Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 93562, 8 pages
doi:10.1155/2007/93562

Research Article

On Sectional Curvatures of (ϵ)-Sasakian Manifolds

Rakesh Kumar, Rachna Rani, and R. K. Nagaich

Received 24 May 2007; Accepted 2 November 2007

Recommended by Mircea-Eugen Craioveanu

We obtain some basic results for Riemannian curvature tensor of (ϵ)-Sasakian manifolds and then establish equivalent relations among ϕ-sectional curvature, totally real sectional curvature, and totally real bisectional curvature for (ϵ)-Sasakian manifolds.

1. Introduction

The index of a metric plays significant roles in differential geometry as it generates variety of vector fields such as space-like, time-like, and light-like fileds. With the help of these vector fields, we establish interesting properties on (ϵ)-Sasakian manifolds, which was introduced by Bejancu and Duggal [1] and further investigated by Xufeng and Xiaoli [2]. Since Sasakian manifolds with indefinite metrics play crucial roles in physics [3], hence the study of these manifolds becomes the central theme in present scenario. Here the next section is concerned with the basic results of Riemannian curvature tensor of (ϵ)-Sasakian manifolds. In Section 3, these results will be used to obtain the equivalent relations among ϕ-sectional curvature, totally real sectional curvature, and totally real bisectional curvature. In [1], authors defined the (ϵ)-Sasakian manifold as follows.

Let M be a real $(2n+1)$-dimensional differentiable manifold endowed with an almost contact structure (ϕ,η,ξ), where ϕ is a tensor field of type $(1,1)$, η is a 1-form, and ξ is a vector field on M satisfying

$$\phi^2X = -X+\eta(X)\xi, \quad \eta(\xi) = 1. \tag{1.1}$$

It follows that

$$\eta(\phi X) = 0, \quad \phi(\xi) = 0, \quad \operatorname{rank}\phi = 2n; \tag{1.2}$$

then M is called an almost contact manifold. If there exists a semi-Riemannian metric g satisfying

$$g(\phi X,\phi Y) = g(X,Y) - \epsilon\eta(X)\eta(Y) \quad \forall X,Y \in \chi(X), \tag{1.3}$$

where $\epsilon = \pm 1$, then (ϕ,η,ξ,g) is called an (ϵ) almost contact metric structure and M is known as an (ϵ) almost contact manifold.

For an (ϵ) almost contact manifold we also have

$$\begin{gathered} \eta(X) = \epsilon g(X,\xi) \quad \forall X \in \chi(X), \\ \epsilon = g(\xi,\xi), \end{gathered} \tag{1.4}$$

hence ξ is never a light-like vector field on M, and according to the casual character of ξ, we have two classes of (ϵ)-Sasakian manifolds. When $\epsilon = -1$ and the index of g is an odd number ($\nu = 2s+1$), then M is a time-like Sasakian manifold and M is a space-like Sasakian manifold when $\epsilon = -1$ and $\nu = 2s$. For $\epsilon = 1$ and $\nu = 0$, we obtain usual Sasakian manifold and for $\epsilon = 1$ and $\nu = 1$, M is a Lorentz-Sasakian manifold.

If $d\eta(X,Y) = g(\phi X,Y)$, then M is said to have (ϵ)-contact metric structure (ϕ,η,ξ,g). If, moreover, this structure is normal, that is, if

$$[\phi X,\phi Y] + \phi^2[X,Y] - \phi[X,\phi Y] - \phi[\phi X,Y] = -2d\eta(X,Y)\xi, \tag{1.5}$$

then the (ϵ)-contact metric structure is called an (ϵ)-Sasakian structure, and manifold endowed with this structure is called an (ϵ)-Sasakian manifold.

Now, let σ be a plane section in tangent space $T_p(M)$ at a point p of M, and let it be spanned by vectors X and Y, then the sectional curvature of σ is given by

$$K(X,Y) = \frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}. \tag{1.6}$$

A plane $\{X,Y\}$, where X and Y are orthonormal to ξ and satisfy $\phi(\{X,Y\}) \perp \{X,Y\}$, is called totally real section, and sectional curvature associated with this section is called a totally real sectional curvature. The totally real bisectional curvature $B(X,Y)$ is defined as

$$B(X,Y) = R(X,\phi X,Y,\phi Y), \tag{1.7}$$

where $\eta(X) = \eta(Y) = g(X,Y) = g(X,\phi Y) = 0$.

A plane section $\{X,\phi X\}$, where X is orthonormal to ξ, is called ϕ-section, and the curvature associated with this is called ϕ-sectional curvature which is denoted by $H(X)$, where

$$H(X) = K(X,\phi X) = R(X,\phi X,X,\phi X). \tag{1.8}$$

If a Sasakian manifold M has constant ϕ-sectional curvature c, then it is called a Sasakian space form and denoted by $M^{2n+1}(c)$.

