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Research Article

On the Generalized Ulam-Gavruta-Rassias Stability of Mixed-Type Linear and Euler-Lagrange-Rassias Functional Equations

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In this paper, the mixed-type linear and Euler-Lagrange-Rassias functional equations introduced by J. M. Rassias is generalized to the following n-dimensional functional equation: $f(\sum_{i=1}^{n} x_i) + (n-2)\sum_{i=1}^{n} f(x_i) = \sum_{1\le i<j\le n} f(x_i - x_j)$ when $n > 2$. We prove the general solutions and investigate its generalized Ulam-Gavruta-Rassias stability.

1. Introduction

In 1940, Ulam [1] proposed the famous Ulam stability problem of linear mappings. In 1941, Hyers [2] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality* $\|f(x+y) - f(x) - f(y)\| \le \varepsilon$ for all $x, y \in E$. It was shown that the limit $L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying $\|f(x) - L(x)\| \le \varepsilon$. In 1982–1998, Rassias [3–9] generalized the result to include the following theorem.

Theorem 1.1. *Let X be a real-normed linear space and let Y be a real-complete-normed linear space. Assume in addition that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \ge 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \ne 1$, and f satisfies the Cauchy-Gavruta-Rassias inequality*

$$||f(x+y) - f(x) - f(y)|| \le \theta \|x\|^p \|y\|^q \tag{1.1}$$

for all $x, y \in X$. Then, there exists a unique additive mapping $L : X \to Y$ satisfying

$$f(x) - L(x) \le \frac{\theta}{|2^r - 2|} \|x\|^r \quad \forall\, x \in X. \tag{1.2}$$

If in addition $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is $\mathbb{R}$-linear mapping.

In 2002, Rassias [10] established the Ulam stability of the following *mixed-type* functional equation:

$$f\left(\sum_{i=1}^{3} x_i\right) + \sum_{i=1}^{3} f(x_i) = \sum_{1 \le i < j \le 3} f(x_i + x_j) \tag{1.3}$$

on restricted domains. In this paper, we will generalize Rassias' work to the following n-dimensional mixed-type functional equation:

$$f\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} f(x_i) = \sum_{1 \le i < j \le n} f(x_i + x_j) \tag{1.4}$$

when $n > 2$, and will investigate its generalized Ulam-Gavruta-Rassias stability.

2. The general solution

Theorem 2.1. *Let $n > 2$ be a positive integer, and let X and Y be vector spaces.*

A function $f : X \to Y$ satisfies the functional equation

$$f\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} f(x_i) = \sum_{1 \le i < j \le n} f(x_i + x_j) \tag{2.1}$$

if and only if the even part of f, defined by $f_e(x) = (1/2)(f(x) + f(-x))$ for all $x \in X$, satisfies the classical quadratic functional equation, which is also a special Euler-Lagrange-Rassias equation [7, 9],

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \tag{2.2}$$

and the odd part of f, defined by $f_o(x) = (1/2)(f(x) - f(-x))$ for all $x \in X$, satisfies the Cauchy functional equation

$$f(x+y) = f(x) + f(y). \tag{2.3}$$

Proof. For the *if* part of the proof, suppose that $f : X \to Y$ satisfies (2.1), we can uniquely express f as $f(x) = f_e(x) + f_o(x)$ for all $x \in X$, where the even part, f_e, and the odd part, f_o, are defined as in the theorem. We will show that f_e satisfies (2.2) and f_o satisfies (2.3).

Setting $(x_1, x_2, \dots, x_n) = (0, 0, \dots, 0)$ in (2.1), we see that $f(0) = 0$. Setting $(x_1, x_2, \dots, x_n) = (x, y, -y, 0, 0, \dots, 0)$ in (2.1), we get

$$\begin{aligned} f(x) + (n-2)\big(f(x) + f(y) + f(-y)\big) &= f(x-y) + f(x+y) \\ &\quad + (n-3)\big(f(x) + f(y) + f(-y)\big), \end{aligned} \tag{2.4}$$

which is simplified to

$$2f(x) + f(y) + f(-y) = f(x+y) + f(x-y) \tag{2.5}$$

for all $x, y \in X$. Replacing x and y with $-x$ and $-y$, respectively, then taking half the sum and half the difference with (2.5), we have

$$\begin{aligned} 2f_e(x) + f_e(y) + f_e(-y) &= f_e(x+y) + f_e(x-y), \\ 2f_o(x) + f_o(y) + f_o(-y) &= f_o(x+y) + f_o(x-y). \end{aligned} \tag{2.6}$$

