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Research Article

Sobriety and Localic Compactness in Categories of L-Bitopological Spaces

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The notions of L-sobriety and L-spatiality are introduced for the category L-BiTop of L-bitopological spaces. Such notions are used to extend the known adjunction between the category L-Top of L-topological spaces and the category Loc of locals to one between the category L-BiTop and BiLoc. Also, the concepts of localic regularity and localic compactness are introduced in the mentioned category.

1. Introduction

Since topological ideas were introduced in fuzzy sets by Chang [1] in 1968, the notion of L-topology has become rather diverse in its topics as well as its methods. Many authors [2–6] constructed a category to play the same role with respect to a given notion of L-topology as that locales play for classical topological spaces.

In [3, 4, 7], Rodabaugh generalized the classical adjunction between the category Top of topological spaces and the category Loc of locales to another adjunction between L-Top (the category of L-topological spaces) and SLoc (the category of semilocales). Also, he introduced the fuzzification of spatiality and sobriety to generalize the equivalence between the categories SobTop (of sober spaces) and SpatLoc (of spatial locales) to the area of L-topology. These constructions allow the replacement of SLoc if L is a frame. Also, [2, 7, 8] yield a class of adjunctions and equivalences indexed by $L \in$ SFrm which set up classes of Stone representation theorems and Stone-Čech compactifications with appropriate restrictions on L. Finally, many of the ideas concerning the class of basic adjunctions and equivalences were anticipated by Höhle [9].

In this paper, the ideas of spatial bilocales and sober L-bitopological spaces are introduced. Such ideas used to extend the above adjunction between L-Top and SLoc to another one between L-BiTop and BiLoc. Also, with the aid of L-spatiality and L-sobriety, we introduce and study the concepts of localic regularity and localic compactness in the category L-BiTop.

In Section 2, we summarize some of needed tools. In Section 3, the known adjunction between L-Top and Loc is extended to another adjunction between L-BiTop and BiLoc. In Section 4, the ideas of sobriety and spatiality will be introduced in the categories L-BiTop and BiLoc. Such ideas allow us to generalize the known equivalence between the categories of ordinary sober bitopological spaces and of spatial biframes to another equivalence between the categories L-SobBiTop (of sober L-bitopological spaces) and L-SpatBiLoc (of spatial bilocales). In Section 5, we will define and relate the concepts of localic regularity and localic compactness in the category L-BiTop. Also, we will show that the subcategory of compact regular distributive objects of BiLoc and the subcategory of all compact regular L-sober objects of L-BiTop are categorically equivalent.

2. Preliminaries

The category SFrm [4, 10] comprises all complete lattices, together with morphisms preserving arbitrary $\bigvee$ and finite $\wedge$, and taken with the usual composition and identities. Objects of SFrm are called semiframes. Frm is a subcategory of SFrm consisting of complete lattices satisfying the first infinite distributive law (of finite meets over arbitrary joins).

The category SLoc is the dual of the category SFrm, that is, SLoc = SFrm$^{\mathrm{op}}$.

For $X \in$ Set and $L \in$ SFrm, recall that an L-topological space is a pair (X,τ), where $\tau \subset L^X$ is a sub-semiframe of the semiframe L^X of all mappings $\mu : X \to L$. If L is a frame, then τ is a frame or locale.

To discuss L-continuity, we need the appropriate powerset operator.

Given a function $f : X \to Y$, the image and preimage operators are defined as follows:

$$\begin{gathered} f_L^{\rightarrow} : L^X \longrightarrow L^Y \quad \text{by } f_L^{\rightarrow}(\mu)(y) = \bigvee_{f(x)=y} \mu(x), \\ f_L^{\leftarrow} : L^Y \longrightarrow L^X \quad \text{by } f_L^{\leftarrow}(\rho) = \rho \circ f. \end{gathered} \tag{2.1}$$

An L-continuous map $f : (X,\tau) \to (Y,\theta)$ is a map $f : X \to Y$ such that for all $\nu \in \theta$, $f_L^{\leftarrow}(\nu) \in \tau$. Note that $(f_L^{\leftarrow})_{|\theta} : \tau \leftarrow \theta$ is a semiframe morphism, and hence that

$$[(f_L^{\leftarrow})_{|\theta}]^{\mathrm{op}} : \tau \longrightarrow \theta \tag{2.2}$$

is a semilocalic morphism. If L is a frame, then $(f_L^{\leftarrow})_{|\theta}$ and $[(f_L^{\leftarrow})_{|\theta}]^{\mathrm{op}}$ are frame and localic morphisms, respectively. Now, for $L \in$ SFrm, the category L-Top comprises all L-topological spaces (see [1]) together with L-continuous maps between them.

An $(X,\tau) \in L$-Top is said to be a localic compact [7] if and only if for all $u \subset \tau$, $\bigvee u = \top$, $\exists$ finite open subcover $v \subset u$, $\bigvee v = \top$.

