos\alpha \quad (z \in \mathcal{U}). \tag{1.15}$$

Write

$$\mathcal{R}_\lambda(0,\rho) := \mathcal{R}_\lambda(\rho). \tag{1.16}$$

Let $\mathcal{P}$ be the family of functions $p \in \mathcal{A}$ satisfying $p(0) = 1$ and $\Re(p(z)) > 0$ $(z \in \mathcal{U})$.

It follows from (1.15) that

$$f \in \mathcal{R}_\lambda(\alpha,\rho) \Longleftrightarrow e^{i\alpha} \frac{\Omega_z^\lambda f(z)}{z} = \left[(1-\rho)p(z) + \rho\right]\cos\alpha + i \sin\alpha, \tag{1.17}$$

where α is real, $|\alpha| < \pi/2$, and $p(z) \in \mathcal{P}$.

We note that

$$\begin{aligned} \mathcal{R}_0(\alpha,\rho) &:= \left\{ f \in \mathcal{A}_0 \mid \Re\left\{ e^{i\alpha} \frac{f(z)}{z} \right\} > \rho\cos\alpha \right\}, \\ \mathcal{R}_1(\alpha,\rho) &:= \left\{ f \in \mathcal{A}_0 \mid \Re\left\{ e^{i\alpha} f'(z) \right\} > \rho\cos\alpha \right\}, \end{aligned} \tag{1.18}$$

and the class $\mathcal{R}_\lambda(\rho)$ has been studied in [8].

It is well known (cf. [2]) that for $f \in \mathcal{S}$ and given by (1.2), the sharp inequality $|a_3 - a_2^2| \leq 1$ holds. This corresponds to the Hankel determinant with $q = 2$ and $n = 1$. For a given family $\mathcal{F}$ of functions in $\mathcal{A}_0$, the more general problem of finding sharp estimates for $|\mu a_2^2 - a_3|$ $(\mu \in \mathbb{R}$ or $\mu \in \mathbb{C})$ is popularly known as the Fekete-Szegö problem for $\mathcal{F}$. The Fekete-Szegö problem for the families $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{CV}$, $\mathcal{K}$ has been completely solved by many authors including [9–12].

In the present paper, we consider the Hankel determinant for $q = 2$ and $n = 2$ and we find the sharp bound for the functional $|a_2 a_4 - a_3^2|$ $(f \in \mathcal{R}_\lambda(\alpha,\rho))$. We also obtain some basic properties of the class $\mathcal{R}_\lambda(\alpha,\rho)$. Our investigation includes a recent result of Janteng et al. [13]. We also generalize some results of Ling and Ding [8].

2. Preliminaries

To establish our results, we recall the following.

Lemma 2.1 (see [2]). *Let the function $p \in \mathcal{P}$ and be given by the series*

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathcal{U}). \tag{2.1}$$

Then, the sharp estimate

$$\left|c_k\right| \leq 2 \quad (k \in \mathbb{N}) \tag{2.2}$$

holds.

Lemma 2.2 (cf. [14, page 254], see also [15]). *Let the function $p \in \mathcal{P}$ be given by the power series* (2.1). *Then,*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right) \tag{2.3}$$

for some x, $|x| \leq 1$, and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \tag{2.4}$$

for some z, $|z| \leq 1$.

Lemma 2.3 (see [16]). *Let F and G be univalent convex functions in $\mathcal{U}$. Then, the Hadamard product $F{*}G$ is also a univalent convex function in $\mathcal{U}$.*

Lemma 2.4 (see [17]). *Let F and G be univalent convex functions in $\mathcal{U}$. Also let $f \prec F$ and $g \prec G$ in $\mathcal{U}$. Then, $f{*}g \prec F{*}G$ in $\mathcal{U}$.*

Lemma 2.5 (see [16], also see [8]). *Let f and g be starlike of order $1/2$. Then, for each function $F(z)$, satisfying $\Re(F(z)) > \alpha$ $(0 \leq \alpha < 1,\ z \in \mathcal{U})$, one has*

$$\Re\left(\frac{f(z){*}F(z)g(z)}{f(z){*}g(z)}\right) > \alpha \quad (z \in \mathcal{U}). \tag{2.5}$$