By the evenness of f_e, we immediately see that f_e satisfies the classical quadratic functional equation given by (2.2). By the oddness of f_o, we see that $2f_o(x) = f_o(x+y) + f_o(x-y)$ which is recognized as the Jensen functional equation. Since $f_o(0) = 0$, if we put $y = x$ in the above equation, then $f(2x) = 2f(x)$. By another substitution, $(x, y) = ((x+y)/2, (x-y)/2)$, we derive the Cauchy functional equation $f_o(x+y) = f_o(x) + f_o(y)$.

Now for the *only if* part of the proof, suppose that the even part and the odd part of $f : X \to Y$ satisfy (2.2) and (2.3), respectively, that is, $f_e(x+y) + f_e(x-y) = 2f_e(x) + 2f_e(y)$ and $f_o(x+y) = f_o(x) + f_o(y)$. We will show that f satisfies (2.1). Noting that a linear combination of two solutions of (2.1) yields just another solution, we will in turn prove that each part of f satisfies (2.1).

First, consider the odd part and make use of the linearity of the Cauchy functional equation. The left-hand side of (2.1) is

$$f_o\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} f_o(x_i) = \sum_{i=1}^{n} f_o(x_i) + (n-2)\sum_{i=1}^{n} f_o(x_i) = (n-1)\sum_{i=1}^{n} f_o(x_i), \tag{2.7}$$

and the right-hand side of (2.1) is

$$\sum_{1\le i<j\le n} f_o(x_i + x_j) = \sum_{1\le i<j\le n} \left(f_o(x_i) + f_o(x_j)\right) = \frac{2}{n}\binom{n}{2}\sum_{i=1}^{n} f_o(x_i) = (n-1)\sum_{i=1}^{n} f_o(x_i). \tag{2.8}$$

Thus, we have established (2.1) on the odd part of f.

For the even part, we will show by mathematical induction that (2.1) holds for every positive integer n. For $n = 1$, we take $\sum_{1\le i<j\le 1} f_e(x_i + x_j)$ as 0; then $f_e(x_1) + (1-2)f_e(x_1) = 0$, which is trivially true. For $n = 2$, we have $f_e(x_1 + x_2) + 0 = f_e(x_1 + x_2)$, which is again trivially true. For $n \ge 3$, we assume that (2.1) holds for every number of variables from 1 to $n-1$, that is,

$$f_e\left(\sum_{i=1}^{k} x_i\right) + (k-2)\sum_{i=1}^{k} f_e(x_i) = \sum_{1\le i<j\le k} f_e(x_i + x_j) \tag{2.9}$$

for $k = 1, 2, \ldots, n-1$. For each $i, j = 1, 2, \ldots, n$ with $i \ne j$, we have

$$f_e(x_i - x_j) + f_e(x_i + x_j) = 2\left(f_e(x_i) + f_e(x_j)\right). \tag{2.10}$$

Then,

$$\sum_{1\le i<j\le n} \left(f_e(x_i - x_j) + f_e(x_i + x_j)\right) = 2\sum_{1\le i<j\le n} \left(f_e(x_i) + f_e(x_j)\right) = \frac{4}{n}\binom{n}{2}\sum_{i=1}^{n} f_e(x_i). \tag{2.11}$$

Thus,

$$\sum_{1\leq i<j\leq n}\left(f_e(x_i-x_j)+f_e(x_i+x_j)\right)=2(n-1)\sum_{i=1}^{n}f_e(x_i). \tag{2.12}$$

For each $j,k=1,2,\ldots,n$ with $j\neq k$, we have

$$f_e\left(\sum_{i=1}^{n}x_i-2x_j\right)+f_e\left(\sum_{i=1}^{n}x_i-2x_k\right)=2f_e\left(\sum_{i=1}^{n}x_i-x_j-x_k\right)+2f_e(x_j-x_k). \tag{2.13}$$

