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Research Article

Dunkl Translation and Uncentered Maximal Operator on the Real Line

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We establish estimates of the Dunkl translation of the characteristic function $\chi_{[-\varepsilon,\varepsilon]}$, $\varepsilon > 0$, and we prove that the uncentered maximal operator associated with the Dunkl operator is of weak type $(1,1)$. As a consequence, we obtain the L^p-boundedness of this operator for $1 < p \leq +\infty$.

1. Introduction

On the real line, the Dunkl operators are differential-difference operators introduced in 1989 by Dunkl [1] and are denoted by Λ_α, where α is a real parameter $> -1/2$. These operators are associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. The Dunkl kernel E_α is used to define the Dunkl transform $\mathcal{F}_\alpha$ which was introduced by Dunkl in [2]. Rösler in [3] shows that the Dunkl kernels verify a product formula. This allows us to define the Dunkl translation τ_x, $x \in \mathbb{R}$. As a result, we have the Dunkl convolution.

The Hardy-Littlewood maximal function was first introduced by Hardy and Littlewood in 1930 for functions defined on the circle (see [4]). Later it was extended to various Lie groups, symmetric spaces, some weighted measure spaces (see [5–10]), and different hypergroups (see [11–14]).

In this paper, we establish an estimate of the Dunkl translation of the characteristic function $\tau_x(\chi_{[-\varepsilon,\varepsilon]})(y)$, $x, y \in \mathbb{R}$, $x \neq 0$, based on the inversion formula which extends some results of [11] to the Dunkl operator on $\mathbb{R}$, and we prove the weak type $(1,1)$ of the uncentered maximal operator M defined for each integrable function f on $(\mathbb{R}, d\mu_\alpha)$ by

$$M(f)(x) = \sup_{\varepsilon>0,\,|z|\in B(x,\varepsilon)} \frac{1}{\mu_\alpha(]-\varepsilon,\varepsilon[)} \left| \int_{-\varepsilon}^{\varepsilon} \tau_z(f)(-y)\,d\mu_\alpha(y) \right|, \quad x \in \mathbb{R}, \tag{1.1}$$

where $B(x,\varepsilon)$ is the interval $[\max\{0,|x|-\varepsilon\},|x|+\varepsilon[$ and μ_α is a weighted Lebesgue measure on $\mathbb{R}$ (see Section 2). Finally, we obtain for $1<p\leq+\infty$ the L^p-boundedness of M. In the case $z=x$, these results are already proved on $\mathbb{R}^d$ in [9] by using the maximal function associated to the Poisson semigroup.

The contents of this paper are as follows.

In Section 2, we collect some basic definitions and results about harmonic analysis associated with Dunkl operator.

In Section 3, we establish estimates of $\tau_x(\chi_{[-\varepsilon,\varepsilon]})(y)$, $x,y\in\mathbb{R}$, $x\neq 0$, and we prove the weak type $(1,1)$ of the uncentered maximal operator M and the L^p-boundedness for $1<p\leq+\infty$ of M.

In the sequel, c represents a suitable positive constant which is not necessarily the same in each occurrence. Furthermore, we denote by

(i) $\mathscr{E}(\mathbb{R})$ the space of C^∞-functions on $\mathbb{R}$,

(ii) $D_*(\mathbb{R})$ the space of even functions in $\mathscr{E}(\mathbb{R})$ with compact support,

(iii) $S_*(\mathbb{R})$ the space of even functions in $\mathscr{E}(\mathbb{R})$ decreasing rapidly.

2. Preliminaries

For a real parameter $\alpha>-1/2$, we consider the differential-difference operator defined by

$$\Lambda_\alpha(f)(x)=\frac{df}{dx}(x)+\frac{2\alpha+1}{x}\left[\frac{f(x)-f(-x)}{2}\right],\quad f\in\mathscr{E}(\mathbb{R}),\tag{2.1}$$

called Dunkl operator.

For $\lambda\in\mathbb{C}$, the initial problem

$$\Lambda_\alpha(f)(x)=\lambda f(x),\quad f(0)=1,\ x\in\mathbb{R},\tag{2.2}$$

has a unique solution $E_\alpha(\lambda)$ called Dunkl kernel and given by

$$E_\alpha(\lambda x)=j_\alpha(i\lambda x)+\frac{\lambda x}{2(\alpha+1)}j_{\alpha+1}(i\lambda x),\quad x\in\mathbb{R},\tag{2.3}$$

where j_α is the normalized Bessel function of the first kind and order α, defined by

$$j_\alpha(\lambda x)=\begin{cases}2^\alpha\Gamma(\alpha+1)\dfrac{J_\alpha(\lambda x)}{(\lambda x)^\alpha} & \text{if } \lambda x\neq 0,\\ 1 & \text{if } \lambda x=0,\end{cases}\tag{2.4}$$

where J_α is the Bessel function of first kind and order α (see [15]).

