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Research Article

Joint Spectra of Generators in Topological Algebras

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Let $\mathcal{A}$ be a complex topological algebra with unit 1 and $\mathcal{U}$ a family of proper closed ideals in $\mathcal{A}$. For an arbitrary $S \subset \mathcal{A}$ we define a globally defined joint spectrum $\sigma_{\mathcal{U}}(S) = \{(\lambda_s)_{s\in S} \in \mathbb{C}^S \mid \exists\, I \in \mathcal{U}(s-\lambda_s) \in I\ \forall s \in S\}$. We prove that for S generating $\mathcal{A}$ the spectrum $\sigma_{\mathcal{U}}(S)$ can be identified with the set $\mathfrak{M}_{\mathcal{U}}$ of continuous multiplicative functionals f on $\mathcal{A}$ such that $\ker f \in \mathcal{U}$. The relation is given by the formula $\sigma_{\mathcal{U}}(S) = \{(f(s))_{s\in S} \mid f \in \mathfrak{M}_{\mathcal{U}}\}$. If $\mathcal{A}$ is a Q-algebra, the set $\sigma_{\mathcal{U}}(S)$ is rationally convex.

1. Introduction

In this paper, we mean by a complex topological algebra a complex Hausdorff topological vector space which is an associative algebra with separately continuous multiplication. Let $\mathcal{A}$ be a complex topological algebra with unit denoted by 1.

We denote $\sigma(a) = \{\mu \in \mathbb{C} \mid a-\lambda \text{ is not invertible in } \mathcal{A}\}$ which is the usual spectrum in $\mathcal{A}$.

Denote by $\mathcal{P}$ the free algebra of k noncommutative indeterminants and unit 1. The elements of $\mathcal{P}$ will be called polynomials.

Let $\mathfrak{M}(\mathcal{A})$ be the space of multiplicative linear continuous functionals on $\mathcal{A}$.

In the case of a finitely generated topological algebra

$$\mathcal{A} = \mathcal{A}(x_1,\dots,x_k), \tag{1.1}$$

it was proved by Sołtysiak [1] that the topological left joint spectrum of generators

$$\sigma_l^{(t)}(x_1,\dots,x_k) = \left\{(\mu_1,\dots,\mu_k) \in \mathbb{C}^k \mid 1 \notin \operatorname{cl} \sum_{i=1}^{k} \mathcal{A}(x_i - \mu_i)\right\} \tag{1.2}$$

can be identified with $\mathfrak{M}(\mathcal{A})$ because

$$\sigma_l^{(t)}(x_1,\dots,x_k) = \{(f(x_1),\dots,f(x_k)) \mid f \in \mathfrak{M}(\mathcal{A})\}. \tag{1.3}$$

The right topological joint spectrum $\sigma_r^{(t)}$ defined in the analogous way has the same description, hence $\sigma_l^{(t)} = \sigma_r^{(t)}$.

Paper [1] is a response of the author to the article of Żelazko [2], where the analogous result is proved for a commutative algebra $\mathcal{A}$.

This phenomenon is of much more general character. The supposition of finitness of the set of generators can be removed and other globally defined joint spectra of generators can be interpreted in terms of properly chosen sets of multiplicative functionals.

Let $\mathcal{A}$ be a complex unital topological algebra. Let $\mathcal{U}$ be a family of left ideals in $\mathcal{A}$. For an arbitrary set $S \subset \mathcal{A}$ consider

$$\sigma_{\mathcal{U}}(S) = \{(\lambda_s)_{s\in S} \in \mathbb{C}^S \mid \exists I \in \mathcal{U},\ (s-\lambda_s) \in I\ \forall s \in S\}. \tag{1.4}$$

If for an arbitrary family of left ideals $\mathcal{U}$ we define $\mathcal{U}'$ as the family of all subideals of all elements of $\mathcal{U}$, then $\sigma_{\mathcal{U}'} = \sigma\mathcal{U}$.

By $\mathfrak{M}_{\mathcal{U}}$ we denote the set of continuous multiplicative functionals φ on $\mathcal{A}$ such that $\ker\varphi \in \mathcal{U}$.

The algebra $\mathcal{A}$ is generated by $S \subset \mathcal{A}$ if the smallest closed subalgebra of $\mathcal{A}$ containing S and 1 is equal to $\mathcal{A}$. This being the case, we denote $\mathcal{A} = \mathcal{A}(S)$.