Write down the above equation for every possible pair (j,k) and note that there are $\binom{n}{2}$ such pairs; so each $f_e(\sum_{i=1}^{n}x_i-2x_j)$ appears $n-1$ times in all $\binom{n}{2}$ equations. Adding up the equations, we get

$$(n-1)\sum_{j=1}^{n}f_e\left(\sum_{i=1}^{n}x_i-2x_j\right)=2\sum_{1\leq j<k\leq n}f_e\left(\sum_{i=1}^{n}x_i-x_j-x_k\right)+2\sum_{1\leq j<k\leq n}f_e(x_j-x_k). \tag{2.14}$$

For each $j=1,2,\ldots,n$, we have

$$f_e\left(\sum_{i=1}^{n}x_i\right)+f\left(\sum_{i=1}^{n}x_i-2x_j\right)=2f_e\left(\sum_{i=1}^{n}x_i-x_j\right)+2f_e(x_j). \tag{2.15}$$

Sum the above equation for all j's and substitute the result from (2.12) and (2.14), then rearrange the resulting equation

$$\begin{aligned}&nf_e\left(\sum_{i=1}^{n}x_i\right)+\frac{2}{n-1}\sum_{1\leq j<k\leq n}f_e\left(\sum_{i=1}^{n}x_i-x_j-x_k\right)\\&\quad=2\sum_{j=1}^{n}f_e\left(\sum_{i=1}^{n}x_i-x_j\right)+\frac{2}{n-1}\sum_{1\leq i<j\leq n}f_e(x_i+x_j)-2\sum_{i=1}^{n}f_e(x_i).\end{aligned} \tag{2.16}$$

Note that $\sum_{j=1}^{n}f_e(\sum_{i=1}^{n}x_i-x_j)$ is the sum of f of x_i's taken $n-1$ variables at a time, and $\sum_{1\leq j<k\leq n}f_e(\sum_{i=1}^{n}x_i-x_j-x_k)$ is the sum of f of x_i's taken $n-2$ variables at a time. From the induction assumption, (2.1) holds for $n-1$ and $n-2$ variables, that is,

$$\begin{aligned}&\sum_{j=1}^{n}f_e\left(\sum_{i=1}^{n}x_i-x_j\right)+(n-1)(n-3)\sum_{i=1}^{n}f_e(x_i)=(n-2)\sum_{1\leq i<j\leq n}f_e(x_i+x_j),\\&\sum_{1\leq j<k\leq n}f_e\left(\sum_{i=1}^{n}x_i-x_j-x_k\right)+\frac{(n-1)(n-2)(n-4)}{2}\sum_{i=1}^{n}f_e(x_i)\\&\qquad=\frac{(n-2)(n-3)}{2}\sum_{1\leq i<j\leq n}f_e(x_i+x_j).\end{aligned} \tag{2.17}$$

Substitute (2.17) into (2.16) and simplify, we will finally establish (2.1) on the even part of f. Thus, f satisfies (2.1) and the proof is complete. □

3. The Ulam-Gavruta-Rassias stability

Rassias [10] established the Ulam stability of (2.1) in the special case when $n = 3$ on restricted domains. The following theorem provides a general condition for which a *true* general solution discussed in Theorem 2.1 exists near an approximate solution. For convenience, we define

$$Df(x_1,x_2,\dots,x_n) = f\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} f(x_i) - \sum_{1\le i<j\le n} f(x_i+x_j). \tag{3.1}$$

From now on, we will refer to the even part and the odd part of a function by subscripts e and o, respectively.

