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Research Article

λ-Rearrangements Characterization of Pringsheim Limit Points

Richard F. Patterson

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Sufficient conditions are given to assure that a four-dimensional matrix A will have the property that any double sequence x with finite P-limit point has- a λ-rearrangement z such that each finite P-limit point of x is a P-limit point of Az.

1. Introduction

In [1] Agnew presented the following theorem: if x is a bounded sequence and A is a regular summability matrix, then there exists a subsequence y of x such that each limit point of x is a limit point of Ay. Fridy [2] extended this result by replacing subsequence with rearrangement. Keagy [3] presented two theorems that strengthened the results of both Agnew and Fridy. This was accomplished by weakening the regularity conditions and replacing finite limit point for bounded sequence. The goal of the paper is to present two multidimensional theorems analogou to Keagy's theorems with λ-rearrangement replacing rearrangement, RH-regularity replacing regularity, and convergent in the Pringsheim sense replacing convergent. Other implications will also be presented.

2. Definitions, notations, and preliminary results

Definition 2.1. Let A denote a four-dimensional summability method that maps the complex double sequences x into the double sequence Ax, where the mnth term to Ax is as follows:

$$(Ax)_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}x_{k,l}. \tag{2.1}$$

Definition 2.2 (see Pringsheim [4]). A double sequence $x = [x_{k,l}]$ has a *Pringsheim limit* L (denoted by P-lim$x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. Such an x will be more briefly described as "P-convergent."

In addition to P-convergent, Pringsheim also presented the following notion of divergent.

Definition 2.3. A double sequence x is called *definite divergent* if for every (arbitrarily large) $G > 0$ there exist two natural numbers n_1 and n_2 such that $|x_{n,k}| > G$ for $n \geq n_1$, $k \geq n_2$.

In [5], Robison presented a four-dimensional notion of regularity for double sequences with an additional assumption of boundedness. This assumption was made because a double sequence which is P-convergent is not necessarily bounded. In addition to this notion, Robison and Hamilton both presented a Silverman-Toeplitz-type multidimensional characterization of regularity in [5, 6]. The definition of the regularity for four-dimensional matrices will be stated next, followed by the Robison-Hamilton characterization of the regularity of four-dimensional matrices.

Definition 2.4. The four-dimensional matrix A is said to be *RH-regular* if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

Theorem 2.5. *The four-dimensional matrix A is RH-regular if and only if*

RH_1: *P*-$\lim_{m,n} a_{m,n,k,l} = 0$ *for each k and l;*

RH_2: *P*-$\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1$;

RH_2: *P*-$\lim_{m,n} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0$ *for each l;*

RH_4: *P*-$\lim_{m,n} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0$ *for each k;*

RH_5: $\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|$ *is P-convergent; and*

RH_6: *there exist positive numbers A and B such that* $\sum_{k,l>B} |a_{m,n,k,l}| < A$.

The following definition of the subsequence of a double sequence was presented in [7].

Definition 2.6. The double sequence $[y]$ is a double *subsequence* of the sequence $[x]$ provided that there exist two increasing double-index sequences $\{n_j\}$ and $\{k_j\}$ such that if $z_j = x_{n_j,k_j}$, then y is formed by

$$\begin{array}{cccc} z_1 & z_2 & z_5 & z_{10} \\ z_4 & z_3 & z_6 & - \\ z_9 & z_8 & z_7 & - \\ - & - & - & - \end{array} \tag{2.2}$$

Using this concept of subsequence, the following definitions for Pringsheim limit points and divergence of double sequences were presented in [7].

Definition 2.7. A number β is called a *Pringsheim limit point* of the double sequence $x = [x_{n,k}]$ provided that there exists a subsequence $y = [y_{n,k}]$ of $[x_{n,k}]$ that has Pringsheim limit β: P-lim $y_{n,k} = \beta$.

Definition 2.8. A double sequence x is *divergent in the Pringsheim sense* (P-divergent) provided that x does not converge in the Pringsheim sense (P-convergent).

