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Research Article

On Further Analogs of Hilbert's Inequality

Yongjin Li, You Qian, and Bing He

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By introducing the function $|\ln x - \ln y|/(x+y+|x-y|)$, we establish new inequalities similar to Hilbert's type inequality for integrals. As applications, we give its equivalent form as well.

1. Introduction

If f, g are real functions such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then we have (see [1])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \pi \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2}, \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is the well-known Hilbert's inequality. Inequality (1.1) had been generalized by Hardy-Riesz (see [2]) in 1925 as the following result.

If f, g are real functions such that $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx\, dy < \frac{\pi}{\sin(\pi/p)} \left\{ \int_0^\infty f^p(x)dx \right\}^{1/p} \left\{ \int_0^\infty g^q(x)dx \right\}^{1/q}, \tag{1.2}$$

where the constant factor $c = \pi/\sin(\pi/p)$ is the best possible. When $p = q = 2$, (1.2) reduces to (1.1). Inequality (1.2) is named Hardy-Hilbert's integral inequality, which is important in analysis and its applications (see [3]), it has been studied and generalized in many directions by a number of mathematicians (see [4–8]).

Under the same condition of (1.2), we have Hardy-Hilbert's type inequality (see [1, Theorems 341 and 342]):

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dx\,dy < 4\left\{\int_0^\infty f^2(x)dx\int_0^\infty g^2(x)dx\right\}^{1/2},$$
$$\int_0^\infty\int_0^\infty \frac{\ln x-\ln y}{x-y}f(x)g(y)dx\,dy < \pi^2\left\{\int_0^\infty f^2(x)dx\int_0^\infty g^2(x)dx\right\}^{1/2}, \tag{1.3}$$

where the constant factors 4 and π^2 are both the best possible.

Recently, Li et al. [9] obtained the following result.

Theorem 1.1. *If f, g are real functions such that $0<\int_0^\infty f^2(x)dx<\infty$ and $0<\int_0^\infty g^2(x)dx<\infty$, then one has*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}}dx\,dy < c\left(\int_0^\infty f^2(x)dx\right)^{1/2}\left(\int_0^\infty g^2(x)dx\right)^{1/2}, \tag{1.4}$$

where the constant factor $c=\sqrt{2}(\pi-2-\arctan\sqrt{2})=1.7408\ldots$.

In this paper, we give a further analogs of Hilbert's type inequality and its applications.

2. Main results and applications

Theorem 2.1. *If $f(x),g(x)\geq 0$, $0<\int_0^\infty f^2(x)dx<\infty$, $0<\int_0^\infty g^2(x)dx<\infty$, then one has*

$$\int_0^\infty\int_0^\infty \frac{|\ln x-\ln y|}{x+y+|x-y|}f(x)g(y)\,dx\,dy < 4\left(\int_0^\infty f^2(x)dx\right)^{1/2}\left(\int_0^\infty g^2(x)dx\right)^{1/2}, \tag{2.1}$$

where the constant factor 4 is the best possible.

Proof. Applying Hölder's inequality, we obtain

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{|\ln x-\ln y|}{x+y+|x-y|}f(x)g(y)\,dx\,dy\\
&\quad=\int_0^\infty\int_0^\infty\left[\left(\frac{|\ln x-\ln y|}{x+y+|x-y|}\right)^{1/2}f(x)\left(\frac{x}{y}\right)^{1/4}\right]\\
&\qquad\times\left[\left(\frac{|\ln x-\ln y|}{x+y+|x-y|}\right)^{1/2}g(y)\left(\frac{y}{x}\right)^{1/4}\right]dx\,dy\\
&\quad\leq\left\{\int_0^\infty\left(\int_0^\infty\frac{|\ln x-\ln y|}{x+y+|x-y|}\left(\frac{x}{y}\right)^{1/2}dy\right)f^2(x)\,dx\right\}\\
&\qquad\times\left\{\int_0^\infty\left(\int_0^\infty\frac{|\ln x-\ln y|}{x+y+|x-y|}\left(\frac{y}{x}\right)^{1/2}dx\right)g^2(y)\,dy\right\}.
\end{aligned} \tag{2.2}$$

Define the weight function $\overline{\omega}(u)$ as

$$\omega(u):=\int_0^\infty\frac{|\ln u-\ln v|}{u+v+|u-v|}\left(\frac{u}{v}\right)^{1/2}dv. \tag{2.3}$$