Lemma 2.6 (see [8]). *Let the function $h(z) = 1 + h_1 z + h_2 z^2 + \cdots$ be univalent convex in $\mathcal{U}$. For $0 \leq \lambda < 1$ if $\Omega_z^\lambda f(z)/z \prec h(z)$ $(z \in \mathcal{U})$, then*

$$\frac{f(z)}{z} \prec \left\{\mathcal{L}(2-\lambda, 2)\left[zh(z)\right]\right\} \quad (z \in \mathcal{U}). \tag{2.6}$$

3. Main results

We prove the following.

Theorem 3.1. *Let the function f given by* (1.2) *be in the class $\mathcal{R}_\lambda(\alpha, \rho)$ $(0 \leq \lambda < 1,\ -\pi/2 < \alpha < \pi/2,$ and $0 \leq \rho \leq 1)$. Then,*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)^2\cos^2\alpha}{9}. \tag{3.1}$$

The estimate (3.1) *is sharp.*

Proof. Let $f \in \mathcal{R}_\lambda(\alpha,\rho)$ $(0 \le \lambda < 1,\ -\pi/2 < \alpha < \pi/2,$ and $0 \le \rho \le 1)$. Then, by (1.17),

$$e^{i\alpha}\frac{\Omega_z^\lambda f(z)}{z} = \left[(1-\rho)p(z)+\rho\right]\cos\alpha + i\sin\alpha \quad (z \in \mathcal{U}), \tag{3.2}$$

where $p \in \mathcal{P}$ and is given by (2.1). Using (1.6), (1.7), and (1.13), we write

$$\Omega_z^\lambda f(z) = z + \sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}a_n z^n, \quad (z \in \mathcal{U}). \tag{3.3}$$

Comparing the coefficients, we get

$$\begin{aligned} e^{i\alpha}\frac{2}{(2-\lambda)}a_2 &= (1-\rho)c_1\cos\alpha, \\ e^{i\alpha}\frac{6}{(2-\lambda)(3-\lambda)}a_3 &= (1-\rho)c_2\cos\alpha, \\ e^{i\alpha}\frac{24}{(2-\lambda)(3-\lambda)(4-\lambda)}a_4 &= (1-\rho)c_3\cos\alpha. \end{aligned} \tag{3.4}$$

Therefore, (3.4) yields

$$\left|a_2a_4 - a_3^2\right| = \frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)(\cos^2\alpha)}{12}\left|\left(\frac{(4-\lambda)c_1c_3}{4} - \frac{(3-\lambda)c_2^2}{3}\right)\right|. \tag{3.5}$$

Since the functions $p(z)$ and $p(e^{i\theta}z)$, $(\theta \in \mathbb{R})$ are members of the class $\mathcal{P}$ simultaneously, we assume without loss of generality that $c_1 > 0$. For convenience of notation, we take $c_1 = c$ $(c \in [0,2])$.

Using (2.3) along with (2.4), we get

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| &= \frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)(\cos^2\alpha)}{12} \\ &\quad \times \left|\frac{(4-\lambda)c}{16}\left\{c^3 + 2(4-c^2)cx - c(4-c^2)x^2 + 2(4-c^2)(1-|x|^2)z\right\}\right| \\ &= \frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)(\cos^2\alpha)}{48} \\ &\quad \times \left|\left(\frac{(4-\lambda)}{4} - \frac{3-\lambda}{3}\right)c^4 + \left(\frac{(4-\lambda)(4-c^2)c^2}{2} - \frac{2c^2(3-\lambda)(4-c^2)}{3}\right)x\right. \\ &\qquad \left. - \left(\frac{(4-\lambda)(4-c^2)c^2}{4} + \frac{(3-\lambda)(4-c^2)^2}{3}\right)x^2 + \frac{(4-\lambda)(4-c^2)c(1-|x|^2)z}{2}\right| \\ &= \frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)(\cos^2\alpha)}{48} \\ &\quad \times \left|\frac{\lambda c^4}{12} + \frac{\lambda(4-c^2)c^2x}{6} - \left(\frac{48-\lambda(16-c^2)}{12}\right)(4-c^2)x^2 + \frac{(4-\lambda)(4-c^2)c(1-|x|^2)z}{2}\right|. \end{aligned} \tag{3.6}$$