2. The principal result

Theorem 2.1. *Let $\mathcal{A} = \mathcal{A}(S)$ be a unital topological algebra. Let $\mathcal{U}$ be an arbitrary family of closed left ideals in $\mathcal{A}$. Then,*

$$\sigma_{\mathcal{U}}(S) = \{(\varphi(s))_{s\in S} \mid \varphi \in \mathfrak{M}_{\mathcal{U}}\}. \tag{2.1}$$

Proof. Let $\varphi \in \mathfrak{M}_{\mathcal{U}}$. Thus, $s - \varphi(s) \in \ker\varphi$. By supposition the two-sided ideal $\ker\varphi$ belongs to $\mathcal{U}$, so the function $(\varphi(s))_{s\in S}$ is an element of $\sigma_{\mathcal{U}}(S)$.

Now, suppose that $(\lambda_s)_{s\in S} \in \sigma_{\mathcal{U}}(S)$. We define a functional on the dense subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ generated by S and 1. Every element of $\mathcal{A}_0$ is of the form $p(s_1,\dots,s_k)$, where $p \in \mathcal{P}$ and $s_1,\dots,s_k \in S$.

Let

$$\varphi_0(p(s_1,\dots,s_k)) = p(\lambda_{s_1},\dots,\lambda_{s_k}). \tag{2.2}$$

First of all, it is necessary to prove that this definition is correct. According to the remainder theorem (see [3]), for every polynomial p there exist polynomials q_i such that

$$p(s_1,\dots,s_k) - p(\lambda_{s_1},\dots,\lambda_{s_k}) = \sum_{i=1}^{k} q_i(s_1,\dots,s_k)(s_i - \lambda_{s_i}). \tag{2.3}$$

If we suppose $p(s_1,\dots,s_k) = 0$, we obtain

$$-p(\lambda_{s_1},\dots,\lambda_{s_k}) = \sum_{i=1}^{k} q_i(s_1,\dots,s_k)(s_i - \lambda_{s_i}). \tag{2.4}$$

The right-hand side is an element of the ideal generated by the elements $s - \lambda_s$, $s \in S$, which by supposition belongs to some element I of $\mathcal{U}$, so in particular it is a proper ideal.

The left-hand side is proportional to unit 1, hence, it must be zero. The definition of φ_0 is correct. Visibly φ_0 is multiplicative.

By the same remainder formula, the kernel of φ_0 consists of elements of the form $\sum_{i=1}^{k} q_i(s_1,\dots,s_k)(s_i - \lambda_{s_i})$ that belong to the same closed ideal $I \in \mathcal{U}$. The kernel of φ_0 is not dense in A_0, so it is continuous and it extends to a multiplicative continuous functional on $\mathcal{A}$ denoted by φ. Every $x \in \mathcal{A}$ decomposes as $x = (x - \varphi(x)) + \varphi(x)$, which leads to the decompositions

$$\mathcal{A}_0 = \ker\varphi_0 + \mathbb{C}\cdot 1 \subset \ker\varphi + \mathbb{C}\cdot 1 = \mathcal{A} \tag{2.5}$$

which are the topological direct sums. Since $\mathcal{A}_0$ is dense in $\mathcal{A}$ and $\ker\varphi_0 \subset \ker\varphi$, it follows that $\ker\varphi_0$ is dense in $\ker\varphi$. It follows that $\ker\varphi \subset I$ and in fact they coincide. Hence, φ belongs to $\mathfrak{M}_{\mathcal{U}}$. □

Theorem 2.1 implies that $\sigma_{\mathcal{U}}(S) \neq \emptyset$ if and only if the family of ideals $\mathcal{U}$ contains two-sided ideals of codimension 1, although we have supposed only that $\mathcal{U}$ consists of left ideals.

The theorem permits us to identify $\sigma_{\mathcal{U}}(S)$ with $\mathfrak{M}_{\mathcal{U}} \subset \mathfrak{M}(\mathcal{A})$. In particular, for an arbitrary generating set $S \subset \mathcal{A}$ we obtain "the same" $\sigma_{\mathcal{U}}(S)$ identified with $\mathfrak{M}_{\mathcal{U}}$. Obviously, the result is more interesting for small generating sets S.

There are many examples which show that this theorem, although proved principally with the arguments used in [1, 2], is not only a formal amusement.