From [3, 4, 7], we recall the definition of the following functors:

(i) the functor

$$\Omega_L : L\text{-Top} \longrightarrow \text{SLoc}, \tag{2.3}$$

where

$$\Omega_L(X,\tau) = \tau, \qquad \Omega_L(f : (X,\tau) \longrightarrow (Y,\theta)) = \left[(f_L^{\leftarrow})_{|\theta}\right]^{\text{op}} : \tau \longrightarrow \theta; \tag{2.4}$$

(ii) the functor

$$LPT : L\text{-Top} \longleftarrow \text{SLoc} \tag{2.5}$$

defined by $A \to (Lpt(A),\ \Phi_L^{\rightarrow}(A))$, where

$$\begin{gathered} Lpt(A) = \{p : A \longrightarrow L : p \in \text{ SFrm}\}, \\ \Phi_L : A \longrightarrow L^{Lpt(A)} \quad \text{by } \Phi_L(a)(p) = p(a), \\ LPT(A) = \big(Lpt(A), \Phi_L^{\rightarrow}(A)\big), \\ LPT(f : A \longrightarrow B) = \left[f^{\text{op}}\right]_L^{\leftarrow}, \quad \text{that is, } LPT(f)(p) = p \circ f^{\text{op}}, \end{gathered} \tag{2.6}$$

where $f^{\text{op}} : B \to A$ is a concrete map in SFrm.

If L is a frame, then the above functors are given in the following form:

$$\begin{gathered} \Omega_L : L\text{-Top} \longrightarrow \text{Loc}, \\ LPT : L\text{-Top} \longrightarrow \text{Loc}. \end{gathered} \tag{2.7}$$

The functors Ω_L and LPT are adjunctions via $(\Omega_L \dashv LPT)$ [3, 4]. The unit of this adjunction is given by $\Psi_L : (X,\tau) \to LPT(\Omega_L(X,\tau))$, where $\Psi_L(x)(\mu) = \mu(x)$. The counit is given by $\varepsilon_A^{\text{op}} : A \to \Omega_L(LPT(A))$, where $\varepsilon_A^{\text{op}}(a) = \Phi_L(a)$.

Lemma 2.1 [3, 4, 10]. *The following holds:*

(i) *for all $(X,\tau) \in L$-Top, (X,τ) is L-sober $\Leftrightarrow \Psi_L$ is an L-homomorphism;*

(ii) *for all $A \in$ SLoc, A is L-spatial $\Leftrightarrow \varepsilon_A^{\text{op}} : A \to \Omega_L(LPT(A))$ is a frame isomorphism.*

Theorem 2.2 [3, 4]. *Let $A, L \in$ SLoc, and $(X,\tau) \in L$-Top. Then*

(i) *(X,τ) is compact $\Leftrightarrow \Omega_L(X,\tau) = \tau$ is compact;*

(ii) *if A is L-spatial, then A is compact $\Leftrightarrow LPT(A)$ is compact.*

We now describe briefly the concept of L-real line and the unit L-interval [9]. Let L be a complete lattice, let $\mathbf{R}_L$ be the set of all order-reversing member $\lambda \in L^{\mathbf{R}}$ such that $\vee\lambda^{\rightarrow}(\mathrm{R}) = \top$, and let $\wedge\lambda^{\rightarrow}(\mathbf{R}) = \bot$.

For $\lambda \in \mathrm{R}_L$ and $t \in \mathbf{R}$, let $\lambda^{+}(t) = \vee\lambda(t,\infty)$ and $\lambda^{-}(t) = \wedge\lambda(-\infty,t)$. Further, define

$$\lambda \sim \mu \Longleftrightarrow \lambda^{+} = \mu^{+}. \tag{2.8}$$

This is an equivalence relation and the set $\mathrm{R}(L)$ of all equivalence classes $[\lambda]$ is called the L-real line [9].

With $[\lambda] \leq [\mu]$, if and only if $\lambda^+ \leq \mu^+$, $\mathbf{R}(L)$ becomes a partially ordered set. There are two L-topologies on $\mathrm{R}(L)$:

(i) $\mathcal{R}_L = \{R_t : t \in \mathbf{R}\} \cup \{1_\varnothing, 1_{\mathbf{R}(L)}\}$,

(ii) $\mathcal{L}_L = \{L_t : t \in \mathbf{R}\} \cup \{1_\varnothing, 1_{\mathbf{R}(L)}\}$,

where $R_t[\lambda] = \lambda^+(t)$ and $L_t[\lambda] = 1 - \lambda^-(t)$ for every $[\lambda] \in \mathbf{R}(L)$.

The smallest L-topology on $\mathrm{R}(L)$, which contains $\mathcal{R}_L \cup \mathcal{L}_L$, is called the natural L-topology on $\mathbf{R}(L)$.

3. Bilocales and L-bitopological spaces

$L \in \mathrm{Frm}$, the known adjunction between L-Top and Loc will be extended to another one between the category of L-bitopological spaces and the category of bilocales. To do so, we begin by recalling some needed concepts about biframes.

Definition 3.1 [11]. A biframe is a triple $A = (A_0, A_1, A_2)$, where A_1 and A_2 are subframes of a frame A_0 such that A_0 is generated by $A_1 \cup A_2$.