Theorem 3.1. *Let $n > 2$ be a positive integer, let X be a real vector space, let Y be a Banach space, let $\phi : X^n \to [0,\infty)$ be an even function. Define $\varphi(x) = \phi(x,x,-x,0,\dots,0)$ for all $x \in X$. If*

$$\sum_{i=0}^{\infty} 2^{-i}\varphi(2^i x) \text{ converges}, \qquad \lim_{m\to\infty} 2^{-m}\phi(2^m x_1,\dots,2^m x_n) = 0 \tag{3.2}$$

or

$$\sum_{i=1}^{\infty} 4^{i}\varphi(2^{-i} x) \text{ converges}, \qquad \lim_{m\to\infty} 4^{m}\phi(2^{-m} x_1,\dots,2^{-m} x_n) = 0 \tag{3.3}$$

for all $x_1,x_2,\dots,x_n \in X$, and a function $f : X \to Y$ satisfies $f(0) = 0$ and

$$\|Df(x_1,x_2,\dots,x_n)\| \le \phi(x_1,x_2,\dots,x_n) \tag{3.4}$$

for all $x_1,x_2,\dots,x_n \in X$, then there exists a unique function $T : X \to Y$ that satisfies functional equation (2.1) and, if condition (3.2) holds,

$$\|f_e(x) - T_e(x)\| \le \frac{1}{4}\sum_{i=0}^{\infty} 4^{-i}\varphi(2^i x), \qquad \|f_o(x) - T_o(x)\| \le \frac{1}{2}\sum_{i=0}^{\infty} 2^{-i}\varphi(2^i x0) \tag{3.5}$$

or, if condition (3.3) holds,

$$\|f_e(x) - T_e(x)\| \le \frac{1}{4}\sum_{i=1}^{\infty} 4^{i}\varphi(2^{-i} x), \qquad \|f_e(x) - T_e(x)\| \le \frac{1}{2}\sum_{i=1}^{\infty} 2^{i}\varphi(2^{-i} x). \tag{3.6}$$

The function T is given by

$$T(x)=\begin{cases}\lim_{m\to\infty}4^{-m}f_e(2^m x)+2^{-m}f_o(2^m x) & \text{if condition (3.2) holds,}\\ \lim_{m\to\infty}4^{m}f_e(2^{-m}x)+2^{m}f_o(2^{-m}x) & \text{if condition (3.3) holds}\end{cases} \tag{3.7}$$

for all $x\in X$.

Proof. We will prove the theorem for a function ϕ satisfying condition (3.2) and accordingly inequality (3.5). A proof for conditions (3.3) and (3.6) can be reproduced in a similar manner. Setting $(x_1,x_2,\dots,x_n)=(x,x,-x,0,0,\dots,0)$ in (3.4) and simplifying, we have $\|3f(x)+f(-x)-f(2x)\|\le\varphi(x)$. Replacing x by $-x$, we have $\|3f(-x)+f(x)-f(-2x)\|\le\varphi(-x)=\varphi(x)$. Then,

$$\begin{aligned}
&\|4f_e(x)-f_e(2x)\|\\
&\quad=\frac{1}{2}\|(3f(x)+f(-x)-f(2x))+(3f(-x)+f(x)-f(-2x))\|\\
&\quad\le\frac{1}{2}\|3f(x)+f(-x)-f(2x)\|+\frac{1}{2}\|3f(-x)+f(x)-f(-2x)\|\\
&\quad\le\frac{1}{2}\varphi(x)+\frac{1}{2}\varphi(x)=\varphi(x),\\
&\|2f_o(x)-f_o(2x)\|\\
&\quad=\frac{1}{2}\|(3f(x)+f(-x)-f(2x))-(3f(-x)+f(x)-f(-2x))\|\\
&\quad\le\frac{1}{2}\|3f(x)+f(-x)-f(2x)\|+\frac{1}{2}\|3f(-x)+f(x)-f(-2x)\|\\
&\quad\le\frac{1}{2}\varphi(x)+\frac{1}{2}\varphi(x)=\varphi(x).
\end{aligned}\tag{3.8}$$

