Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 89713, 8 pages
doi:10.1155/2007/89713

Research Article

On the Common Index Divisors of a Dihedral Field of Prime Degree

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Received 11 September 2006; Revised 6 January 2007; Accepted 6 January 2007

Recommended by Dihua Jiang

A criterion for a prime to be a common index divisor of a dihedral field of prime degree is given. This criterion is used to determine the index of families of dihedral fields of degrees 5 and 7.

1. Introduction

Let L be an algebraic number field of degree n. Let O_L denote the ring of integers of L. The element $\alpha \in O_L$ is called a generator of L if $L = \mathbb{Q}(\alpha)$. The index of α is the positive integer $\operatorname{ind}\alpha$ given by

$$D(\alpha) = (\operatorname{ind}\alpha)^2 d(L), \tag{1.1}$$

where $d(L)$ is the discriminant of L and $D(\alpha)$ is the discriminant of the minimial polynomial of α. The index of L is

$$i(L) = \gcd\{\operatorname{ind}\alpha \mid \alpha \text{ is a generator of } L\}. \tag{1.2}$$

A positive integer > 1 dividing $i(L)$ is called a common index divisor of L. If O_L possesses an element β such that $\{1, \beta, \beta^2, \dots, \beta^{n-1}\}$ is an integral basis for L, then L is said to be monogenic. If L is monogenic, then $i(L) = 1$. Thus a field possessing a common index divisor is nonmonogenic.

Let $f(x)$ be an irreducible polynomial in $\mathbb{Z}[x]$ of odd prime degree q and suppose that $\operatorname{Gal}(f(x)) \simeq D_q$ (the dihedral group of order $2q$). We note that $D_q = \langle \sigma, \tau \rangle$ with $\sigma^q = \tau^2 = (\sigma\tau)^2 = 1$. Let M be the splitting field of $f(x)$. Let θ be a root of $f(x)$ and set

$L = \mathbb{Q}(\theta)$ so that the degree of L over $\mathbb{Q}$ is equal to q. We denote the unique quadratic subfield of M by K.

We prove in Section 2 the following theorem which gives a criterion for a prime p to be a common index divisor of L.

Theorem 1.1. *Let $f(x) \in \mathbb{Z}[x]$ be irreducible,* $\deg(f(x)) = q$ *(an odd prime), and* $\mathrm{Gal}(f(x)) \simeq D_q$. *Let M be the splitting field of $f(x)$. Let $\theta \in \mathbb{C}$ be a root of $f(x)$. Set $L = \mathbb{Q}(\theta)$ so that $[L : \mathbb{Q}] = q$. Let K be the unique quadratic subfield of M. If p is a prime satisfying*

$$p < \frac{1}{2}(q+1), \quad p \mid d(K), \tag{1.3}$$

then

$$p = R_1 R_2^2 \cdots R_{(q+1)/2}^2 \tag{1.4}$$

for distinct prime ideals $R_1, R_2, \ldots, R_{(q+1)/2}$ of O_L, and p is a common index divisor of L.

As an application of Theorem 1.1, we determine in Section 3 the index of a field defined by a dihedral quintic trinomial of the form $x^5 + ax + b$, $a, b \in \mathbb{Z}$.

In Section 4, we determine the index of an infinite family of fields defined by dihedral polynomials of degree 7.

Finally in Section 5, we consider a dihedral field of degree 11 and use Theorem 1.1 to show that it is nonmonogenic.

We note that a method for calculating a generator of K, and hence $d(K)$, directly from $f(x)$ is given in [1].