A biframe map (or homomorphism) $h : A \to B$, between biframes $A = (A_0, A_1, A_2)$ and $B = (B_0, B_1, B_2)$, is a frame map $h_0 : A_0 \to B_0$, for which $h(A_i) \subseteq B_i$ $(i = 1, 2)$.

A biframe $A = (A_0, A_1, A_2)$ is said to be symmetric [12] if and only if $A_0 = A_1 = A_2$.

We refer to A_0 as the total part of $A = (A_0, A_1, A_2)$, A_1 and A_2 the first and second parts, respectively.

A biframe homomorphism $h : A \to B$ is called as follows [11]:

(i) onto if both $h|_{A_1}$ and $h|_{A_2}$ are both onto (and hence $h|_{A_0}$ is also onto);

(ii) one-to-one if $h_0 : A_0 \to B_0$ is one-to-one;

(iii) isomorphism if $h_0 : A_0 \to B_0$ is both injective and onto;

(iii) dense if $h(a) = 0$ implies that $a = 0$, for all $a \in A_0$.

By BiFrm, we mean the category of biframes as objects and biframe homomorphisms as morphisms.

The category BiLoc is the opposite (dual) of the category of biframes, that is, BiLoc = BiFrm$^{\mathrm{op}}$.

The objects in the category L-BiTop are triples (X, τ_1, τ_2), where X is a nonempty set and τ_1, τ_2 are L-topologies on X. The morphisms are maps $f : X \to Y$ such that $f : (X, \tau_1) \to (Y, \rho_1)$ and $f : (X, \tau_2) \to (Y, \rho_2)$ are both L-continuous. In this case, we say that f is L-bicontinuous and we write $f : (X, \tau_1, \tau_2) \to (Y, \rho_1, \rho_2)$.

Between the category L-Top and L-BiTop there is a faithful functor

$$S : L\text{-}\mathrm{Top} \longleftarrow L\text{-}\mathrm{BiTop}, \tag{3.1}$$

which we now describe.

If $X = (X, \tau_1, \tau_2) \in L$-BiTop, then $SX = (X, \tau_1 \vee \tau_2)$, where $\tau_1 \vee \tau_2$ is the coarsest L-topology finer than both τ_1 and τ_2, $S(f) = f$.

The left adjoint of S is the functor

$$D : L\text{-}\mathrm{Top} \longrightarrow L\text{-}\mathrm{BiTop} \tag{3.2}$$

given by $D(X, \tau) = (X, \tau, \tau)$, $D(f) = f$.

One notes that since D embeds L-Top in L-BiTop, then we will regard the constructions in L-BiTop as extensions of the constructions in the category L-Top.

There is a similar adjoint pair of faithful functors (not defined here) between BiFrm and Frm. The right adjoint is the embedding of Frm into BiFrm, and allows us to talk of biframe notions as extensions of frame notions.

We define the functor

$$\Omega_L : L\text{-}\mathrm{BiTop} \longrightarrow \mathrm{BiLoc}, \tag{3.3}$$

where

$$\begin{gathered} \Omega_L(X,\tau_1,\tau_2) = (\tau_1 \vee \tau_2, \tau_1, \tau_2), \\ \Omega_L(f : (X,\tau_1,\tau_2) \longrightarrow (Y,\theta_1,\theta_2)) = \left[(f_L^{\leftarrow})|_{\theta_i}\right]^{\mathrm{op}} : \tau_i \longrightarrow \theta_i, \quad i = 1,2. \end{gathered} \tag{3.4}$$

Now, we will introduce some ideas needed to define a functor in the opposite direction.

For a biframe $A = (A_0, A_1, A_2)$, let $Lpt\ (A) = \{p : A_0 \to L : p \in \mathrm{Frm}\} = Lpt(A_0)$. Also, we define a biframe map

$$\Phi_L : (A_0, A_1, A_2) \longrightarrow \left(L^{Lpt(A_0)}, L^{Lpt(A_0)}, L^{Lpt(A_0)}\right) \tag{3.5}$$

such that

(1) $\Phi_L : A_0 \to L^{Lpt(A_0)}$ is a frame map, where $\Phi_L(a)(p) = p(a)$;
(2) $\Phi_L^{\rightarrow}(A_1) \subseteq L^{Lpt(A_0)}$;
(3) $\Phi_L^{\rightarrow}(A_2) \subseteq L^{Lpt(A_0)}$.

So we have the functor

$$LPT : L\text{-}\mathrm{BiTop} \longleftarrow \mathrm{BiLoc} \tag{3.6}$$

defined by

$$(A_0, A_1, A_2) \longrightarrow \left(Lpt(A_0), \Phi_L^{\rightarrow}(A_1), \Phi_L^{\rightarrow}(A_2)\right), \tag{3.7}$$

where

$$LPT(f : A \longrightarrow B) = [f]^{\mathrm{op}}, \quad \text{that is, } LPT(f)(p) = p \circ f^{\mathrm{op}},\ f^{\mathrm{op}} : B \longrightarrow A \tag{3.8}$$

is a concrete map in BiFrm.

It is clear that $\{\Phi_L(a) : a_i \in A_i, i = 1,2\}$ is an L-topology on $Lpt(A)$ and, therefore, we have $(Lpt(A_0), \Phi_L^{\rightarrow}(A_1), \Phi_L^{\rightarrow}(A_2)) \in L$-BiTop.