For fixed u, letting $v = ut$, we have

$$\begin{aligned}\omega(u) &= \int_0^\infty \frac{|\ln u - \ln tu|}{u + tu + |u - tu|}\left(\frac{1}{t}\right)^{1/2} u\,dt = \int_0^\infty \frac{|\ln t|}{1 + t + |1 - t|}\left(\frac{1}{t}\right)^{1/2} dt \\ &= -\int_0^1 \frac{\ln t}{2}\left(\frac{1}{t}\right)^{1/2} dt + \int_1^\infty \frac{\ln t}{2t}\left(\frac{1}{t}\right)^{1/2} dt \\ &= -\int_0^1 (\ln t)\left(\frac{1}{t}\right)^{1/2} dt = -4\int_0^1 \ln s\,ds(t^{1/2} = s) = 4.\end{aligned} \tag{2.4}$$

Thus

$$\int_0^\infty\int_0^\infty \frac{|\ln x - \ln y|}{x + y + |x - y|} f(x)g(y)\,dx\,dy \le 4\left(\int_0^\infty f^2(x)dx\right)^{1/2}\left(\int_0^\infty g^2(x)dx\right)^{1/2}. \tag{2.5}$$

If (2.5) takes the form of the equality, then there exist constants c and d, such that they are not all zero and

$$c\frac{|\ln x - \ln y|}{x + y + |x - y|} f^2(x)\left(\frac{x}{y}\right)^{1/2} = d\frac{|\ln x - \ln y|}{x + y + |x - y|} g^2(y)\left(\frac{y}{x}\right)^{1/2}, \quad \text{a.e. on } (0,\infty)\times(0,\infty). \tag{2.6}$$

Then we have

$$cxf^2(x) = dyg^2(y), \quad \text{a.e. on } (0,\infty)\times(0,\infty). \tag{2.7}$$

Hence we have

$$cxf^2(x) = dyg^2(y) = \text{constant}, \quad \text{a.e. on } (0,\infty)\times(0,\infty). \tag{2.8}$$

Without losing the generality, suppose $c \neq 0$, then

$$\int_0^\infty f^2(x)dx = \int_0^\infty \frac{1}{x}\frac{\text{const}}{c}dx = \frac{\text{const}}{c}\int_0^\infty \frac{1}{x}dx, \tag{2.9}$$

which contradicts the facts that $0 < \int_0^\infty f^2(x)dx < \infty$. Hence (2.5) takes the form of strict inequality. So we have (2.1).

Assume that the constant factor 4 in (2.1) is not the best possible, then there exists a positive number K with $K < 4$ and $a > 0$; we have

$$\int_a^\infty\int_0^\infty \frac{|\ln x - \ln y|}{x + y + |x - y|} f(x)g(y)dx\,dy < K\left(\int_a^\infty f^2(x)dx\right)^{1/2}\left(\int_a^\infty g^2(x)dx\right)^{1/2}. \tag{2.10}$$

For $0 < \varepsilon < 1$, setting $f_\varepsilon(x) = x^{(-\varepsilon-1)/2}$, for $x \in [1,\infty)$; $f_\varepsilon(x) = 0$, for $x \in (0,1)$, $g_\varepsilon(y) = y^{(-\varepsilon-1)/2}$, for $y \in [1,\infty)$; $g_\varepsilon(y) = 0$, for $y \in (0,1)$.

Since

$$K\left(\int_a^\infty f^2(x)dx\right)^{1/2}\left(\int_a^\infty g^2(x)dx\right)^{1/2} = K\int_a^\infty x^{-1-\varepsilon}dx = K\cdot\frac{1}{\varepsilon a^\varepsilon}, \tag{2.11}$$

setting $y = ux$, we find

$$\begin{aligned}
&\int_a^\infty \int_0^\infty \frac{|\ln x - \ln y|}{x+y+|x-y|} f_\varepsilon(x) g_\varepsilon(y) dx\, dy \\
&\quad = \int_a^\infty \int_b^\infty \frac{|\ln x - \ln y|}{x+y+|x-y|} x^{-(1+\varepsilon)/2} y^{-(1+\varepsilon)/2} dx\, dy \\
&\quad = \int_a^\infty \int_{b/x}^\infty \frac{|\ln x - \ln ux|}{x+ux+|1-ux|} x \cdot x^{-(1+\varepsilon)} u^{-(1+\varepsilon)/2} dx\, du \\
&\quad = \int_a^\infty \int_{b/x}^\infty \frac{x^{-(1+\varepsilon)} u^{-(1+\varepsilon)/2} |\ln u|}{1+u+|1-u|} dx\, du.
\end{aligned} \tag{2.12}$$