An application of triangle inequality and replacement of $|x|$ by μ give

$$\begin{aligned}\left|a_2a_4-a_3^2\right| &\le \frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)\left(\cos^2\alpha\right)}{48}\\
&\quad\times\left[\frac{\lambda c^4}{12}+\frac{\lambda\left(4-c^2\right)c^2\mu}{6}+\frac{\left(4-c^2\right)\left[48-\lambda\left(16-c^2\right)\right]\mu^2}{12}+\frac{(4-\lambda)\left(4-c^2\right)c}{2}\right.\\
&\qquad\left.-\frac{(4-\lambda)\left(4-c^2\right)c\mu^2}{2}\right]\\
&=\frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)\left(\cos^2\alpha\right)}{48}\\
&\quad\times\left[\frac{\lambda c^4}{12}+\frac{(4-\lambda)\left(4-c^2\right)c}{2}+\frac{\lambda\left(4-c^2\right)c^2\mu}{6}\right.\\
&\qquad\left.+\frac{\lambda\left[c^2-6(4-\lambda)c/\lambda+16(3-\lambda)/\lambda\right]\left(4-c^2\right)\mu^2}{12}\right]\\
&=\frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)\left(\cos^2\alpha\right)}{48}\\
&\quad\times\left[\frac{\lambda c^4}{12}+\frac{(4-\lambda)\left(4-c^2\right)c}{2}+\frac{\lambda\left(4-c^2\right)c^2\mu}{6}+\frac{\lambda\left(c-\beta_1\right)\left(c-\beta_2\right)\left(4-c^2\right)\mu^2}{12}\right]\\
&:=F(c,\mu)\ \text{(say)},\end{aligned}\tag{3.7}$$

where

$$\beta_1=2,\quad \beta_2=\frac{8(3-\lambda)}{\lambda},\quad 0\le c\le 2,\quad 0\le\mu\le 1.\tag{3.8}$$

We next maximize the function $F(c,\mu)$ on the closed square $[0,2]\times[0,1]$. Since

$$\frac{\partial F}{\partial\mu}=\frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)\cos^2\alpha}{48}\left[\frac{\lambda\left(4-c^2\right)c^2}{6}+\frac{\lambda\left(4-c^2\right)(c-2)(c-8(3-\lambda)/\lambda)\mu}{6}\right],\tag{3.9}$$

$c-2<0$, and $c-8(3-\lambda)/\lambda<0$, we have $\partial F/\partial\mu>0$ for $0<c<2$, $0<\mu<1$. Thus $F(c,\mu)$ cannot have a maximum in the interior of the closed square $[0,2]\times[0,1]$. Moreover, for fixed $c\in[0,2]$,

$$\max_{0\le\mu\le1}F(c,\mu)=F(c,1)=G(c)\ \text{(say)}.\tag{3.10}$$

Next,

$$G'(c)=\frac{-(1-\rho)^2(2-\lambda)^2(3-\lambda)\left(c^2-(7\lambda-12)\right)c\cos^2\alpha}{72},\tag{3.11}$$

so that $G'(c)<0$ for $0<c<2$ and has real critical point at $c=0$. Also $G(c)>G(2)$. Therefore, $\max_{0\le c\le2}$ occurs at $c=0$. Therefore, the upper bound of (3.7) corresponds to $\mu=1$ and $c=0$. Hence,

$$\left|a_2a_4-a_3^2\right|\le\frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)^2\cos^2\alpha}{9}\tag{3.12}$$

which is the assertion (3.1). Equality holds for the function

$$f(z) = \Phi(2-\lambda, 2; z) * e^{-i\alpha}\left[z\left(\frac{1+(1-2\rho)z^2}{1-z^2}\cos\alpha + i\sin\alpha\right)\right]. \tag{3.13}$$

The proof of Theorem 3.1 is complete. □

The choice of $\alpha = 0$ yields what follows.