For every $(X, \tau_1, \tau_2) \in L$-BiTop, define the mapping $\Psi_L : X \to LPT(\Omega_L(X))$ as follows. For all $x \in X$, $\mu \in \Omega_L(X)$, $\Psi_L(x)(\mu) = \mu(x)$.

Lemma 3.2. *For all $A = (A_0, A_1, A_2) \in BiFrm$, then $S[LPT(A_0, A_1, A_2)] = LPT(A_0)$, where*

$$S : L\text{-}Top \longleftarrow L\text{-}BiTop. \tag{3.9}$$

Proof. For a biframe (A_0,A_1,A_2), it is clear that

$$\begin{aligned} &S[LPT(A_0,A_1,A_2)] \\ &\quad = S[Lpt(A_0),\Phi_L^{\rightarrow}(A_1),\Phi_L^{\rightarrow}(A_2)] = (Lpt(A_0),\Phi_L^{\rightarrow}(A_1)\vee\Phi_L^{\rightarrow}(A_2)) \\ &\quad = (Lpt(A_0),\Phi_L^{\rightarrow}(A_1\vee A_2)) = (Lpt(A_0),\Phi_L^{\rightarrow}(A_0)) = LPT(A_0), \end{aligned} \tag{3.10}$$

and this completes the proof. □

Lemma 3.3. *The mapping*

$$\Psi_L : (X,\tau_1,\tau_2) \longrightarrow (Lpt(\tau_1\vee\tau_2),\Phi_L^{\rightarrow}(\tau_1),\Phi_L^{\rightarrow}(\tau_2)) \tag{3.11}$$

is L-bicontinuous and pairwise L-open w.r.t. its range in $(Lpt(\tau_1\vee\tau_2),\Phi_L^{\rightarrow}(\tau_1),\Phi_L^{\rightarrow}(\tau_2))$.

Proof. To prove that the mapping Ψ_L is L-bicontinuous and pairwise L-open, it suffices to prove that both the mappings

$$\Psi_L : (X,\tau_1) \longrightarrow (Lpt(\tau_1\vee\tau_2),\Phi_L^{\rightarrow}(\tau_1)), \qquad \Psi_L : (X,\tau_2) \longrightarrow (Lpt(\tau_1\vee\tau_2),\Phi_L^{\rightarrow}(\tau_2)) \tag{3.12}$$

are L-continuous and L-open w.r.t. their respective ranges.

(i) L-continuity: For $i\in\{1,2\}$, for all $\mu\in\Phi_L^{\rightarrow}(\tau_i)$, and for all $x\in X$, there exists $\rho\in\tau_i$ such that

$$\Phi_L(\rho)=\mu, \qquad \Psi_L^{\leftarrow}(\mu)(x)=\Psi_L^{\leftarrow}(\Phi_L(\rho))(x)=\rho(x), \quad \text{that is, } \Psi_L^{\leftarrow}(\mu)\in\tau_i. \tag{3.13}$$

Hence Ψ_L is L-bicontinuous.

(ii) Openness: In fact, for $v\in\tau_i$, $i\in\{1,2\}$, and $p\in Lpt(\tau_1\vee\tau_2)$,

$$\begin{aligned} \Psi_L^{\rightarrow}(v)(p) &= \sup\nolimits_{x\in X}\{v(x):\Psi_L(x)=p\} \\ &= \sup\nolimits_{x\in X}\{\Psi_L(x)(v):\Psi_L(x)=p\} \\ &= p(v)=\Phi_L^{\rightarrow}(v)(p). \end{aligned} \tag{3.14}$$

Now, $\Phi_L(v)\in\Phi_L^{\rightarrow}(\tau_i)$, the L-topology on $Lpt(\tau_1\vee\tau_2)$, and it follows that $\Psi_L^{\rightarrow}(v)=\Phi_L(v)$, that is, $\Psi_L^{\rightarrow}(v)|_{\Psi_L^{\rightarrow}(X)}=\Phi_L(v)|_{\Psi_L^{\rightarrow}(X)}$.

Thus $\Psi_L^{\rightarrow}(v)$ is open w.r.t. the subspace topology of $\Psi_L^{\rightarrow}(X)$ induced from $Lpt(\tau_1\vee\tau_2)$, that is, Ψ_L is a pairwise L-open map. □

For a biframe $A=(A_0,A_1,A_2)$, we define the biframe map $\varepsilon_A^{\mathrm{op}}:A\to\Omega_L(LPT(A))$, where $\varepsilon_A(A)=\Phi_L(A)$.

Theorem 3.4. *The functor*

$$LPT : L\text{-}BiTop \longleftarrow BiLoc \tag{3.15}$$

is a right adjoint of the functor

$$\Omega_L : L\text{-}BiTop \longrightarrow BiLoc \tag{3.16}$$

with unit $\Psi_L : X\to LPT^{\rightarrow}(\Omega_L(X,\tau_1,\tau_2))$ *and counit* $\varepsilon_A : A\leftarrow\Omega_L(LPT(A))$.