2. Riemannian curvature tensor

THEOREM 2.1 [1]. *An (ϵ) almost contact metric structure (ϕ,η,ξ,g) is (ϵ)-Sasakian if and only if*

$$(\nabla_X\phi)Y = g(X,Y)\xi - \epsilon\eta(Y)X, \quad \forall X,Y \in \chi(M), \tag{2.1}$$

where ∇ is the Levi-Civita connection with respect to g. Also one has

$$\nabla_X\xi = -\epsilon\phi X, \quad \forall X \in \chi(M). \tag{2.2}$$

For an (ϵ)-Sasakian manifold, using (2.1) we have

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.3}$$

where R denotes the Riemannian curvature tensor on M, and also from above we have

$$R(X,\xi)Y = -\epsilon g(X,Y)\xi + \eta(Y)X. \tag{2.4}$$

Using (2.1) and (2.2), we have

$$R(X,Y)\phi Z = \phi R(X,Y)Z + \epsilon\{g(Z,\phi X)Y - g(Z,\phi Y)X \quad + g(X,Z)\phi Y - g(Y,Z)\phi X\}. \tag{2.5}$$

And by using (2.5), we obtain the following set of equations:

$$R(X,Y)Z = -\phi R(X,Y)\phi Z + \epsilon\{g(Y,Z)X - g(X,Z)Y + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X\}, \tag{2.6}$$

$$\begin{aligned} g(R(X,Y)\phi Z,\phi W) = g(R(X,Y)Z,W) &\\ + \epsilon\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z) &\\ - g(\phi Z,X)g(\phi W,Y) + g(\phi Z,Y)g(\phi W,X)\}, & \end{aligned} \tag{2.7}$$

$$\begin{aligned} g(R(\phi X,\phi Y)\phi Z,\phi W) = g(R(X,Y)Z,W) + \eta(W)\eta(Y)g(X,Z) &\\ - \eta(W)\eta(X)g(Y,Z) + \eta(Z)\eta(X)g(Y,W) &\\ - \eta(Z)\eta(Y)g(X,W). & \end{aligned} \tag{2.8}$$

Now, we can write (2.5) as

$$\begin{aligned} g(R(X,Y)\phi Z,W) = g(\phi R(X,Y)Z,W) &\\ + \epsilon\{g(Z,\phi X)g(Y,W) - g(Z,\phi Y)g(X,W) &\\ + g(X,Z)g(\phi Y,W) - g(Y,Z)g(\phi,W)\}, & \end{aligned} \tag{2.9}$$

or

$$g(R(X,Y)\phi Z,W)=g(\phi R(X,Y)Z,W)-\epsilon P(X,Y;Z,W), \tag{2.10}$$

where

$$\begin{aligned}P(X,Y;Z,W)&=g(Y,Z)g(\phi X,W)-g(\phi X,Z)g(Y,W)\\&\quad+g(\phi Y,Z)g(X,W)-g(X,Z)g(\phi Y,W).\end{aligned} \tag{2.11}$$

Clearly $P(X,Y;Z,W)=-P(Z,W;X,Y)$, and if $\{X,Y\}$ is an orthonormal pair orthogonal to ξ, and if we set $g(\phi X,Y)=\cos\theta, 0\le\theta\le\pi$, then

$$P(X,Y;X,\phi Y)=-\sin^2\theta. \tag{2.12}$$

If we put $D(X)=Q(X,\phi X)$ for any vector X orthogonal to ξ and $Q(X,Y)=g(R(X,Y)Y,X)$ for any vectors X and Y, then we have the following lemma.

Lemma 2.2. *For any vectors X and Y orthogonal to ξ, one obtains*

$$\begin{aligned}Q(X,Y)=\frac{1}{32}\{&3D(X+\phi Y)+3D(X-\phi Y)-D(X+Y)\\&-D(X-Y)-4D(X)-4D(Y)-24\epsilon P(X,Y;X,\phi Y)\}.\end{aligned} \tag{2.13}$$

Proof. For X, Y orthogonal to ξ, we have

$$\begin{aligned}D(X+Y)+D(X-Y)=2\{&D(X)+D(Y)+2R(X,\phi X,Y,\phi Y)\\&+2R(X,\phi Y,Y,\phi X)+R(X,\phi Y,X,\phi Y)+R(Y,\phi X,Y,\phi X)\},\end{aligned} \tag{2.14}$$

and using (2.8), we have

$$\begin{aligned}R(\phi X,\phi Y,\phi X,\phi Y)&=R(X,Y,X,Y),\\R(X,\phi Y,X,\phi Y)&=R(Y,\phi X,Y,\phi X).\end{aligned} \tag{2.15}$$