Corollary 3.2. *Let the function f given by* (1.2) *be a member of the class $\mathcal{R}_\lambda(\rho)$. Then,*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)^2}{9}. \tag{3.14}$$

Equality holds for the function

$$f(z) = \mathcal{L}(2-\lambda, 2) * \frac{z\left(1+(1-2\rho)z^2\right)}{1-z^2}. \tag{3.15}$$

Remark 3.3. Taking $\lambda \to 1$, $\alpha = 0$, and $\rho = 0$, we get a recent result due to Janteng et al. [13].

Theorem 3.4. *Suppose $-\pi/2 < \alpha < \pi/2$, $0 \leq \rho < 1$, and $0 \leq \mu < \lambda < 1$. Then,*

$$\mathcal{R}_\lambda(\alpha, \rho) \subset \mathcal{R}_\mu(\alpha, \rho). \tag{3.16}$$

Proof. Let

$$f \in \mathcal{R}_\lambda(\alpha, \rho) \quad \left(0 \leq \mu < \lambda < 1, \ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}, \ 0 \leq \rho \leq 1\right). \tag{3.17}$$

Using the associative and commutative properties of the operator $\mathcal{L}$, we write

$$\begin{aligned} \Omega_z^\mu f(z) &= \mathcal{L}(2, 2-\mu) f(z) \\ &= \mathcal{L}(2-\lambda, 2)\mathcal{L}(2, 2-\lambda)\mathcal{L}(2, 2-\mu) f(z) \\ &= \mathcal{L}(2-\lambda, 2-\mu)\Omega_z^\lambda f(z) \\ &= \Phi(2-\lambda, 2-\mu; z) * \Omega_z^\lambda f(z), \end{aligned} \tag{3.18}$$

where the function Φ is defined by (1.7). Therefore,

$$\begin{aligned} \frac{e^{i\alpha}\Omega_z^\mu f(z)}{z} &= \frac{\Phi(2-\lambda, 2-\mu; z) * \left(e^{i\alpha}\Omega_z^\lambda f(z)/z\right) \cdot z}{\Phi(2-\lambda, 2-\mu; z) * z} \\ &= \frac{f(z) * F(z) g(z)}{f(z) g(z)}, \end{aligned} \tag{3.19}$$

where $f(z) = \Phi(2-\lambda, 2-\mu; z)$, $g(z) = z$, $F(z) = e^{i\alpha}\Omega_z^\lambda f(z)/z$. We note that $g \in \mathcal{S}^*(1/2)$, and $\Re(F(z)) > \rho\cos\alpha \, (0 \leq \rho \leq 1, \ -\pi/2 < \alpha < \pi/2)$. Moreover, it is well known (cf. [18]) that

$\Phi(2-\lambda, 2-\mu; z) \in \mathcal{S}^*(1/2)$. Therefore, by Lemma 2.5,

$$\Re\left(\frac{e^{i\alpha}\Omega_z^{\mu} f(z)}{z}\right) > \rho\cos\alpha \quad \left(-\frac{\pi}{2} < \alpha < \frac{\pi}{2},\ z \in \mathcal{U},\ 0 \le \rho \le 1\right). \tag{3.20}$$

Hence, $f(z) \in \mathcal{R}_{\mu}(\alpha, \rho)$, and the proof of Theorem 3.4 is complete. □

Theorem 3.5. *Let* $f \in \mathcal{S}^*(1/2)$ *and* $g \in \mathcal{R}_{\lambda}(\alpha, \rho)$ $(0 \le \rho \le 1,\ -\pi/2 < \alpha < \pi/2,\ 0 \le \lambda < 1)$. *Then the Hadamard product*

$$f * g \in \mathcal{R}_{\lambda}(\alpha, \rho). \tag{3.21}$$

Proof. Since the Hadamard product is associative and commutative, we have

$$\Omega_z^{\lambda}(f * g)(z) = f(z) * \Omega_z^{\lambda} g(z). \tag{3.22}$$

Therefore,

$$\frac{e^{i\alpha}\Omega_z^{\lambda}(f * g)(z)}{z} = \frac{f(z) * \left(e^{i\alpha}\Omega_z^{\lambda} g(z)/z\right) \cdot z}{f(z) * z}. \tag{3.23}$$