We have for all $x\in\mathbb{R}$ that

$$\begin{gathered}\text{the function } \lambda\longrightarrow j_\alpha(\lambda x) \text{ is even on } \mathbb{R},\\ \left|E_\alpha(-i\lambda x)\right|\leq 1.\end{gathered}\tag{2.5}$$

Let A_α be the function defined on $\mathbb{R}$ by

$$A_\alpha(x)=\frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)},\quad x\in\mathbb{R},\tag{2.6}$$

and let μ_α be the weighted Lebesgue measure on $\mathbb{R}$ given by

$$d\mu_\alpha(x) = A_\alpha(x)dx. \tag{2.7}$$

For every $1 \leq p \leq +\infty$, we denote by $L^p(\mu_\alpha)$ the space $L^p(\mathbb{R}, d\mu_\alpha)$ and we use $\|\cdot\|_{p,\alpha}$ as a shorthand for $\|\cdot\|_{L^p(\mu_\alpha)}$.

The Dunkl transform $\mathcal{F}_\alpha$ which was introduced by Dunkl in [2] is defined for $f \in L^1(\mu_\alpha)$ by

$$\mathcal{F}_\alpha(f)(x) = \int_{\mathbb{R}} E_\alpha(-ixy)f(y)d\mu_\alpha(y), \quad x \in \mathbb{R}. \tag{2.8}$$

According to [16], we have the following results:

(i) for all $f \in L^1(\mu_\alpha)$, we have $\|\mathcal{F}_\alpha(f)\|_{\infty,\alpha} \leq \|f\|_{1,\alpha}$;

(ii) for all $f \in L^1(\mu_\alpha)$ such that $\mathcal{F}_\alpha(f) \in L^1(\mu_\alpha)$, we have the inversion formula

$$f(x) = \int_{\mathbb{R}} E_\alpha(i\lambda x)\mathcal{F}_\alpha(f)(\lambda)d\mu_\alpha(\lambda), \quad \text{a.e } x \in \mathbb{R}; \tag{2.9}$$

(iii) for every $f \in L^2(\mu_\alpha)$, we have

$$\left|\left|\mathcal{F}_\alpha(f)\right|\right|_{2,\alpha} = \|f\|_{2,\alpha}. \tag{2.10}$$

In the sequel, we consider the signed measure $\gamma_{x,y}$ on $\mathbb{R}$ given by

$$d\gamma_{x,y}(z) = \begin{cases} W_\alpha(x,y,z)d\mu_\alpha(z) & \text{if } x,y \in \mathbb{R}\backslash\{0\}, \\ d\delta_x(z) & \text{if } y = 0, \\ d\delta_y(z) & \text{if } x = 0, \end{cases} \tag{2.11}$$

where W_α (see [3]) is an even function satisfying the following properties:

$$\begin{gathered} W_\alpha(x,y,z) = W_\alpha(y,x,z) = W_\alpha(-x,z,y) = W_\alpha(-z,y,-x), \\ \int_{\mathbb{R}} |\, W_\alpha(x,y,z)\,|\, d\mu_\alpha(z) \leq 4. \end{gathered} \tag{2.12}$$

We have

$$\operatorname{supp}(\gamma_{x,y}) = S_{x,y} \cup (-S_{x,y}) \quad \text{with } S_{x,y} = \big[\,\big|\,|x| - |y|\,\big|, |x| + |y|\big]. \tag{2.13}$$

For $x, y \in \mathbb{R}$ and f a continuous function on $\mathbb{R}$, the Dunkl translation operator τ_x given by

$$\tau_x(f)(y) = \int_{\mathbb{R}} f(z)d\gamma_{x,y}(z) \tag{2.14}$$

satisfies the following properties (see [17]):

(i) τ_x is a continuous linear operator from $\mathscr{E}(\mathbb{R})$ into itself;

(ii) for all $f \in \mathscr{E}(\mathbb{R})$, we have

$$\tau_x(f)(y) = \tau_y(f)(x), \qquad \tau_0(f)(x) = f(x). \tag{2.15}$$

The Dunkl convolution $f *_\alpha g$, of two continuous functions f and g on $\mathbb{R}$ with compact support, is defined by

$$(f *_\alpha g)(x) = \int_{\mathbb{R}} \tau_x(f)(-y) g(y) d\mu_\alpha(y), \quad x \in \mathbb{R}. \tag{2.16}$$

The convolution $*_\alpha$ is associative and commutative (see [3]). The following results are shown in [18].