By (2.10) and for $b \to 0^+$, we have

$$\int_a^\infty \int_0^\infty \frac{x^{-(1+\varepsilon)} u^{-(1+\varepsilon)/2} |\ln u|}{1+u+|1-u|} dx\, dy \leq K \cdot \frac{1}{\varepsilon a^\varepsilon}, \tag{2.13}$$

or

$$\frac{1}{\varepsilon a^\varepsilon} \int_0^\infty \frac{u^{-(1+\varepsilon)/2} |\ln u|}{1+u+|1-u|} du \leq K \cdot \frac{1}{\varepsilon a^\varepsilon}, \tag{2.14}$$

that is

$$\int_0^\infty \frac{u^{-(1+\varepsilon)/2} |\ln u|}{1+u+|1-u|} du \leq K. \tag{2.15}$$

When $\varepsilon \to 0^+$, we have

$$\int_0^\infty \frac{u^{-(1+\varepsilon)/2} |\ln u|}{1+u+|1-u|} du = \int_0^\infty \frac{u^{-1/2} |\ln u|}{1+u+|1-u|} du + o(1) = 4 + o(1). \tag{2.16}$$

This contradicts the hypothesis. Hence the constant factor 4 in (2.1) is the best possible. □

THEOREM 2.2. *Suppose $f \geq 0$ and $0 < \int_0^\infty f^2(x) dx < \infty$. Then*

$$\int_0^\infty \left[\int_0^\infty \frac{|\ln x - \ln y|}{x+y+|x-y|} f(x) \right]^2 dy < 16 \int_0^\infty f^2(x) dx, \tag{2.17}$$

where the constant factor 16 is the best possible. Inequality (2.17) is equivalent to (2.1).

Proof. Letting $g(y)=\int_0^\infty(|\ln x-\ln y|/(x+y+|x-y|))f(x)dx$, then by (2.5) we get

$$\begin{aligned}
0&<\int_0^\infty g^2(y)dy\\
&=\int_0^\infty\left[\int_0^\infty\frac{|\ln x-\ln y|}{x+y+|x-y|}f(x)dx\right]^2dy\\
&=\int_0^\infty\int_0^\infty\frac{|\ln x-\ln y|}{x+y+|x-y|}f(x)g(y)dx\,dy\\
&\leq 4\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(y)dy\right\}^{1/2}.
\end{aligned}\tag{2.18}$$

Hence we obtain

$$0<\int_0^\infty g^2(y)dy=16\int_0^\infty f^2(x)dx<\infty.\tag{2.19}$$

By (2.1), both (2.18) and (2.19) take the form of strict inequality, so we have (2.17). On the other hand, suppose that (2.17) is valid. By Hölder's inequality, we find

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\frac{|\ln x-\ln y|}{x+y+|x-y|}f(x)g(y)dx\,dy\\
&\quad=\int_0^\infty\left[\int_0^\infty\frac{|\ln x-\ln y|}{x+y+|x-y|}f(x)dx\right]g(y)dy\\
&\quad\leq\left\{\int_0^\infty\left[\int_0^\infty\frac{|\ln x-\ln y|}{x+y+|x-y|}f(x)dx\right]^2dy\right\}^{1/2}\left\{\int_0^\infty g^2(x)dx\right\}^{1/2}.
\end{aligned}\tag{2.20}$$

By (2.17) we have (2.1). Thus (2.1) and (2.17) are equivalent.

If the constant 16 in (2.17) is not the best possible, by (2.20), we may get a contradiction that the constant factor in (2.1) is not the best possible. This completes the proof. □

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Yongjin Li: Institute of Logic and Cognition, Department of Mathematics, Sun Yat-Sen University, Guangzhou 510275, China
Email address: stslyj@mail.sysu.edu.cn

You Qian: Department of Mathematics, Sun Yat-Sen University, Guangzhou 510275, China
Email address: vincin@126.com

Bing He: Department of Mathematics, Guangdong Education College, Guangzhou 510303, China
Email address: hzs314@163.com