Rewrite the inequality on f_e as $\|f_e(x)-4^{-1}f_e(2x)\|\le 4^{-1}\varphi(x)$ for all $x\in X$. Suppose that $\|f_e(x)-4^{-m}f_e(2^m x)\|\le(1/4)\sum_{i=0}^{m-1}4^{-i}\varphi(2^i x)$ for a positive integer m. Then,

$$\begin{aligned}
&\|f_e(x)-4^{-(m+1)}f_e(2^{m+1}x)\|\\
&\quad\le\|f_e(x)-4^{-m}f_e(2^m x)\|+\|4^{-m}f_e(2^m x)-4^{-(m+1)}f_e(2^{m+1}x)\|\\
&\quad\le\|f_e(x)-4^{-m}f_e(2^m x)\|+4^{-m}\|f_e(2^m x)-4^{-1}f_e(2\cdot 2^m x)\|\\
&\quad\le\frac{1}{4}\sum_{i=0}^{m-1}4^{-i}\varphi(2^i x)+4^{-m}\varphi(2^m x)=\frac{1}{4}\sum_{i=0}^{m}4^{-i}\varphi(2^i x).
\end{aligned}\tag{3.9}$$

Hence, $\|f_e(x)-4^{-m}f_e(2^m x)\|\le(1/4)\sum_{i=0}^{m-1}4^{-i}\varphi(2^i x)$ for every positive integer m.

If we rewrite the inequality for f_o as $\|f_o(x)-2^{-1}f_o(2x)\|\le 2^{-1}\varphi(x)$ and repeat the same steps as in the case of f_e, we will have $\|f_o(x)-2^{-m}f_o(2^m x)\|\le(1/2)\sum_{i=0}^{m-1}2^{-i}\varphi(2^i x)$ for every positive integer m.

The convergence of the sequence $\{4^{-m}f_e(2^m x)\}$ can be settled as follows. For every positive integer p,

$$\begin{aligned}||4^{-(m+p)}f_e(2^{m+p}x)-4^{-m}f_e(2^m x)|| &= 4^{-m}||4^{-p}f_e(2^p\cdot 2^m x)-f_e(2^m x)||\\ &\leq 4^{-m}\cdot\frac{1}{4}\sum_{i=0}^{p-1}4^{-i}\varphi(2^i\cdot 2^m x)\\ &\leq \frac{1}{4}\sum_{i=0}^{\infty}4^{-(i+m)}\varphi(2^{i+m}x).\end{aligned} \tag{3.10}$$

By the definition of ϕ and condition (3.2), the right-hand side approaches 0 as m goes to infinity, hence, we have a Cauchy sequence in a Banach space. Let $T_e(x) = \lim_{m\to\infty}4^{-m}f_e(2^m x)$ for all $x\in X$, and thus $\|f_e(x)-T_e(x)\|\leq(1/4)\sum_{i=0}^{\infty}4^{-i}\varphi(2^i x)$. We can similarly show that $\{2^{-m}f_o(2^m x)\}$ converges; so let $T_o(x)=\lim_{m\to\infty}2^{-m}f_o(2^m x)$ for all $x\in X$, and thus $\|f_o(x)-T_o(x)\|\leq(1/2)\sum_{i=0}^{\infty}2^{-i}\varphi(2^i x)$. Define $T(x)=T_e(x)+T_o(x)$ for all $x\in X$.

In order to show that T satisfies (2.1), we will in turn show that T_e and T_o satisfy (2.1). For convenience, define Df_e and Df_o as the even part and the odd part of Df in (3.1), respectively. For T_e, consider

$$\begin{aligned}&4^{-m}||Df_e(2^m x_1,\ldots,2^m x_n)||\\ &\quad= 4^{-m}\cdot\frac{1}{2}||Df(2^m x_1,\ldots,2^m x_n)+Df(-2^m x_1,\ldots,-2^m x_n)||\\ &\quad\leq 4^{-m}\phi(2^m x_1,\ldots,2^m x_n).\end{aligned} \tag{3.11}$$

As m tend to infinity, the left-hand side approaches $\|DT_e(x_1,\ldots,x_n)\|$ and, by condition (3.2), the right-hand side approaches 0. Thus,

$$DT_e(x_1,x_2,\ldots,x_n) = T_e\left(\sum_{i=1}^{n}x_i\right)+(n-2)\sum_{i=1}^{n}T_e(x_i)-\sum_{1\leq i<j\leq n}T_e(x_i+x_j)=0, \tag{3.12}$$

which shows that T_e satisfies (2.1).