In addition to these definitions, the author also presented the following theorem in [8].

Theorem 2.9. *If each of T and A is an RH-regular matrix, x is any bounded double-complex sequence, and ϵ is any bounded positive term double sequence with $P\text{-}\lim_{k,l}\epsilon_{k,l} = 0$, then there exists a subsequence y of x such that $T(Ay)$ exists and each P-limit of x is a P-limit of $T(Ay)$.*

In [9], Patterson and Rhoades presented the following definition for rearrangement of double sequences.

Definition 2.10. Fix $\lambda > 1$. The double sequence $y^{(\pi,\lambda)}$ is called a "*λ-rearrangement*" of the double sequence x provided that there is a one-to-one function π from the positive integers into themselves such that

(1) $\{z^i_{(m,n)}\}$ is a one-dimensional sequence constructed from the double sequence $\{x_{m,n}\}$ as follows:

$$\begin{array}{lllll} z^1_{(1,1)} = x_{1,1}, & z^2_{(1,2)} = x_{1,2}, & z^3_{(2,2)} = x_{2,2}, & z^4_{(2,1)} = x_{2,1}, & z^5_{(1,3)} = x_{1,3}, \\ z^6_{(2,3)} = x_{2,3}, & z^7_{(3,3)} = x_{3,3}, & z^8_{(3,2)} = x_{3,2}, & z^9_{(3,1)} = x_{3,1}, & z^{10}_{(1,4)} = x_{1,4}\cdots; \end{array} \tag{2.3}$$

(2) let $z^{i_j}_{(m,n)}$ be a subsequence of $z^i_{(m,n)}$ consisting of all elements $z^i_{(m,n)}$ such that $1/\lambda \le m/n \le \lambda$;

(3) let $z^{i_{\pi_j}}_{(m',n')}$ be a rearrangement of $z^{i_j}_{(m,n)}$;

(4)

$$y^{(\pi,\lambda)} = y^{(\pi,\lambda)}_{p,q} = \begin{cases} z^{i_{\pi_j}}_{(m',n')}, & \text{if } \dfrac{1}{\lambda} \le \dfrac{m'}{n'} \le \lambda,\ m' = p,\ n' = q, \\ z^i_{(m,n)}, & \text{if } \dfrac{1}{\lambda} > \dfrac{m}{n},\ \dfrac{m}{n} > \lambda,\ m = p,\ n = q. \end{cases} \tag{2.4}$$

The word λ-permutation will be reserved to indicate the reordering of a finite double sequence. In addition, we will say that the ordered pair (k,l) is in the λ-wedge [10] of x if $1/\lambda \le k/l \le \lambda$.

3. Main results

The theorems presented in this section are multidimensional analogs of Keagy theorems in [3]. Throughout the proofs of the main results, we will use the ordering presented in Definition 2.6.

Theorem 3.1. *Let A be a four-dimensional matrix transformation with P-null pairwise rows and pairwise columns. Also let x be a bounded double-complex sequence. If y is a double Pringsheim subsequence of x such that Ay exists and has a finite P-limit point, then there exists a λ-rearrangement z of x such that Az exists and each P-limit point of Ay is a P-limit point of Az.*

Proof. In [8], Theorem 2.9 grants us a double sequence v constructed from u such that each term of u is a P-limit point of $T(Ay)$ and each P-limit point is a term of v. In addition, v has the property that each term of u is not only a P-limit point and/or a term of $T(Ay)$ but also a term in the λ-wedge of v. Let us choose the order pairs (m_1,n_1) and (α_1,β_1). Also let us denote the λ-permutation of $\{x_{k,l}\}$'s with $1 \le k \le \alpha_1$ and $1 \le l \le \beta_1$ as B_{α_1,β_1} (i.e., $\alpha_1\beta_1$-block). In addition, let $(\bar{k},\bar{l})$ be the first $(\beta_2+\alpha_2-\beta_1-\alpha_1-1)$-terms of $\{x_{i,j}\}$ with the following properties:

(1) $(\bar{k},\bar{l})$ is in the λ-wedge of $\{x_{i,j}\}$,

(2) $(\bar{k},\bar{l})$ is not in the $\alpha_1\beta_1$-block of $\{x_{i,j}\}$,

(3) $(\bar{k},\bar{l})$ is not in $\{y_{i,j} : \alpha_1 < i < \infty \cup \beta_1 < j < \infty\}$,

where the first $(\beta_2+\alpha_2-\beta_1-\alpha_1+1)$-terms has an ordering as in Definition 2.6. Also let us choose (m_2,n_2) and (α_2,β_2) such that $m_2 > m_1$, $n_2 > n_1$, $\alpha_2 > \alpha_1$, and $\beta_2 > \beta_1$, where (α_1,β_2) and (α_2,β_1) are in the λ-wedge of v such that

$$\sum_{k,l=1,1}^{\alpha_1,\beta_1} \left|a_{m_2,n_2,k,l}z_{k,l}\right| < \frac{1}{2^4}, \qquad \sum_{k>\alpha_1\&1\le l<\beta_1} \left|a_{m_2,n_2,k,l}x_{k,l}\right| < \frac{1}{2^4},$$

$$\sum_{1<k<\alpha_1\&l>\beta_1} \left|a_{m_2,n_2,k,l}x_{k,l}\right| < \frac{1}{2^4}, \qquad \sum_{k>\alpha_1\&l>\beta_1} \left|a_{m_2,n_2,k,l}y_{k,l} - v_{1,1}\right| < \frac{1}{2^4},$$

$$\left|\sum_{k,l=\alpha_1+1,\beta_1+1}^{\alpha_2-1,\beta_2-1} a_{m_2,n_2,k,l}y_{k,l} - v_{2,2}\right| < \frac{1}{2^4}, \qquad \left|\sum_{k,l\ge\alpha_2,\beta_2} a_{m_2,n_2,k,l}y_{k,l}\right| < \frac{1}{2^4},$$

$$\sum_{k\ge\alpha_2\&\beta_1\le l\le\beta_2} \left|a_{m_2,n_2,k,l}x_{k,l}\right| < \frac{1}{2^4}, \qquad \sum_{l\ge\beta_2\&\alpha_1\le l\le\alpha_2} \left|a_{m_2,n_2,k,l}x_{k,l}\right| < \frac{1}{2^4},$$

$$\sup_{m,n}\sum_{\beta_1\le j<\beta_2} \left|a_{m,n,\alpha_1,j}\right|\left|z_{\alpha_1,j} - y_{\alpha_1,j}\right| < \frac{1}{2^4}, \qquad \sup_{m,n}\sum_{\alpha_1<i<\alpha_2} \left|a_{m,n,i,\beta_1}\right|\left|z_{i,\beta_1} - y_{i,\beta_1}\right| < \frac{1}{2^4}. \tag{3.1}$$

Let us define z as follows: $z_{k,l} = y_{k,l}$ if $\alpha_1 < k < \alpha_2$ and $\beta_1 < l < \beta_2$, $z_{\alpha_1,j} = x_{\bar{k},\bar{l}}$ if $\beta_1 \le j \le \beta_2$, $z_{i,\beta_1} = x_{\bar{k},\bar{l}}$ if $\alpha_1 \le j < \alpha_2$, and $z_{k,l} = x_{k,l}$ otherwise. Suppose in general that the double-index sequences $(\alpha_{s-1},\beta_{t-1})$ and (m_{s-1},n_{t-1}) have been chosen with $\alpha_{s-1} > \alpha_{s-2}$, $\beta_{t-1} > \beta_{t-2}$, $m_{s-1} > m_{s-2}$, and $n_{t-1} > n_{t-2}$. Also let us denote the λ-permutation of $B_{\alpha_{s-1}\beta_{t-1}}$ of x by $\{z_{i,j}\}_{i,j=1,1}^{\alpha_{s-1},\beta_{t-1}}$. In addition, let $(\bar{k},\bar{l})$ be the first $(\beta_t+\alpha_s-\beta_t-\alpha_s-1)$-terms of $\{x_{i,j}\}$ with the following properties:

(1) $(\bar{k},\bar{l})$ is in the λ-wedge of $\{x_{i,j}\}$,

(2) $(\bar{k},\bar{l})$ is not in the $\alpha_s\beta_t$-block of $\{x_{i,j}\}$,

(3) $(\bar{k},\bar{l})$ is not in $\{y_{i,j} : \alpha_s < i < \infty \cup \beta_t < j < \infty\}$,

where the first $(\beta_t+\alpha_s-\beta_{t-1}-\alpha_{s-1}-1)$-terms have an ordering as in Definition 2.6. Now let us choose (m_s,n_t) and (α_s,β_t) such that $m_s > m_{s-1}$, $n_t > n_{t-1}$, $\alpha_s > \alpha_{s-1}$, and $\beta_t > \beta_{t-1}$, where (α_{s-1},β_t) and (α_s,β_{t-1}) are in the λ-wedge of v with the following properties:

$$\sum_{k,l=1,1}^{\alpha_{s-1},\beta_{t-1}} \left|a_{m_s,n_t,k,l}z_{k,l}\right| < \frac{1}{2^{s+t}}, \qquad \sum_{k>\alpha_{s-1}\&1\le l<\beta_{t-1}} \left|a_{m_s,n_t,k,l}x_{k,l}\right| < \frac{1}{2^{s+t}},$$

$$\sum_{1\le k\le\alpha_{s-1}\&l>\beta_{t-1}} \left|a_{m_s,n_t,k,l}x_{k,l}\right| < \frac{1}{2^{s+t}}, \qquad \sum_{k>\alpha_{s-1}\&l>\beta_{t-1}} \left|a_{m_s,n_t,k,l}y_{k,l}-v_{s,t}\right| < \frac{1}{2^{s+t}},$$

$$\left|\sum_{k,l=\alpha_{s-1}+1,\beta_{t-1}+1}^{\alpha_s-1,\beta_t-1} a_{m_s,n_t,k,l}y_{k,l}-v_{s,t}\right| < \frac{1}{2^{s+t}}, \qquad \left|\sum_{k,l\ge\alpha_s,\beta_t} a_{m_s,n_t,k,l}y_{k,l}\right| < \frac{1}{2^{s+t}},$$

$$\sum_{k\ge\alpha_s\&\beta_{t-1}<l\le\beta_t} \left|a_{m_s,n_t,k,l}x_{k,l}\right| < \frac{1}{2^{s+t}}, \qquad \sum_{\alpha_{s-1}<k\le\alpha_s\&l\ge\beta_s} \left|a_{m_s,n_t,k,l}x_{k,l}\right| < \frac{1}{2^{s+t}},$$

$$\sup_{m,n}\sum_{\beta_{t-}\le j<\beta_t} \left|a_{m,n,\alpha_{s-1},j}\right|\left|z_{\alpha_{s-1},j}-y_{\alpha_{s-1},j}\right| < \frac{1}{2^{s+1}},$$

$$\sup_{m,n}\sum_{\alpha_{s-1}<i<\alpha_s} \left|a_{m,n,i,\beta_{t-1}}\right|\left|z_{i,\beta_{t-1}}-y_{i,\beta_{t-1}}\right| < \frac{1}{2^{t+1}}. \tag{3.2}$$