Substituting (2.15) in (2.14), we get

$$\begin{aligned}D(X+Y)+D(X-Y)=2\{&D(X)+D(Y)+2R(X,\phi X,Y,\phi Y)\\&+2R(X,\phi Y,Y,\phi X)+2Q(X,\phi Y)\}.\end{aligned} \tag{2.16}$$

Replacing Y by ϕY in (2.16), we get

$$\begin{aligned}D(X+\phi Y)+D(X-\phi Y)=2\{&D(X)+D(Y)-2R(X,\phi X,\phi Y,Y)\\&-2R(X,Y,\phi Y,\phi X)+2Q(X,Y)\}.\end{aligned} \tag{2.17}$$

Using (2.16) and (2.17), we have

$$\begin{aligned}3D(X+\phi Y)+3D(X-\phi Y)-D(X+Y)-D(X-Y)\ &-4D(X)-4D(Y)\\ =12Q(X,Y)-4Q(X,\phi Y)+8R(X,\phi X,Y,\phi Y)&+12R(X,Y,\phi X,\phi Y)\\ +R(X,\phi Y,\phi X,Y).&\end{aligned} \tag{2.18}$$

Replacing W by ϕX and Z by Y in (2.9), we have

$$R(X,Y,\phi X,\phi Y)=R(X,Y,X,Y)+\epsilon P(X,Y;X,\phi Y). \tag{2.19}$$

Again replacing Y by ϕY, W by Y, and Z by X in (2.9), we have

$$R(X,\phi Y,Y,\phi X)=R(X,\phi Y,X,\phi Y)+\epsilon P(X,Y;X,\phi Y). \tag{2.20}$$

By using Bianchi's first identity (2.19) and (2.20), we have

$$R(X,\phi X,Y,\phi Y)=Q(X,Y)+Q(X,\phi Y)+24\epsilon P(X,Y;X,\phi Y). \tag{2.21}$$

Thus using the last four equations, we have the result. □

Now, it should be noted that $D(X)=H(X)$ if and only if X is a unit vector, and $Q(X,Y)=K(X,Y)$ if and only if $\{X,Y\}$ is an orthonormal pair. Then, as an application of lemma, we have the following lemma.

Lemma 2.3. *Let $\{X,Y\}$ be an orthonormal pair of the tangent space of an (ϵ)-Sasakian manifold M orthogonal to ξ. If one puts $g(X,\phi Y)=\cos\theta, 0\le\theta\le\pi$, then*

$$\begin{aligned}K(X,Y)=\frac{1}{8}\Big\{3(1+\cos\theta)^2H\left(\frac{X+\phi Y}{|X+\phi Y|}\right)&\\ +3(1-\cos\theta)^2H\left(\frac{X-\phi Y}{|X-\phi Y|}\right)-H\left(\frac{X+Y}{|X+Y|}\right)&\\ -H\left(\frac{X-Y}{|X-Y|}\right)-H(X)-H(Y)+6\epsilon\sin^2\theta\Big\}.&\end{aligned} \tag{2.22}$$

Proof. It follows from Lemma (2.2).

Since the ϕ-sectional curvature determines the curvature of a Sasakian manifold, then it can be easily verified that if the ϕ-sectional curvature $H(X)$ is independent of the choice of a vector X at any point and has value c, then c is constant on M and the curvature tensor

R of (ϵ)-Sasakian manifold satisfies

$$\begin{aligned} R(X,Y,Z,W) = {} & \frac{(c+3\epsilon)}{4}\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\} \\ & + \frac{(c-\epsilon)}{4}\{\eta(X)\eta(Z)g(Y,W)-\eta(Y)\eta(Z)g(X,W) \\ & \quad + \eta(Y)\eta(W)g(X,Z)-\eta(X)\eta(W)g(Y,Z) \\ & \quad + g(\phi Y,Z)g(\phi X,W)-g(\phi X,Z)g(\phi Y,W) \\ & \quad + 2g(X,\phi Y)g(\phi Z,W)\}. \end{aligned} \tag{2.23}$$

□

Now, our next aim of this paper is as follows.

Theorem 2.4. *Let (M^{2n+1},ϕ,η,ξ) be an (ϵ)-Sasakian manifold of dimension ≥ 7, then the following relations are equivalent.*

(i) *M has constant ϕ-sectional curvature c; that is, $H(X)$ is constant.*
(ii) *M has constant totally real sectional curvature; that is, for any totally real section $\{X,Y\}$, $K(X,Y)$ is constant.*
(iii) *M has constant totally real bisectional curvature; that is, $B(X,Y)$ is constant.*

3. Proof of the main Theorem 2.4

In the proof, we assume that X, Y, and Z are unit vector fields.