We can similarly show that T_o satisfies (2.1) by considering

$$\begin{aligned}&2^{-m}||Df_o(2^m x_1,\ldots,2^m x_n)||\\ &\quad= 2^{-m}\cdot\frac{1}{2}||Df(2^m x_1,\ldots,2^m x_n)-Df(-2^m x_1,\ldots,-2^m x_n)||\\ &\quad\leq 2^{-m}\phi(2^m x_1,\ldots,2^m x_n),\end{aligned} \tag{3.13}$$

and take the limit as $m\to\infty$. Hence, $T=T_e+T_o$ satisfies (2.1) as desired.

To prove the uniqueness of T, suppose that there exists another function $S : X \to Y$ such that S satisfies (2.1) and satisfies the inequality (3.5) with T replaced by S. Then,

$$\begin{aligned} \|S(x) - T(x)\| &\le \|S(x) - f(x)\| + \|T(x) - f(x)\| \\ &\le \|S_e(x) - f_e(x)\| + \|S_o(x) - f_o(x)\| \\ &\quad + \|T_e(x) - f_e(x)\| + \|T_o(x) - f_o(x)\|. \end{aligned} \tag{3.14}$$

It is straightforward to show that every solution of the *quadratic* functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ has the *quadratic* property $f(nx) = n^2 f(x)$ and every solution of the *linear* functional equation $f(x+y) = f(x)+f(y)$ has the *linear* property $f(nx) = nf(x)$ for every positive integer n and for every x in the domain. We thus obtain

$$\begin{aligned} \|S(x) - T(x)\| &\le 4^{-m}\|S_e(2^m x) - f_e(2^m x)\| + 2^{-m}\|S_o(2^m x) - f_o(2^m x)\| \\ &\quad + 4^{-m}\|T_e(2^m x) - f_e(2^m x)\| + 2^{-m}\|T_o(2^m x) - f_o(2^m x)\| \\ &\le 2\left(4^{-m} \cdot \frac{1}{4}\sum_{i=0}^{\infty} 4^{-i}\varphi(2^i \cdot 2^m x) + \frac{1}{2^m} \cdot \frac{1}{2}\sum_{i=0}^{\infty} 2^{-i}\varphi(2^i \cdot 2^m x)\right) \\ &= \frac{1}{2}\sum_{i=0}^{\infty} 4^{-(i+m)}\varphi(2^{i+m}x) + \sum_{i=0}^{\infty} 2^{-(i+m)}\varphi(2^{i+m}x) \end{aligned} \tag{3.15}$$

for all $x \in X$. As m goes to infinity, the right-hand side approaches 0, and $S(x) = T(x)$ for all $x \in X$. This completes the proof. □

The following corollary proves the Hyers-Ulam stability of (2.1).

Corollary 3.2. *If a function $f : X \to Y$ satisfies $f(0) = 0$ and the functional equation*

$$\|Df(x_1, x_2, \dots, x_n)\| \le \varepsilon \tag{3.16}$$

for some $\varepsilon > 0$ and for all $x_1, x_2, \dots, x_n \in X$, then there exists a unique function $T : X \to Y$ that satisfies functional equation (2.1) and, for all $x \in X$,

$$\|f_e(x) - T_e(x)\| \le \frac{\varepsilon}{3}, \qquad \|f_o(x) - T_o(x)\| \le \varepsilon. \tag{3.17}$$

Proof. Let $\phi(x_1, x_2, \dots, x_n) = \varepsilon$, then condition (3.2) in Theorem 3.1 holds. Hence, it follows from the theorem that there exists a unique function $T : X \to Y$ such that

$$\|f_e(x) - T_e(x)\| \le \frac{1}{4}\sum_{i=0}^{\infty} 4^{-i} \cdot \varepsilon = \frac{\varepsilon}{3}, \qquad \|f_o(x) - T_o(x)\| \le \frac{1}{2}\sum_{i=0}^{\infty} 2^{-i}\varepsilon = \varepsilon. \tag{3.18}$$

□

The following corollary proves the Hyers-Ulam-Rassias stability of (2.1).