2. Proof of Theorem 1.1

As $p \mid d(K)$, we have $p = \wp^2$ for some prime ideal $\wp$ of O_K. Suppose that $\wp$ is inert in M/K. Then $p = \wp^2$ in $M/\mathbb{Q}$. This contradicts [2, Theorem 10.1.26, part (6)]. Hence $\wp$ is not inert in M/K. Suppose $\wp$ totally ramifies in M/K. Then $\wp = Q^q$ for some prime ideal Q of M. Thus $p = \wp^2 = Q^{2q}$ in M. Hence, by [2, Theorem 10.1.26, part (9)], we have $p \mid q$. But p and q are primes so $p = q$. This contradicts the assumption $p < (1/2)(q+1)$. Hence $\wp$ does not totally ramify in M. Then, as M is normal over K of prime degree q, we have

$$\wp = Q_1 Q_2 \cdots Q_q \tag{2.1}$$

for distinct prime ideals $Q_1, Q_2, \ldots, Q_q$ of M. Thus

$$p = \wp^2 = Q_1^2 Q_2^2 \cdots Q_q^2. \tag{2.2}$$

Hence, by [2, Theorem 10.1.26, part (6)], we have

$$p = R_1 R_2^2 \cdots R_{(q+1)/2}^2 \tag{2.3}$$

for distinct prime ideals $R_1, R_2, \ldots, R_{(q+1)/2}$ of L, which is (1.4). We note that the decomposition of p in L can be checked directly by studying the $\mathrm{Gal}(M/L)$ action on the coset space D_q/D, where D is a decomposition subgroup at p.

Let $g(x)$ be any defining polynomial for L, so that $\deg(g(x)) = q$. Let ϕ be a root of $g(x)$ such that $\mathbb{Q}(\phi) = L$. Suppose $p \nmid \operatorname{ind}(\phi)$. The inertial degree $f = 1$ in the extension $M/\mathbb{Q}$ (using the tower $M/K/\mathbb{Q}$), hence in $L/\mathbb{Q}$, so that all the irreducible factors of $g(x)$ modulo p are linear. Thus $g(x)$ has at most p irreducible factors modulo p. Hence, by Dedekind's theorem, p factors into at most p prime ideals in L. Thus by (1.4) we have $(1/2)(q+1) \le p$. This contradicts $p < (1/2)(q+1)$. Hence $p \mid \operatorname{ind}(\phi)$ for all defining polynomials g. Thus p is a common index divisor of L.

3. Dihedral quintic trinomials

Let $f(x) = x^5 + ax + b \in \mathbb{Z}[x]$ have Galois group D_5. Then there exist coprime integers m and n and $i, j \in \{0,1\}$ such that

$$\begin{aligned} a &= 2^{2-4i}5^{1-4j}d_2(m^2 - mn - n^2)E^2F, \\ b &= 2^{4-5i}5^{-5j}d_1(2m-n)(m+2n)E^3F, \end{aligned} \tag{3.1}$$

where d_1^2 is the largest square dividing m^2+n^2, d_2^5 is the largest fifth power dividing $m^2 + mn - n^2$, and

$$E = \frac{m^2+n^2}{d_1^2}, \qquad F = \frac{m^2+mn-n^2}{d_2^5}. \tag{3.2}$$

This result is due to Roland et al. [3, page 138], see also [4, page 139]. The discriminant of x^5+ax+b is

$$D(f) = 2^{16-20i}5^{6-20j}\left(2m^6+4m^5n+5m^4n^2-5m^2n^4+4mn^5-2n^6\right)^2E^{10}F^4, \tag{3.3}$$

see [3, equation (3), page 139]. As $\gcd(m,n) = 1$, we have $3 \nmid m^2+n^2$ and $3 \nmid m^2+mn-n^2$ so $3 \nmid E$ and $3 \nmid F$. If $3 \mid n$, then $3 \nmid m$, and so $3 \nmid 2m^6+4m^5n+5m^4n^2-5m^2n^4+4mn^5-2n^6$. If $3 \nmid n$, then as the polynomial $2x^6+4x^5+5x^4-5x^2+4x-2$ is irreducible (mod 3), we deduce that $3 \nmid 2m^6+4m^5n+5m^4n^2-5m^2n^4+4mn^5-2n^6$. Hence $3 \nmid D(f)$. Thus $3 \nmid \operatorname{ind}(\theta)$, where $L = \mathbb{Q}(\theta)$, $f(\theta) = 0$. Hence $3 \nmid i(L)$. By Engstrom [5, page 234] as $[L:\mathbb{Q}] = 5$, the only primes that can divide $i(L)$ are 2 and 3. We use our theorem to show that $2 \mid i(L)$. From Spearman and Williams [4, pages 149, 150], the discriminant $d(K)$ of the unique quadratic subfield of the splitting field of $f(x)$ satisfies