(i) For all $x \in \mathbb{R}$, the operator τ_x extends to $L^p(\mu_\alpha)$, $p \geq 1$, and we have for $f \in L^p(\mu_\alpha)$ that

$$\|\tau_x(f)\|_{p,\alpha} \leq 4\|f\|_{p,\alpha}. \tag{2.17}$$

(ii) For all $x, \lambda \in \mathbb{R}$ and $f \in L^1(\mu_\alpha)$, we have

$$\mathscr{F}_\alpha(\tau_x(f))(\lambda) = E_\alpha(i\lambda x)\mathscr{F}_\alpha(f)(\lambda). \tag{2.18}$$

(iii) Assume that $p, q, r \in [1, +\infty[$ satisfyies $1/p + 1/q = 1 + 1/r$ (the Young condition). Then, the map $(f, g) \to f *_\alpha g$ defined on $C_c(\mathbb{R}) \times C_c(\mathbb{R})$ extends to a continuous map from $L^p(\mu_\alpha) \times L^q(\mu_\alpha)$ to $L^r(\mu_\alpha)$, and we have

$$\|f *_\alpha g\|_{r,\alpha} \leq 4\|f\|_{p,\alpha}\|g\|_{q,\alpha}. \tag{2.19}$$

(iv) For all $f \in L^1(\mu_\alpha)$ and $g \in L^2(\mu_\alpha)$, we have

$$\mathscr{F}_\alpha(f *_\alpha g) = \mathscr{F}_\alpha(f)\mathscr{F}_\alpha(g). \tag{2.20}$$

3. Estimates for Dunkl translation and weak type (1, 1) of the uncentered maximal operator

In this section, we establish estimates of $\tau_x(\chi_{[-\varepsilon,\varepsilon]})(y)$, $x, y \in \mathbb{R}$, $x \neq 0$, where $\chi_{[-\varepsilon,\varepsilon]}$ is the characteristic function of the interval $[-\varepsilon, \varepsilon]$, and we prove the weak-type $(1,1)$ of the uncentered maximal operator M and the L^p-boundedness for $1 < p \leq +\infty$ of M.

We observe that for $x, y \in \mathbb{R}\backslash\{0\}$ and $\varepsilon > 0$,

$$|\tau_x(\chi_{[-\varepsilon,\varepsilon]})(y)| \leq c\check{\tau}_{|x|}(\chi_{[0,\varepsilon]})(|y|), \tag{3.1}$$

where for a continuous function f on $[0, +\infty[$ and $r, s > 0$, $\check{\tau}_r$ denotes the translation of the Bessel hypergroup given by

$$\check{\tau}_r(f)(s) = \frac{2^{2-\alpha}(\Gamma(\alpha+1))^2}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_0^{+\infty} f(z)\Delta_\alpha(r, s, t) d\mu_\alpha(t) \tag{3.2}$$

with

$$\Delta_\alpha(r,s,t) = \begin{cases} \dfrac{\big([(r+s)^2-t^2][t^2-(r-s)^2]\big)^{\alpha-1/2}}{(rst)^{2\alpha}} & \text{if } |r-s|<t<r+s, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

On the other hand, we have from (2.3), (2.5), and (2.8) that

$$\big|\mathcal{F}_\alpha(\chi_{[-\varepsilon,\varepsilon]})(\lambda)\big| \le \frac{\varepsilon}{\alpha+1}A_\alpha(\varepsilon) \quad \text{for } \varepsilon>0,\ \lambda\in\mathbb{R}, \tag{3.4}$$

and by (2.4),

$$\big|\mathcal{F}_\alpha(\chi_{[-\varepsilon,\varepsilon]})(\lambda)\big| \le c\varepsilon^{\alpha+1/2}\lambda^{-\alpha-3/2} \quad \text{for } \lambda\in[\varepsilon^{-1},+\infty[. \tag{3.5}$$

Then, using (3.4), (3.5), and the fact that $|E_\alpha(i\lambda x)| \le c(A_\alpha(x))^{-1/2}|\lambda|^{-\alpha-1/2}$, for $|x|>2\varepsilon$, $\lambda\in\mathbb{R}\backslash\{0\}$, the next lemma follows closely the argumentations of [11, Proposition 4.6 and Lemma 5.1].