If $H(X)$ is constant and equal to c, then for a totally real section $\{X,Y\}$, (2.23) gives $K(X,Y) = -(c+3\epsilon)/4$ and $B(X,Y) = -(c+7\epsilon)/2$; this gives (i)⇒(ii) and (i)⇒(iii) respectively.

Now, let $\{X,Y\}$ be a totally real section, then $\{(X+Y)/\sqrt{2},(-\phi X+\phi Y)/\sqrt{2}\}$ is also a totally real section, and assume that M has constant totally real sectional curvature (say k); then

$$K\left(\frac{X+Y}{\sqrt{2}},\frac{-\phi X+\phi Y}{\sqrt{2}}\right) = k; \tag{3.1}$$

this gives

$$4k = H(X)+H(Y)+K(X,\phi Y)+K(Y,\phi X)-4R(X,\phi Y,Y,\phi X)-2R(X,Y,\phi X,\phi Y), \tag{3.2}$$

or

$$H(X)+H(Y) = 8k+6. \tag{3.3}$$

Since the dimension of M is $(2n+1), n = 3$, therefore there exists a unit vector Z orthonormal to $\{X,Y\}$ such that

$$H(X)+H(Z) = 8k+6. \tag{3.4}$$

Therefore, using (3.3) and (3.4), we conclude that

$$H(X) = H(Y). \tag{3.5}$$

Thus, we have (ii)⇒(i).

Next, we prove that (iii)⇒(i).

Since

$$B(X,Y) = R(X,\phi X,Y,\phi Y), \tag{3.6}$$

where $\eta(X) = \eta(Y) = g(X,Y) = g(X,\phi Y) = 0$, then using (2.19) and (2.20), we have

$$B(X,Y) = K(X,Y) + K(X,\phi Y) - 2\epsilon. \tag{3.7}$$

Now, let M have constant totally real bisectional curvature (say t), then

$$K(X,Y) + K(X,\phi Y) = t + 2\epsilon. \tag{3.8}$$

Also $\{(X+Y)/\sqrt{2},(-\phi X+\phi Y)/\sqrt{2}\}$ is a totally real section for a totally real section $\{X,Y\}$ then

$$B\left(\frac{X+Y}{\sqrt{2}},\frac{-\phi X+\phi Y}{\sqrt{2}}\right) = t; \tag{3.9}$$

this gives

$$H(X) + H(Y) + 2R(X,\phi X,Y,\phi Y) - 4R(X,\phi Y,X,\phi Y) = 4t - 2\epsilon, \tag{3.10}$$

or

$$H(X) + H(Y) - 4K(X,\phi Y) = 2t - 2\epsilon. \tag{3.11}$$

Replacing Y by ϕY, we get

$$H(X) + H(Y) - 4K(X,Y) = 2t - 2\epsilon. \tag{3.12}$$

Using (3.8) in addition to (3.11) and (3.12), we have

$$H(X) + H(Y) = 4t + 2\epsilon. \tag{3.13}$$

Since there can exist a unit vector Z orthogonal to $\{X,Y\}$, then

$$H(X) + H(Z) = 4t + 2\epsilon. \tag{3.14}$$

Using (3.13) and (3.14), we have

$$H(X) = H(Y). \tag{3.15}$$

Hence, the result is given.

References

[1] A. Bejancu and K. L. Duggal, "Real hypersurfaces of indefinite Kaehler manifolds," *International Journal of Mathematics and Mathematical Sciences*, vol. 16, no. 3, pp. 545–556, 1993.
[2] X. Xufeng and C. Xiaoli, "Two theorems on ϵ-Sasakian manifolds," *International Journal of Mathematics and Mathematical Sciences*, vol. 21, no. 2, pp. 249–254, 1998.
[3] K. L. Duggal, "Space time manifolds and contact structures," *International Journal of Mathematics and Mathematical Sciences*, vol. 13, no. 3, pp. 545–553, 1990.

Rakesh Kumar: Department of Mathematics, University College of Engineering, Punjabi University Patiala 147002, India
Email address: dr_rk37c@yahoo.co.in

Rachna Rani: Department of Mathematics, Punjabi University, Patiala 147002, India
Email address: rachna_ucoe@yahoo.co.in

R. K. Nagaich: Department of Mathematics, Punjabi University, Patiala 147002, India
Email address: dr_nagaich@yahoo.co.in