(1) In the case of a topological unital algebra $\mathcal{A}$, by choosing as $\mathcal{U}$ the family of all closed ideals in $\mathcal{A}$, we obtain for an arbitrary generating set $S \subset \mathcal{A}$ the identification

$$\sigma_l^{(t)}(S) = \mathfrak{M}(\mathcal{A}). \tag{2.6}$$

In particular for $S = \mathcal{A}$, Theorem 2.1 provides the description of $\mathfrak{M}(\mathcal{A})$ as of the set of functions $f : \mathcal{A} \to \mathbb{C}$ such that the left ideal generated by the elements $s - f(s)$, $s \in S$, is not dense in $\mathcal{A}$.

(2) If we take $\mathcal{U} = \{J\}$ a single ideal, then the condition $\sigma_{\mathcal{U}}(S)$ is satisfied for some generating set $S \subset \mathcal{A}$ if and only if J is a two-sided ideal of codimension 1.

(3) Also in the general case, let $\mathcal{U}$ be the family of all closed finitely generated left ideals. If for some generating set $S \subset \mathcal{A}$, we have $\sigma_{\mathcal{U}}(S) \neq \emptyset$, it follows by Theorem 2.1 that $\mathcal{A}$ contains finitely generated ideals of codimension 1 which are elements of $\mathcal{U}$.

(4) If we assume that $\mathcal{A}$ is a Q-algebra, we obtain the same results for the usual left joint spectrum σ_l in place of $\sigma_l^{(t)}$.

(5) Consider a commutative Banach algebra $\mathcal{A}$ and take $\mathcal{U}$ equal to the set of all ideals consisting of joint topological zero divisors. The corresponding joint spectrum $\sigma_{\mathcal{U}}$ is the aproximate point spectrum τ. In this case, according to Theorem 2.1, for an arbitrary generating set $S \subset \mathcal{A}$,

$$\tau(S) = \text{ cortex } (\mathcal{A}). \tag{2.7}$$

(6) Also in the case of acommutative Banach algebra we can consider $\mathcal{U}$ to be the set of all ideals in $\mathcal{A}$ consisting of topological zero divisors. The corresponding joint spectrum was studied in [4] and denoted by ω. The result proved there jointly with the present Theorem 2.1 gives the following.

For an arbitrary generating set $S \subset \mathcal{A}$

$$\omega(S) = \text{ the } \mathcal{A}\text{-rationally convex hull of cortex } (\mathcal{A}). \tag{2.8}$$

(7) Let X be a Banach space and $\mathcal{A}$ a closed commutative subalgebra of bounded linear operators in X generated by a set $S \subset \mathcal{A}$. Let $\mathcal{U}$ be the family of all ideals in $\mathcal{A}$ not containing Fredholm operators. The corresponding $\sigma_{\mathcal{U}}$ is the essential joint spectrum σ_e. Theorem 2.1 establishes one-to-one correspondence between $\sigma_e(S)$ and the space of multiplicative functionals that never vanish on Fredholm operators.

The monograph [5] provides a great number of examples of joint spectra defined by a family of ideals. In each case we can deduce from Theorem 2.1 a similar result.

The right ideals and two-sided ideals versions of Theorem 2.1 are obviously valid.

Let us note that our assumption that the family $\mathcal{U}$ consists of closed ideals was necessary only for proving that every element of $\sigma_{\mathcal{U}}(S)$ defines a multiplicative character which is continuous.

Skiping the question of the continuity, by the same proof we obtain a theorem of strictly algebraic character.

Theorem 2.2. *Let $\mathcal{A} = \mathcal{A}(S)$ be a complex unital algebra. Let $\mathcal{U}$ be an arbitrary family of left or right ideals in $\mathcal{A}$. Then*

$$\sigma_{\mathcal{U}}(S) = \{(\varphi(s))_{s\in S} \mid \varphi \in \mathfrak{M}_{\mathcal{U}}^*\}, \tag{2.9}$$

where $\mathfrak{M}_{\mathcal{U}}^*$ denotes the set of multiplicative functionals on $\mathcal{A}$ whose kernels belong to $\mathcal{U}$.

Obviously in this case, $\mathcal{A} = \mathcal{A}(S)$ means that $\mathcal{A}$ is exactly the smallest subalgebra containing S and e.