$$\begin{aligned} 2^2 \,\|\, d(K) &\quad \text{if } m \equiv n+1 \pmod 2, \\ 2^3 \,\|\, d(K) &\quad \text{if } m \equiv n \equiv 1 \pmod 2. \end{aligned} \tag{3.4}$$

Thus $2 \mid d(K)$. Hence, by Theorem 1.1, 2 is a common index divisor of L. From Engstrom [5, Table, page 234], as $2 = R_1R_2^2R_3^2$ by Theorem 1.1, we deduce, $i(L) = 2$. As $i(L) \neq 1$, this gives an infinite family of nonmonogenic dihedral quintic fields. In [6], an infinite family of monogenic dihedral quintic fields was exhibited.

4. A class of dihedral polynomials of degree 7

We recall a family of polynomials of degree 7 due to Smith [7, page 790]. This family is $f_t(x)$ $(t \in \mathbb{Z})$, where $f_t(x)$ is given by

$$\begin{aligned} f_t(x) = x^7 - (7t^3+35t^2+21t+1)\big[&21x^5+(98t+70)x^4 \\ &-(1029t^3+4557t^2+343t-105)x^3 \\ &-28(7t+1)(49t^3+147t^2+63t-3)x^2 \\ &+7(7t^2+42t-1)(7t^2+14t-5)(7t+1)^2x \\ &+235298t^7+1236858t^6+1138074t^5 \\ &+562226t^4+11270t^3-4914t^2-322t+6\big]. \end{aligned} \tag{4.1}$$

Smith showed that the Galois group of $f_t(x)$ over $\mathbb{Q}(t)$ is D_7. We are interested in determining integers t for which the Galois group of $f_t(x)$ (considered as a polynomial in $\mathbb{Z}[x]$) over $\mathbb{Q}$ is D_7. MAPLE gives the discriminant of $f_t(x)$ as

$$\begin{aligned} D(f_t) = 2^{46}7^{12}t^{15}&(7t^2-14t-9)^6(7t^3+35t^2+21t+1)^6 \\ &\times(63t^2+266t-25)^2(49t^4-196t^3-1694t^2-140t-3)^2. \end{aligned} \tag{4.2}$$

Lemma 4.1. (i) *If* $t \equiv 1 \pmod 3$, *then* $3 \nmid D(f_t)$.
(ii) *If* $t \equiv 1, 2$ *or* $4 \pmod 5$, *then* $5 \nmid D(f_t)$.

The proof follows from (4.2).

Lemma 4.2. *If* $t \in \mathbb{Z}$ *is such that*

$$2 \mid t, \quad 7t^3+35t^2+21t+1 \text{ is square-free } > 1, \tag{4.3}$$

then $f_t(x)$ *is irreducible over* $\mathbb{Q}$.

Proof. Set $a(t) = 7t^3+35t^2+21t+1$ and $b(t) = -235298t^7-1236858t^6-1138074t^5-562226t^4-11270t^3+4914t^2+322t-6$. Then, from (4.1), we see that

$$f_t(x) \equiv x^7 \pmod{a(t)}, \tag{4.4}$$

$$f_t(0) = a(t)b(t). \tag{4.5}$$

The resultant of $a(t)$ and $b(t)$ as polynomials in t is (by MAPLE) $2^{45}7^7$. Clearly $7 \nmid a(t)$ and (as $2 \mid t$) $2 \nmid a(t)$. Thus $\gcd_{\mathbb{Z}}(a(t), b(t)) = 1$. Let q be any prime dividing $a(t)$ (so $q \neq 2, 7$). Then $q \| a(t)$ and $q \nmid b(t)$. Thus, by (4.1) and (4.4), q divides the coefficients of x^i $(i = 0,1,2,3,4,5,6)$ in $f_t(x)$ and by (4.5) $q \| f_t(0)$. Hence, by Eisenstein's criterion, $f_t(x)$ is irreducible over $\mathbb{Q}$. □