Corollary 3.3. *Let p be a positive real number with $0 < p < 1$ or $p > 2$. If a function $f : X \to Y$ satisfies the inequality*

$$||Df(x_1,x_2,\dots,x_n)|| \le \varepsilon\sum_{i=1}^{n}||x_i||^p \tag{3.19}$$

for some $\varepsilon > 0$ and for all $x_1,x_2,\dots,x_n \in X$, then there exists a unique function $T : X \to Y$ that satisfies functional equation (2.1) and, for all $x \in X$,

$$||f_e(x) - T_e(x)|| \le \frac{3\varepsilon}{4|1-2^{p-2}|}\|x\|^p, \qquad ||f_o(x) - T_o(x)|| \le \frac{3\varepsilon}{2|1-2^{p-1}|}\|x\|^p. \tag{3.20}$$

Proof. Substituting $x_1 = x_2 = \cdots = x_n = 0$ into (3.19), we get

$$f(0) + (n-2)\cdot nf(0) = \binom{n}{2}f(0). \tag{3.21}$$

Since $n > 2$, it follows that $1 + n(n-2) > \binom{n}{2}$, hence, $f(0) = 0$.

Let $\phi(x_1,x_2,\dots,x_n) = \varepsilon\sum_{i=1}^{n}\|x_i\|^p$. If $0 < p < 1$, then condition (3.2) in Theorem 3.1 holds and it follows that

$$\begin{aligned} ||f_e(x) - T_e(x)|| &\le \frac{1}{4}\sum_{i=0}^{\infty}4^{-i}(3\varepsilon\cdot 2^{ip}\|x\|^p) = \frac{3\varepsilon}{4(1-2^{p-2})}\|x\|^p, \\ ||f_o(x) - T_o(x)|| &\le \frac{1}{2}\sum_{i=0}^{\infty}2^{-i}(3\varepsilon\cdot 2^{ip}\|x\|^p) = \frac{3\varepsilon}{2(1-2^{p-1})}\|x\|^p. \end{aligned} \tag{3.22}$$

If $p > 1$, we apply Theorem 3.1 with condition (3.3) to get a similar result. □

The following corollary proves the Ulam-Gavruta-Rassias stability of (2.1).

Corollary 3.4. *Let $p_1,p_2,\dots,p_n$ be nonnegative real numbers and $r = \sum_{i=1}^{n}p_i$ with $0 < r < 1$ or $r > 2$. If a function $f : X \to Y$ satisfies the inequality*

$$||Df(x_1,x_2,\dots,x_n)|| \le \varepsilon\prod_{i=1}^{n}||x_i||^{p_i} \tag{3.23}$$

for some $\varepsilon > 0$ and for all $x_1,x_2,\dots,x_n \in X$, then there exists a unique function $T : X \to Y$ that satisfies functional equation (2.1) and, for $n = 3$,

$$||f_e(x) - T_e(x)|| \le \frac{\varepsilon}{4|1-2^{r-2}|}\|x\|^r, \qquad ||f_o(x) - T_o(x)|| \le \frac{\varepsilon}{2|1-2^{r-1}|}\|x\|^r \tag{3.24}$$

for all $x \in X$.

Proof. We can show that $f(0) = 0$ by the same substitution used in the proof of Corollary 3.3. Let $\phi(x_1,x_2,\dots,x_n) = \varepsilon\prod_{i=1}^{n}\|x_i\|^{p_i}$. According to Theorem 3.1, if $0 < r < 1$, then condition (3.2) holds, and if $r > 2$, then condition (3.3) holds. If $n > 3$, then the desired result immediately follows. However, for $n = 3$, we have

$$\begin{aligned} ||f_e(x) - T_e(x)|| &\leq \frac{1}{4}\sum_{i=0}^{\infty} 4^{-i}(\varepsilon \cdot 2^{ir}\|x\|^r) = \frac{\varepsilon}{4(1-2^{r-2})}\|x\|^r, \\ ||f_o(x) - T_o(x)|| &\leq \frac{1}{2}\sum_{i=0}^{\infty} 2^{-i}(\varepsilon \cdot 2^{ir}\|x\|^r) = \frac{\varepsilon}{2(1-2^{r-1})}\|x\|^r \end{aligned} \tag{3.25}$$

when $0 < r < 1$, and a similar result when $r > 1$. □

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