Proof. To prove that the functor LPT is a right adjoint of Ω_L (i.e., $\Omega_L \dashv LPT$), we need to prove that for all $A = (A_0, A_1, A_2) \in$ BiLoc and for all $f : X \to LPT(A_0, A_1, A_2)$, there exists, uniquely, a biframe map $f^* : \Omega_L(X) \to (A_0, A_1, A_2)$ such that $f = LPT(f^*) \circ \Psi_L$.

To prove the existence, let $f^* = f^{\leftarrow} \circ \Phi_L^{\rightarrow}$, then f^* is obviously a biframe map and for all $x \in X$ and $a \in A_0$,

$$\begin{aligned} &LPT(f^*) \circ \Psi_L(x)(a) \\ &\quad = \Psi_L(x)(f^*(a)) = \Psi_L(x)(f^{\leftarrow} \circ \Phi_L^{\rightarrow}(a)) = f^{\leftarrow} \circ \Phi_L^{\rightarrow}(a)(x) = \Phi_L^{\rightarrow}(a)(f(x)) = f(x)(a). \end{aligned} \tag{3.17}$$

Hence $LPT(f^*) \circ \Psi_L = f$.

Uniqueness follows immediately from the condition that for all $x \in X$ and $a \in A_0$, $f^*(a)(x) = f(x)(a)$. □

4. L-sobriety and L-sobrifications

In this section, the notions of L-sobriety and L-spatiality are introduced. Such ideas allow us to generalize the equivalence between the subcategories of sober objects in L-Top and L-spatial objects in Loc to the equivalence between the categories L-SobBiTop and L-SpatBiLoc.

Definition 4.1. An $(X, \tau_1, \tau_2) \in L$-BiTop is said to be pairwise L-T_0 (i.e., fulfills the T_0-axiom) if and only if for every pair $(x, y) \in X \times X$ with $x \neq y$, there exists $\mu \in \tau_1 \vee \tau_2$ such that $\mu(x) \neq \mu(y)$.

By L-T_0BiTop, we mean a full subcategory of L-BiTop consisting of those L-BiTop objects, which are pairwise L-T_0.

As a consequence of the above definition, we have the following easily established proposition.

Proposition 4.2. *$(X, \tau_1, \tau_2) \in L$-$T_0$BiTop $\Leftrightarrow S(X, \tau_1, \tau_2) = (X, \tau_1 \vee \tau_2)$ is L-T_0.*

Now, we will write an example of the pairwise L-T_0-axiom.

Example 4.3. The fuzzy real line $(\mathrm{R}(L))$ with the two L-topologies $\mathscr{L}_L$ and $\mathscr{R}_L$ is pairwise L-T_0.

Proof. Since $S[(\mathrm{R}(L), \mathscr{L}_L, \mathscr{R}_L)] = (\mathrm{R}(L), \mathscr{L}_L \vee \mathscr{R}_L)$, and since $(\mathrm{R}(L), \mathscr{L}_L \vee \mathscr{R}_L)$ is L-T_0 (see [3, 4, Corollary 3.1.2]), then $(\mathrm{R}(L), \mathscr{L}_L, \mathscr{R}_L)$ is pairwise L-T_0. □

Proposition 4.4. *An $(X, \tau_1, \tau_2) \in L$-BiTop is pairwise L-T_0 if and only if the mapping*

$$\Psi_L : (X, \tau_1, \tau_2) \longrightarrow (Lpt(\tau_1 \vee \tau_2), \Phi_L^{\rightarrow}(\tau_1), \Phi_L^{\rightarrow}(\tau_2)) \tag{4.1}$$

is pairwise L-embedding.

Proof. First, suppose that $(X, \tau_1, \tau_2) \in L$-BiTop is pairwise L-T_0, then for $x \neq y \in X$, there exists $\mu \in \tau_1 \vee \tau_2$ such that $\mu(x) \neq \mu(y)$. Therefore, $\Psi_L(x)(\mu) = \mu(x) \neq \mu(y) = \Psi_L(y)(\mu)$, that is, the mapping Ψ_L is injective. Also, since the mapping Ψ_L is pairwise L-continuous and L-open (see Lemma 3.3), then Ψ_L is L-embedding. The second part is trivial. □

Now, we will introduce the concept of L-sobriety of objects in the category L-BiTop.

Definition 4.5. An $(X,\tau_1,\tau_2) \in L$-BiTop is L-sober if and only if the mapping

$$\Psi_L : X \longrightarrow LPT^{\rightarrow}(\Omega_L(X,\tau_1,\tau_2)) \tag{4.2}$$

is bijective.

By L-SobBiTop, we mean the full subcategory of L-BiTop of all L-sober objects.

Lemma 4.6. *An $(X,\tau_1,\tau_2) \in L$-BiTop is L-sober if and only if the mapping*

$$\Psi_L : (X,\tau_1,\tau_2) \longrightarrow (Lpt(\tau_1 \vee \tau_2), \Phi_L^{\rightarrow}(\tau_1), \Phi_L^{\rightarrow}(\tau_2)) \tag{4.3}$$

is a pairwise homomorphism.