3. The spectral mapping property

By the definition of a joint spectrum of the form $\sigma_{\mathcal{U}}$ and the remainder formula, it follows that for every n-tuple of polynomials $p = (p_1,\ldots,p_n) \in \mathcal{P}^n$ of k variables the following formula called one-way spectral mapping formula holds:

$$p(\sigma_{\mathcal{U}}(a_1,\ldots,a_k)) \subset \sigma_{\mathcal{U}}(p(a_1,\ldots,a_k)). \tag{3.1}$$

One of the basic problems of the theory of joint spectra is the question when these sets are equal. This being the case we say that $\sigma_{\mathcal{U}}$ has the spectral mapping property.

For an arbitrary algebra $\mathcal{A}$ and a family $\widetilde{\mathfrak{M}}$ of multiplicative functionals on $\mathcal{A}$ the formula

$$\widetilde{\sigma}(S) = \{(\varphi(s))_{s\in S} \mid \varphi \in \widetilde{\mathfrak{M}}\} \tag{3.2}$$

defines on $\mathcal{A}$ a joint spectrum that has the spectral mapping property.

This observation leads to the following result.

Theorem 3.1. *Let $\mathcal{A}$ be an associative unital topological algebra and suppose that $\sigma_{\mathcal{U}}(S) \neq \emptyset$ for some family $\mathcal{U}$ of closed ideals in $\mathcal{A}$ and for some set S generating $\mathcal{A}$. Then the joint spectrum*

$$\widetilde{\sigma}_{\mathcal{U}}(C) = \{(\varphi(s))_{s\in C} \mid \varphi \in \mathfrak{M}_U\} \tag{3.3}$$

has the spectral mapping property. It satisfies $\widetilde{\sigma}_{\mathcal{U}}(C) \subset \sigma_{\mathcal{U}}(C)$ for arbitrary $C \subset \mathcal{A}$ and $\widetilde{\sigma}_{\mathcal{U}}(G) = \sigma_{\mathcal{U}}(G)$ if G generates $\mathcal{A}$.

The spectrum $\widetilde{\sigma}_{\mathcal{U}}$ is also of the form $\sigma_{\widetilde{\mathcal{U}}}$ with $\widetilde{\mathcal{U}} = \{\ker\varphi \mid \varphi \in \mathfrak{M}_{\mathcal{U}}\}$.

4. Rational convexity of the spectrum of generators

If a complex Banach algebra is finitely generated, then the joint spectrum of the generators is a polynomially convex set. As proved by Brooks [6], it is no longer true even for m-convex algebras.

In this section, we study the rational convexity of the spectrum $\sigma_{\mathcal{U}}$ of generators for an arbitrary family of closed ideals $\mathcal{U}$ in a Q-algebra.

A set $B \subset \mathbb{C}^k$ is rationally convex if it coincides with its rationally convex hull

$$r(B) = \{z \in \mathbb{C}^k \mid \text{ for every polynomial } p,\ p(z) = c \Longrightarrow c \in p(B)\}. \tag{4.1}$$

Let us extend this concept to subsets of $\mathbb{C}^S$. Let $F \subset \mathbb{C}^S$ and let $\mathbf{z} = (z_s)_{s\in S} \in \mathbb{C}^S$.

We say that $\mathbf{z} \in r(F)$ if for every polynomial p of k variables and for every set $\{s_1,\ldots,s_k\} \subset S$,

$$p(z_{s_1},\ldots,z_{s_k}) = c \Longrightarrow \exists \mathbf{b} \in F, \quad c = p(b_{s_1},\ldots,b_{s_k}). \tag{4.2}$$

The set $F \subset \mathbb{C}^S$ is rationally convex if $r(F) = F$.

Let us equip $\mathbb{C}^S$ with the topology of pointwise convergence.

The algebra $\mathcal{A}$ is a Q-algebra if the set $G(\mathcal{A})$ of invertible elements is open in $\mathcal{A}$. Notice that in Q-algebras $\sigma_l^{(t)}$ coincides with σ_l. On the other hand, for $a_1,\dots,a_k \in \mathcal{A}$ the spectrum $\sigma_l(a_1,\dots,a_k)$ is compact.