Let θ denote one of the roots of $f_t(x)$. Let $\alpha_1 = \theta, \alpha_2, \dots, \alpha_7$ be all the roots of $f_t(x)$. Set $L = \mathbb{Q}(\theta)$. Under condition (4.3), we have $[L:\mathbb{Q}] = 7$.

Lemma 4.3. *For $t \in \mathbb{Z}$, set*

$$P_{f_t}(x) = \prod_{1 \le i < j \le 7} (x - (\alpha_i + \alpha_j)). \tag{4.6}$$

Then $P_{f_t}(x) \in \mathbb{Z}[x]$ and

$$P_{f_t}(x) = F_t(x) G_t(x) H_t(x), \tag{4.7}$$

where $F_t(x)$, $G_t(x)$, and $H_t(x)$ are distinct polynomials of degree 7 in $\mathbb{Z}[x]$, which satisfy

$$\begin{gathered} F_t(x) \equiv G_t(x) \equiv H_t(x) \equiv x^7 \pmod{a(t)}, \\ F_t(0) = -32a(t)c(t), \\ G_t(0) = -32a(t)d(t), \\ H_t(0) = 32a(t)e(t), \end{gathered} \tag{4.8}$$

where

$$\begin{aligned} c(t) &= 27783t^6 + 43218t^5 - 300615t^4 + 131516t^3 + 17241t^2 - 14t - 25, \\ d(t) &= 8575t^6 - 52822t^5 + 34153t^4 + 27244t^3 + 2737t^2 - 406t - 25, \\ e(t) &= 1029t^6 - 4802t^5 - 9457t^4 - 5292t^3 - 973t^2 + 14t + 25. \end{aligned} \tag{4.9}$$

Proof. The assertion $P_{f_t}(x) \in \mathbb{Z}[x]$ follows from [8, Lemma 11.1.3, page 359] and the fact that $\alpha_1, \alpha_2, \dots, \alpha_7$ are algebraic integers. The remaining assertions of the lemma can be verified using MAPLE. □

Lemma 4.4. *If $t \in \mathbb{Z}$ is such that*

$$2 \mid t, \quad 7t^3 + 35t^2 + 21t + 1 \text{ is square-free} > 1 \tag{4.10}$$

then the polynomials $F_t(x)$, $G_t(x)$, and $H_t(x)$ are irreducible over $\mathbb{Q}$.

Proof. The resultants of $a(t)$ and $c(t)$ (resp., $a(t)$ and $d(t)$, $a(t)$ and $e(t)$) regarded as polynomials in t are by MAPLE $-2^{30}7^6$ (resp., $-2^{30}7^6$, $2^{30}7^6$). Exactly as in the proof of Lemma 4.2, making use of Lemma 4.3, we find by Eisenstein's criterion that the polynomials $F_t(x)$, $G_t(x)$, and $H_t(x)$ are irreducible over $\mathbb{Q}$. □

Lemma 4.5. *If $t \in \mathbb{Z}$ is such that*

$$2 \mid t, \quad 7t^3+35t^2+21t+1 \text{ is square-free } > 1, \quad t \text{ is not a perfect square}, \tag{4.11}$$

then

$$\operatorname{Gal}(f_t(x)) \simeq D_7. \tag{4.12}$$

Proof. Jensen and Yui [8, Theorem II.1.2, page 359] have shown that a monic polynomial $f(x) \in \mathbb{Q}[x]$ of degree p, where p is a prime $\equiv 3 \pmod 4$, has $\operatorname{Gal}(f) \simeq D_p$ if and only if

(i) $f(x)$ is irreducible over $\mathbb{Q}$,

(ii) $D(f)$ is not a perfect square,

(iii) $P_f(x)$ factors as a product of $(p-1)/2$ distinct irreducible polynomials of degree p over $\mathbb{Q}$.