Lemma 3.1. *There exists a positive constant c such that for any $x,y\in\mathbb{R}$, $x\neq 0$, and $\varepsilon>0$, one has*

$$\big|\tau_x(\chi_{[-\varepsilon,\varepsilon]})(y)\big| \le c\frac{A_\alpha(\varepsilon)}{A_\alpha(x)}. \tag{3.6}$$

Notation 3.2. For $x\in\mathbb{R}$ and $\varepsilon>0$, we denote by $B(x,\varepsilon)$ the interval $[\max\{0,|x|-\varepsilon\}, |x|+\varepsilon[$.

Lemma 3.3. *There exists a positive constant c such that for any $x,y\in\mathbb{R}$ and $\varepsilon>0$, one has*

$$\big|\tau_x(\chi_{[-\varepsilon,\varepsilon]})(y)\big| \le c\frac{\mu_\alpha(]-\varepsilon,\varepsilon[)}{\mu_\alpha(B(x,\varepsilon))}. \tag{3.7}$$

Proof. On the one hand, we have for $|x|\le\varepsilon$ that

$$\mu_\alpha(B(x,\varepsilon)) = \int_{B(x,\varepsilon)} d\mu_\alpha(y) = \int_0^{|x|+\varepsilon} d\mu_\alpha(y) \le c\mu_\alpha(]-\varepsilon,\varepsilon[), \tag{3.8}$$

since

$$\frac{1}{4}\big|\tau_x(\chi_{[-\varepsilon,\varepsilon]})(-y)\big| \le 1, \quad x,y\in\mathbb{R}, \tag{3.9}$$

then we obtain (3.7) for $|x|\le\varepsilon$.

On the other hand, we have for $|x|>\varepsilon$,

$$\begin{aligned} \mu_\alpha(B(x,\varepsilon)) &= \int_{|x|-\varepsilon}^{|x|+\varepsilon} d\mu_\alpha(y) \le c(|x|+\varepsilon)^{2\alpha+1}\int_{|x|-\varepsilon}^{|x|+\varepsilon} dy \\ &\le c\mu_\alpha(]-\varepsilon,\varepsilon[)\frac{A_\alpha(x)}{A_\alpha(\varepsilon)}. \end{aligned} \tag{3.10}$$

Then by (3.6), we obtain (3.7) for $|x|>\varepsilon$, which proves the result. □

According to [7, Lemma 1.6] (see also [11, Lemma 4.21]), we have the following Vitali covering lemma.

Lemma 3.4. *Let E be a measurable subset of $\mathbb{R}_+$ (with respect to μ_α) which is covered by the union of a family of bounded intervals $\{B_j\}$, where $B_j = B(x_j, r_j)$. Then from this family, one can select a disjoints subsequence, $B_1, B_2, \ldots, B_h, \ldots$, (which may be finite) such that*

$$\sum_h \mu_\alpha(B_h) \geq c\mu_\alpha(E). \tag{3.11}$$

Theorem 3.5. *The uncentered maximal operator M is of weak type* $(1,1)$.

Proof. For $\varepsilon > 0$, $x \in \mathbb{R}$, $|z| \in B(x,\varepsilon)$, and $f \in L^1(\mu_\alpha)$, we have

$$\int_{-\varepsilon}^{\varepsilon} \tau_z(f)(-y)\,d\mu_\alpha(y) = \left(f *_\alpha \chi_{[-\varepsilon,\varepsilon]}\right)(z) = \int_{\mathbb{R}} f(y)\tau_z\left(\chi_{[-\varepsilon,\varepsilon]}\right)(-y)\,d\mu_\alpha(y), \tag{3.12}$$

then using (2.13), (2.14), and (3.7), we obtain

$$\begin{aligned} \left| \int_{-\varepsilon}^{\varepsilon} \tau_z(f)(-y)\,d\mu_\alpha(y) \right| &\leq \int_{|y|\in B(z,\varepsilon)} \left| \tau_z\left(\chi_{[-\varepsilon,\varepsilon]}\right)(-y) \right| |f(y)|\,d\mu_\alpha(y) \\ &\leq c\left(\int_{|y|\in B(z,\varepsilon)} |f(y)|\,d\mu_\alpha(y) \right) \frac{\mu_\alpha(]-\varepsilon,\varepsilon[)}{\mu_\alpha(B(z,\varepsilon))}, \end{aligned} \tag{3.13}$$

hence we deduce that

$$M(f)(x) \leq c\widetilde{M}(f)(x), \tag{3.14}$$

where $\widetilde{M}(f)$ is defined by

$$\widetilde{M}(f)(x) = \sup_{\varepsilon>0,\,|z|\in B(x,\varepsilon)} \frac{1}{\mu_\alpha(B(z,\varepsilon))} \int_{|y|\in B(z,\varepsilon)} |f(y)|\,d\mu_\alpha(y). \tag{3.15}$$