Theorem 4.1. *For an arbitrary complex topological algebra $\mathcal{A} = \mathcal{A}(S)$ the set $\sigma_l^{(t)}(S)$ is rationally convex. If $\mathcal{A} = \mathcal{A}(S)$ is a complex Q-algebra and if $\mathfrak{U}$ is a family of left closed ideals in $\mathcal{A}$, then $r(\sigma_{\mathfrak{U}}(S)) = \overline{\sigma_{\mathfrak{U}}(S)}$. In particular, if $\mathfrak{M}_{\mathfrak{U}}$ is closed, $\sigma_{\mathfrak{U}}(S)$ is rationally convex.*

Proof. If $\sigma_{\mathfrak{U}}(S) = \emptyset$, we have nothing to prove. So we suppose that $\sigma_{\mathfrak{U}}(S) \neq \emptyset$. Let $\mathbf{z} \in r(\sigma_{\mathfrak{U}}(S))$. Define a functional on the subalgebra $\mathcal{A}_0 \subset \mathcal{A}$ of elements of the form $p(s_1,\dots,s_k)$, $s_i \in S$, setting

$$f(p(s_1,\dots,s_k)) = p(z_{s_1},\dots,z_{s_k}). \tag{4.3}$$

The correctness of this definition is proved by arguments used in the proof of Theorem 2.1. Suppose that $p(s_1,\dots,s_n) = 0$.

There exists $\boldsymbol{\lambda} \in \sigma_{\mathfrak{U}}(S)$ such that $p(\lambda_{s_1},\dots,\lambda_{s_k}) = p(z_{s_1},\dots,z_{s_k})$. Then by the remainder formula,

$$-p(z_{s_1},\dots,z_{s_k}) = p(s_1,\dots,s_k) - p(\lambda_{s_1},\dots,\lambda_{s_k}) = \sum_{i=1}^{k} q_i(s_1,\dots,s_k)(s_i - \lambda_{s_i}). \tag{4.4}$$

The right-hand side belongs to an ideal $I \in \mathfrak{U}$, while on the right we have element proportional to the unit. Hence,

$$f(p(s_1,\dots,s_k)) = p(z_{s_1},\dots,z_{s_k}) = 0. \tag{4.5}$$

The multiplicative functional f is well defined on $\mathcal{A}_0$. By arguments used in the proof of Theorem 2.1, it follows that f is continuous and it extends to a multiplicative, continuous functional on $\mathcal{A}$. The proof can be ended here if the spectrum in question is $\sigma_l^{(t)}$. In this case, by Theorem 2.1, $\mathbf{z} = f \mid S \in \sigma_l^{(t)}(S)$, so $r(\sigma_l^{(t)}(S)) = \sigma_l^{(t)}(S)$.

In the case of a generic $\mathfrak{U}$, we claim that the functional f belongs to $\overline{\mathfrak{M}_{\mathfrak{U}}}$.

The Gelfand transform associates to $a \in \mathcal{A}$ the function $\hat{a}(\varphi) = \varphi(a)$. The function $\hat{a}$ is continuous on $\mathfrak{M}(\mathcal{A})$ and in the case of a Q-algebra, the mapping $\mathcal{A} \ni a \to C(\mathfrak{M}(\mathcal{A}))$ is continuous (see [7]).

Consider the algebra A that consists of the functions $\hat{a} \mid \overline{\mathfrak{M}_{\mathfrak{U}}}$. This is a subalgebra of $C(\overline{\mathfrak{M}_{\mathfrak{U}}})$. Denote by ψ the superposition of the Gelfand transform with the operator of restriction to $\overline{\mathfrak{M}_{\mathfrak{U}}}$. Then $A = \psi(\mathcal{A}_0)$. The image $J = \psi(\ker f)$ is an ideal in the commutative algebra A. As the calculus above shows, every element of J vanishes at some point of $\mathfrak{M}_{\mathfrak{U}}$. By [8, Theorem 3.3], it follows that there exists $g \in \overline{\mathfrak{M}_{\mathfrak{U}}}$ such that $I \in \ker g$. The kernels of the multiplicative functionals f and g in $\mathcal{A}_0$ coincide. Hence, $f = g$. We have obtained $f \mid S = \mathbf{z} \in \overline{\sigma_{\mathfrak{U}}(S)}$. □

The rational convexity of $\sigma_l^{(t)}(S)$ for general topological algebras $\mathcal{A}(S)$ was observed by Andrzej Sołtysiak during the Warsaw Workshop on Topological Algebras in November 2006.

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