Now applying Lemma 2.5, we get

$$\Re\left(\frac{e^{i\alpha}\Omega_z^{\lambda}(f * g)(z)}{z}\right) > \rho\cos\alpha. \tag{3.24}$$

Hence, $f * g \in \mathcal{R}_{\lambda}(\alpha, \rho)$, and the proof of Theorem 3.5 is complete. □

Theorem 3.6. *Let* $f \in \mathcal{R}_{\lambda}(\alpha, \rho)$ $(0 \le \lambda < 1,\ -\pi/2 < \alpha < \pi/2,\ 0 \le \rho \le 1)$. *Then, the function* $\mathcal{I}(f)$ *defined by the integral transform*

$$\mathcal{I}(f)(z) = \frac{\gamma + 1}{z^{\gamma}} \int_0^z t^{\gamma - 1} f(t)\,dt \quad (z \in \mathcal{U},\ \gamma > -1) \tag{3.25}$$

is also in $\mathcal{R}_{\lambda}(\alpha, \rho)$.

Proof. The Integral transform $\mathcal{I}(f)$ can be written in terms of Carlson-Shaffer operator as

$$(\mathcal{I}(f))(z) = (\mathcal{L}(\gamma + 1, \gamma + 2) f)(z). \tag{3.26}$$

Hence,

$$(\Omega_z^{\lambda}\mathcal{I}(f))(z) = \mathcal{L}(\gamma + 1, \gamma + 2)\Omega_z^{\lambda} f(z) = \Phi(\gamma + 1, \gamma + 2; z) * \Omega_z^{\lambda} f(z). \tag{3.27}$$

Therefore,

$$\frac{e^{i\alpha}(\Omega_z^{\lambda}\mathcal{I}(f))(z)}{z} = \frac{\Phi(\gamma + 1, \gamma + 2; z) * \left(e^{i\alpha}\Omega_z^{\lambda} f(z)/z\right) z}{\Phi(\gamma + 1, \gamma + 2; z) * z}. \tag{3.28}$$

Using a result of Bernardi [19], it can be verified that $\Phi(\gamma + 1, \gamma + 2; z) \in \mathcal{S}^*(1/2)$. Thus by applying Lemma 2.5, the proof of Theorem 3.6 is complete. □

Theorem 3.7. *Let* $f \in \mathcal{R}_\lambda(\alpha,\rho)$, $(0 \le \lambda < 1,\ -\pi/2 < \alpha < \pi/2,\ 0 \le \rho \le 1)$. *Then,*

$$\frac{f(z)}{z} \prec \mathcal{G}(z) \quad (z \in \mathcal{U}), \tag{3.29}$$

where

$$\mathcal{G}(z) = \frac{e^{-i\alpha}}{z}\left\{\Phi(2-\lambda,2;z)*\left[zh(z)\right]\right\},$$
$$h(z) = \left(\frac{1+(1-2\rho)z}{1-z}cos\,\alpha + i\sin\alpha\right), \tag{3.30}$$

and Φ *is defined by* (1.7). *Moreover,* $\mathcal{G}$ *is a univalent convex function in* $\mathcal{U}$.

Proof. Since $\Omega_z^\lambda f(z)/z \prec e^{-i\alpha}h(z)$, by an application of Lemma 2.6, we get

$$\frac{f(z)}{z} \prec \frac{e^{-i\alpha}}{z}\left\{\mathcal{L}(2-\lambda,2)*\left[zh(z)\right]\right\} = \mathcal{G}(z). \tag{3.31}$$

The assertion (3.29) is proved.

It is well known (cf. [18]) that $\Phi(2-\lambda,2;z)/z$ is a univalent convex function. Therefore, by Lemma 2.4, $\mathcal{G}(z)$ is univalent convex function. □

Remark 3.8. For $\alpha = 0$, Theorem 3.7(i) gives a result of Ling and Ding [8, Theorem 2].

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