Proof. L-sobriety of an $(X,\tau_1,\tau_2) \in L$-BiTop is equivalent to the fact of bijectivity of the mapping

$$\Psi_L : (X,\tau_1,\tau_2) \longrightarrow (Lpt(\tau_1 \vee \tau_2), \Phi_L^{\rightarrow}(\tau_1), \Phi_L^{\rightarrow}(\tau_2)). \tag{4.4}$$

Also, the mapping Ψ_L is pairwise L-continuous and L-open (see Lemma 3.3), and this is equivalent to the fact that Ψ_L is pairwise L-homomorphism. □

We now recall the definition of a spatial biframe from [11], and we call it L-spatial in this paper.

Definition 4.7 [11]. A biframe $A = (A_0,A_1,A_2)$ is called L-spatial if and only if the total part A_0 is L-spatial frame.

By L-SpatBiLoc, we mean the full subcategory of BiLoc of all L-spatial bilocales.

Lemma 4.8. *For all $A = (A_0,A_1,A_2) \in$ BiLoc, $A = (A_0,A_1,A_2)$ is L-spatial if and only if the mapping*

$$\varepsilon_A^{\mathrm{op}} : (A_0,A_1,A_2) \longrightarrow \Omega_L(LPT(A_0,A_1,A_2)) \tag{4.5}$$

is a biframe isomorphism.

Proof. Let $A = (A_0,A_1,A_2)$ be a L-spatial biframe. Then, by the definition, the total part A_0 is L-spatial, and this is equivalent to the fact that the map

$$\varepsilon_A^{\mathrm{op}} : A_0 \longrightarrow \Omega_L(LPT(A_0)) \tag{4.6}$$

is a frame isomorphism, and this implies that the map

$$\varepsilon_A^{\mathrm{op}} : (A_0,A_1,A_2) \longrightarrow \Omega_L(LPT(A_0,A_1,A_2)) \tag{4.7}$$

is a biframe isomorphism. □

Lemma 4.9. *For all $(X,\tau_1,\tau_2) \in L$-BiTop and for all $A \in$ BiLoc, then*

(i) $\Omega_L(X,\tau_1,\tau_2) = (\tau_1 \vee \tau_2, \tau_1, \tau_2)$ *is L-spatial,*

(ii) $LPT(A_0,A_1,A_2) = (Lpt(A_0), \Phi_L^{\rightarrow}(A_1), \Phi_L^{\rightarrow}(A_2))$ *is L-Sober.*

Proof. As to (i), clearly, the map $\varepsilon^{\mathrm{op}}_{\tau_1\vee\tau_2} : (\tau_1 \vee \tau_2) \to \Omega_L(LPT(\tau_1 \vee \tau_2)) = \Phi_L^{\to}(\tau_1 \vee \tau_2)$ is a frame isomorphism, which implies that $\tau_1 \vee \tau_2$ is an L-spatial frame and, therefore, the biframe $\Omega_L(X,\tau_1,\tau_2) = (\tau_1 \vee \tau_2, \tau_1, \tau_2)$ is L-spatial.

To prove (ii), by definition, it suffices to prove that the mapping

$$\Psi_L : LPT(A) \longrightarrow LPT(\Omega_L(LPT(A))) = LPT(\Phi_L^{\to}(A_1) \vee \Phi_L^{\to}(A_2), \Phi_L^{\to}(A_1), \Phi_L^{\to}(A_2)) \tag{4.8}$$

is bijective. To this end, we have the following.

(a) Ψ_L is one-to-one.

For all $p_1, p_2 \in Lpt(A_0)$ with $p_1 \neq p_2$, there exist some $a \in A_0$ with $p_1(a) \neq p_2(a)$, and this implies that $\Psi_L(p_1)(\Phi_L^{\to}(a)) = \Phi_L^{\to}(a)(p_1) = p_1(a) \neq p_2(a) = \Psi_L(p_2)(\Phi_L^{\to}(a))$.

Hence Ψ_L is one-to-one.

(b) Ψ_L is onto.

For all $q \in Lpt(\Phi_L^{\to}(A_1 \vee A_2))$, let $p = q \circ \Phi_L^{\to} : A_0 \to \Phi_L^{\to}(A_0) \to L$, then $p \in Lpt(A_0)$ and $a \in A_0$. We have $\Psi_L(p)(\Phi_L^{\to}(a)) = \Phi_L^{\to}(a)(p) = p(a) = q(\Phi_L^{\to}(a))$.

Hence $\Psi_L(p) = q$, that is, Ψ_L is onto. From (a) and (b), it follows that Ψ_L is bijective, and this completes the proof. □

As a consequence of the above lemma, we have the following proposition.

Proposition 4.10. *The following functors are valid:*

(i) $\Omega_L : L\text{-}BiTop \to L\text{-}SpatBiLoc$,

(ii) $LPT : L\text{-}SobBiTop \leftarrow BiLoc$,

(iii) $\Omega_L \circ LPT : BiLoc \to L\text{-}SpatBiLoc$,

(iv) $LPT \circ \Omega_L : L\text{-}BiTop \to L\text{-}SobBiTop$.