Observe that we have

$$\widetilde{M}(f)(-x) = \widetilde{M}(f)(x), \quad x \in \mathbb{R}. \tag{3.16}$$

For $\lambda > 0$, put

$$\begin{aligned} \widetilde{E}_\lambda &= \{x \in \mathbb{R};\ \widetilde{M}(f)(x) > \lambda\}, \\ \widetilde{E}_\lambda^+ &= \{x \in \mathbb{R}_+;\ \widetilde{M}(f)(x) > \lambda\}, \\ \widetilde{E}_\lambda^- &= \{x \in \mathbb{R}_-^*;\ \widetilde{M}(f)(x) > \lambda\}. \end{aligned} \tag{3.17}$$

By (3.16) we obtain

$$\mu_\alpha(\widetilde{E}_\lambda^+) = \mu_\alpha(\widetilde{E}_\lambda^-), \qquad \mu_\alpha(\widetilde{E}_\lambda) = 2\mu_\alpha(\widetilde{E}_\lambda^+). \tag{3.18}$$

Now, for each $x \in \widetilde{E}_\lambda^+$, there exist $\varepsilon > 0$ and $z \in \mathbb{R}$ such that

$$|z| \in B(x,\varepsilon), \qquad \int_{|y|\in B(z,\varepsilon)} |f(y)|\, d\mu_\alpha(y) > \lambda\mu_\alpha(B(z,\varepsilon)). \tag{3.19}$$

Furthermore, note that $x \in B(z,\varepsilon)$, then when x runs through the set $\widetilde{E}_\lambda^+$, the union of the corresponding $B(z,\varepsilon)$ covers $\widetilde{E}_\lambda^+$. Thus, using Lemma 3.4, we can select a disjoint subsequence $B(z_1,\varepsilon_1),\dots,B(z_h,\varepsilon_h),\dots,$ (which may be finite) such that

$$\sum_h \mu_\alpha(B(z_h,\varepsilon_h)) \geq c\mu_\alpha(\widetilde{E}_\lambda^+). \tag{3.20}$$

We have

$$\int_{|y|\in\bigcup_h B(z_h,\varepsilon_h)} |f(y)|\, d\mu_\alpha(y) \geq \sum_h \int_{|y|\in B(z_h,\varepsilon_h)} |f(y)|\, d\mu_\alpha(y). \tag{3.21}$$

Applying (3.19) and (3.20) to each of the mutually disjoint intervals, we get

$$\int_{|y|\in\bigcup_h B(z_h,\varepsilon_h)} |f(y)|\, d\mu_\alpha(y) > \lambda\sum_h \mu_\alpha(B(z_h,\varepsilon_h)) \geq \lambda c\mu_\alpha(\widetilde{E}_\lambda^+). \tag{3.22}$$

But since the first member of this inequality is majorized by $\|f\|_{1,\alpha}$, we obtain

$$\mu_\alpha(\widetilde{E}_\lambda^+) \leq c\frac{\|f\|_{1,\alpha}}{\lambda}, \tag{3.23}$$

and by (3.18), we deduce that

$$\mu_\alpha(\widetilde{E}_\lambda) \leq c\frac{\|f\|_{1,\alpha}}{\lambda}, \tag{3.24}$$

which gives that $\widetilde{M}$ is of weak type $(1,1)$, and hence from (3.14), the same is true for M. □

As consequence of Theorem 3.5, we obtain the following corollary.

Corollary 3.6. *If $1 < p \leq +\infty$ and $f \in L^p(\mu_\alpha)$, then one has*

$$M(f) \in L^p(\mu_\alpha), \quad \big\|M(f)\big\|_{p,\alpha} \leq c\|f\|_{p,\alpha}. \tag{3.25}$$

Proof. Using the Theorem 3.5, [15, Corollary 21.72], and proceeding in the same manner as in the proof on [2, 1.3.Theorem 1], we obtain the desired results. □

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