By Lemma 4.2, $f_t(x)$ is irreducible over $\mathbb{Q}$. As t is not a perfect square, we see by (4.2) that $D(f_t)$ is not a perfect square. Finally, by Lemmas 4.3 and 4.4, $P_{f_t}(x)$ factors as a product of 3 distinct irreducible polynomials of degree 7 over $\mathbb{Q}$. Hence, by the Jensen-Yui criterion, $\operatorname{Gal}(f_t(x)) \simeq D_7$. □

Theorem 4.6. (i) *There exist infinitely many integers t satisfying*

$$2 \| t, \quad t \equiv 1 \pmod 3, \quad t \equiv 1, 2 \text{ or } 4 \pmod 5, \quad 7t^3+35t^2+21t+1 \text{ is square-free} > 1, \tag{4.13}$$

and for these values of t,

$$i(L) = 2^4. \tag{4.14}$$

(ii) *There exist infinitely many integers t satisfying*

$$2 \| t,\ 3 \| t, \quad t \equiv 1, 2 \text{ or } 4 \pmod 5, \quad 7t^3+35t^2+21t+1 \text{ is square-free} > 1. \tag{4.15}$$

and for these values of t,

$$i(L) = 2^4 3. \tag{4.16}$$

Proof. The infinitude of integers of the required forms follows from a result of Erdös [9].

Under conditions (4.13) and (4.15), L is a dihedral field of degree 7, by Lemma 4.5. With the notation of Theorem 1.1, we see from (4.2) that $K = \mathbb{Q}(\sqrt{t})$. Clearly $2 \mid d(K)$. By Theorem 1.1, 2 is a common index divisor of L. Also from Theorem 1.1, we see that $2 = R_1R_2^2R_3^2R_4^2$ for distinct prime ideals R_1, R_2, R_3, R_4 of L. Hence, by Engstrom [5, Table, page 235], we see that $2^4 \| i(L)$. For both (4.13) and (4.15) we have by Lemma 4.1(ii) $5 \nmid D(f_t)$

so $5 \nmid i(L)$. For (4.13) by Lemma 4.1(i) we have $3 \nmid D(f_t)$, so $3 \nmid i(L)$. As $[L:\mathbb{Q}] = 7$, by [5, page 224], the only possible prime divisors of $i(L)$ are 2, 3, and 5. Hence $i(L) = 2^4$ in case (i). For case (ii), by Theorem 1.1, 3 is a common index divisor of L. Also, by Theorem 1.1, we see that $3 = R_1 R_2^2 R_3^2 R_4^2$ for distinct prime ideals R_1, R_2, R_3, R_4 of L. Hence, by Engstrom [5, Table, page 235], we see that $3 \| i(L)$. Finally, as the only possible prime divisors of $i(L)$ are 2, 3, and 5, we deduce that $i(L) = 2^4 3$ in case (ii). □

5. A dihedral field of degree 11

Let

$$\begin{aligned} f(x) = x^{11} - 2x^{10} - 51x^9 - x^8 + 536x^7 \\ + 3x^6 - 1999x^5 + 281x^4 + 2571x^3 \\ - 485x^2 - 680x + 69. \end{aligned} \tag{5.1}$$

By MAPLE, $f(x)$ is irreducible over $\mathbb{Q}$. Let θ be a root of $f(x)$ and set $L = \mathbb{Q}(\theta)$, so that $[L:\mathbb{Q}] = 11$. Let M be the splitting field of $f(x)$. It is known that M is the Hilbert class field of $K = \mathbb{Q}(\sqrt{10401})$ [10] so that L is a dihedral extension of $\mathbb{Q}$. By Theorem 1.1, 3 is a common index divisor of L, hence L is not monogenic.

Acknowledgments

The research of the first and second authors was supported by grants from the Natural Sciences and Engineering Research Council of Canada.

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