As a consequence of the preceding proposition, we give the definition of L-sobrification and L-spatialization functors, respectively. This is given as follows.

Definition 4.11. The compositions

$$LPT \circ \Omega_L : L\text{-}\mathrm{BiTop} \longrightarrow L\text{-}\mathrm{SobBiTop} \tag{4.9}$$

are called the L-sobrification functors.

Definition 4.12. The compositions

$$\Omega_L \circ LPT : \mathrm{BiLoc} \longrightarrow L\text{-}\mathrm{SpatBiLoc} \tag{4.10}$$

are called the L-spatialization functors.

The equivalence between the categories L-SobBiTop of L-sober bitopological spaces and L-SpatBiLoc of L-spatial bilocales is proven as follows.

Theorem 4.13. *For all $L \in$ Frm, L-SobBiTop $\approx$ L-SpatBiLoc.*

Proof. The categorical equivalence L-SobBiTop $\approx$ L-SpatBiLoc follows directly from the adjunction $\Omega_L \dashv LPT$ and the fact that both the unit and counit are isomorphisms in the categories L-SobBiTop and L-SpatBiLoc, respectively. □

5. Regularity and compactness

The purpose of this section is to define and relate the concepts of localic regularity and localic compactness of objects in the categories BiFrm and L-BiTop.

Now, we recall technical tools needed for this section.

Let $(A_0,A_1,A_2) \in$ BiFrm and $a,b \in A_i$, a is said to be well inside b (w.r.t A_i) [5, 7, 11, 12] and denoted by $a \prec_i b \Leftrightarrow \exists c \in A_k (k \neq i)$ such that $a \wedge c = \perp$ and $c \vee b = \top$.

Definition 5.1 (See [5, 11, 12]). An $(A_0,A_1,A_2) \in$ BiFRM is said to be regular if and only if

$$\forall a \in A_i, \quad a = \vee\{b \in A_i, b \prec_i a(w.r.tA_i)\}. \tag{5.1}$$

By RegBiFrm, we mean the full subcategory of BiFrm of regular objects, and RegBiLoc is the dual of RegBiFrm.

Proposition 5.2 (See [11]). *If the biframe $A = (A_0,A_1,A_2)$ is regular, then the frame A_0 is regular.*

Lemma 5.3 (See [11]). *If the BiFrm morphism $h : A \to B$ is surjective and $A = (A_0,A_1,A_2)$ $\in$ BiFrm is regular, so $B = (B_0,B_1,B_2)$ is regular.*

Now, we will define the localic regularity for a certain L-BiTop object using the corresponding regularity of BiFrm objects.

Definition 5.4. For $L \in$ Frm, an (X,τ_1,τ_2) is regular $\Leftrightarrow \Omega_L(X,\tau_1,\tau_2) \in$ RegBiLoc.

By L-RegBiTop, we mean the full subcategory of L-BiTop of regular objects.

Proposition 5.5. *If an $A \in$ BiFrm is regular $\Rightarrow LPT(A)$ is regular and L-sober. The converse holds if A is L-spatial.*

Proof. Let $A = (A_0,A_1,A_2) \in$ RegBiLoc. Since the map $\varepsilon_A^{\mathrm{op}} : A \to \Omega_L LPT(A)$ is surjective, so that (by Lemma 5.3) $\Omega_L LPT(A)$ is regular and, therefore, $LPT(A)$ is regular. By Lemma 4.9, $LPT(A)$ is L-sober. If $LPT(A)$ (resp., $\Omega_L LPT(A)$) is regular, then the biframe $A = (A_0,A_1,A_2)$ becomes regular if the map $\varepsilon_A^{\mathrm{op}} : A \to \Omega_L LPT(A)$ is a biframe isomorphism or, equivalently, $A = (A_0,A_1,A_2)$ is an L-spatial biframe. □

As the preceding proposition offers the preserving of the regular axiom under the functor

$$LPT : L\text{-}\mathrm{BiTop} \longleftarrow \mathrm{BiLoc} \tag{5.2}$$

and with the aid of Definition 5.4, we have the following easily established proposition.

Proposition 5.6. *The following functors holds:*

$$\begin{aligned} \Omega_L : L\text{-}RegBiTop &\longrightarrow RegBiLoc, \\ LPT : L\text{-}RegBiTop &\longleftarrow RegBiLoc. \end{aligned} \tag{5.3}$$

The above statements offer the study of the concept of localic regularity in the categories BiFrm and L-BiTop, respectively. In the sequel, we will introduce the concept of localic compactness in the same categories.

We begin by recalling that an $A \in$ Frm is compact (see [13]) $\Leftrightarrow$ for all $S \subset A$, $\bigvee S = \top$, $\exists F(\text{finite}) \subset S$, $\bigvee F = \top$.

Definition 5.7 (See [5, 12]). An $A = (A_0, A_1, A_2) \in$ BiFrm is said to be compact if and only if the total part A_0 is compact.

By K-BiFrm (resp., K-BiLoc), we mean the full subcategory of BiFrm (resp., BiLoc) of compact objects, where K-BiLoc $= K$-BiFrm$^{\text{op}}$.

Definition 5.8. An $(X, \tau_1, \tau_2) \in L$-BiTop is said to be compact if $S(X, \tau_1, \tau_2) = (X, \tau_1 \vee \tau_2)$ is compact.

By L-KBiTop, we mean the full subcategory of L-BiTop of compact objects.

Theorem 5.9. *Let $L \in Frm$, $A \in BiFrm$, and $(X, \tau_1, \tau_2) \in L$-BiTop. Then*

(1) *(X, τ_1, τ_2) is compact $\Leftrightarrow \Omega_L(X, \tau_1, \tau_2) = (\tau_1 \vee \tau_2, \tau_1, \tau_2)$ is compact,*

(2) *if A is L-spatial, then A is compact $\Leftrightarrow LPT(A_0, A_1, A_2)$ is compact.*

Proof. As to (i), if (X, τ_1, τ_2) is a compact object of L-BiTop, that is, for all $S \subseteq (\tau_1 \vee \tau_2)$, $\vee S = \top$, $\exists F(\text{finite}) \subseteq S$, $\vee F = \top \Leftrightarrow (\tau_1 \vee \tau_2)$ is a compact frame $\Leftrightarrow (\tau_1 \vee \tau_2, \tau_1, \tau_2)$ is a compact biframe.

As to (ii), let $A = (A_0, A_1, A_2)$ be an L-spatial, then the mapping

$$\varepsilon_A^{\text{op}} : A \longrightarrow \Omega_L(LPT(A_0, A_1, A_2)) \tag{5.4}$$

is a biframe isomorphism, that is, $A \approx \Phi_L^{\rightarrow}(A)$.

Compactness of $(A_0, A_1, A_2) \Leftrightarrow A_0$ is compact

$$\begin{aligned} &\Longleftrightarrow LPT(A_0) = (Lpt(A_0), \Phi_L^{\rightarrow}(A_0)) \text{ is compact} \\ &\Longleftrightarrow (Lpt(A_0), \Phi_L^{\rightarrow}(A_1) \vee \Phi_L^{\rightarrow}(A_2)) \text{ is compact} \\ &\Longleftrightarrow LPT(A) = (Lpt(A_0), \Phi_L^{\rightarrow}(A_1), \Phi_L^{\rightarrow}(A_2)) \text{ is compact,} \end{aligned} \tag{5.5}$$

and this completes the proof. □

The following proposition shows that the compact regular distributive objects of BiLoc are categorically equivalent with compact regular L-sober objects of L-BiTop.

Proposition 5.10. *For all distributive $L \in Frm$, under the duality induced by*

$$\begin{aligned} \Omega_L : L\text{-}RegBiTop &\longrightarrow L\text{-}RegBiLoc, \\ LPT : L\text{-}RegBiTop &\longleftarrow L\text{-}RegBiLoc, \end{aligned} \tag{5.6}$$

the following equivalence holds:

$$K\text{-}RegBiLoc \approx L\text{-}KRegSobBiTop. \tag{5.7}$$

Proof. Let $A = (A_0, A_1, A_2) \in L$-RegBiLoc. Then by Proposition 5.5 and Theorem 5.9(ii), $LPT(A)$ is compact, regular and L-sober, that is, $LPT(A) \in L$-KRegSobBiTop.

Conversely, let $(X, \tau_1, \tau_2) \in L$-$K$RegSobBiTop, then, by definitions, $\Omega_L(X, \tau_1, \tau_2)$ is a compact regular biframe.

It remains to be shown that the unit

$$\Psi_L : X \longrightarrow LPT^{\rightarrow}(\Omega_L(X, \tau_1, \tau_2)) \tag{5.8}$$

and the counit

$$\varepsilon_A^{\mathrm{op}} : (A_0, A_1, A_2) \longrightarrow \Omega_L(LPT(A_0, A_1, A_2)) \tag{5.9}$$

of the adjunctions are isomorphisms.

On the one hand, let $(X, \tau_1, \tau_2) \in L$-$K$-RegSobBiTop, then $\Psi_L : X \to LPT^{\rightarrow}(\Omega_L(X, \tau_1, \tau_2))$ is an isomorphism in L-BiTop.

On the other hand, let $A = (A_0, A_1, A_2)$ be a compact regular biframe. The biframe map $\varepsilon_A^{\mathrm{op}} : (A_0, A_1, A_2) \to \Omega_L(LPT(A_0, A_1, A_2))$ is given by the following commutative diagram:

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d] & \Phi_L^{\rightarrow}(A_1) \arrow[d] \\
A_0 \arrow[r] & \Omega_L LPT(A_0) \\
A_2 \arrow[r] \arrow[u] & \Phi_L^{\rightarrow}(A_2). \arrow[u]
\end{tikzcd} \tag{5.10}$$

As seen above, the frame map $\varepsilon_{A_0}^{\mathrm{op}} : A_0 \to \Omega_L(LPT(A_0))$ is an isomorphism; therefore, the biframe map $\varepsilon_A^{\mathrm{op}} : (A_0, A_1, A_2) \to \Omega_L(LPT(A_0, A_1, A_2))$ is an isomorphism in the category BiLoc. □

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