Let us define z as follows: $z_{k,l}=y_{k,l}$, if $\alpha_{s-1}<k<\alpha_s$ and $\beta_{t-1}<l<\beta_t$, $z_{\alpha_s,i}=x_{\bar{k},\bar{l}}$ if $\beta_{t-1}\le i\le\beta_t$, $z_{j,\beta_t}=x_{\bar{k},\bar{l}}$ if $\alpha_{s-1}\le j<\alpha_s$, and $z_{k,l}=x_{k,l}$ otherwise. Let us consider the following:

$$\begin{aligned}
\left|(Az)_{m_s,n_t}-v_{s,t}\right| = \Bigg| &\sum_{k,l=1,1}^{\alpha_{s-1},\beta_{t-1}} a_{m_s,n_t,k,l}z_{k,l} + \sum_{\alpha_{s-1}<k<\alpha_s\&\beta_{t-1}<l<\beta_t} a_{m_s,n_t,k,l}y_{k,l}-v_{s,t} \\
&+ \sum_{\alpha_{s-1}<k<\infty\&1\le l\le\beta_{t-1}} a_{m_s,n_t,k,l}x_{k,l} + \sum_{1\le k\le\alpha_{s-1}\&\beta_{t-1}<l<\infty} a_{m_s,n_t,k,l}x_{k,l} \\
&+ \sum_{\alpha_s\le k<\infty\&\beta_{t-1}<l\le\beta_t} a_{m_s,n_t,k,l}x_{k,l} + \sum_{\alpha_{s-1}<k\le\alpha_s\&\beta_t\le l<\infty} a_{m_s,n_t,k,l}x_{k,l} \\
&+ \sum_{k>\alpha_s\&l>\beta_t} a_{m_s,n_t,k,l}z_{k,l}\Bigg| \\
\le &\sum_{k,l=1,1}^{\alpha_{t-1},\beta_{s-1}} \left|a_{m_s,n_t,k,l}z_{k,l}\right| + \left|\sum_{\alpha_{s-1}<k<\alpha_s\&\beta_{t-1}<l<\beta_t} a_{m_s,n_t,k,l}y_{k,l}-v_{s,t}\right| \\
&+ \sum_{\alpha_{s-1}<k<\infty\&1\le l\le\beta_{t-1}} \left|a_{m_s,n_t,k,l}x_{k,l}\right| + \sum_{1\le k\le\alpha_{s-1}\&\beta_{t-1}<l<\infty} \left|a_{m_s,n_t,k,l}x_{k,l}\right| \\
&+ \sum_{\alpha_s<k<\infty\&\beta_{t-1}<l\le\beta_t} \left|a_{m_s,n_t,k,l}x_{k,l}\right| + \sum_{\alpha_{s-1}<k\le\alpha_s\&\beta_t\le l<\infty} \left|a_{m_s,n_t,k,l}x_{k,l}\right| \\
&+ \sum_{k>\alpha_s\&l>\beta_t} \left|a_{m_s,n_t,k,l}z_{k,l}\right|.
\end{aligned} \tag{3.3}$$

Thus

$$\begin{aligned}
\left|(Az)_{m_s,n_t} - \nu_{s,t}\right| \leq{} & \sum_{k,l=1,1}^{\alpha_{t-1},\beta_{s-1}} \left|a_{m_s,n_t,k,l} z_{k,l}\right| + \left|\sum_{\alpha_{s-1}<k<\alpha_s \& \beta_{t-1}<l<\beta_t} a_{m_s,n_t,k,l} y_{k,l} - \nu_{s,t}\right| \\
& + \sum_{\alpha_{s-1}<k<\infty \& 1\leq l\leq \beta_{t-1}} \left|a_{m_s,n_t,k,l} x_{k,l}\right| + \sum_{1\leq k\leq \alpha_{s-1} \& \beta_{t-1}<l<\infty} \left|a_{m_s,n_t,k,l} x_{k,l}\right| \\
& + \sum_{\alpha_s<k<\infty \& \beta_{t-1}<l\leq \beta_t} \left|a_{m_s,n_t,k,l} x_{k,l}\right| + \sum_{\alpha_{s-1}<k\leq \alpha_s \& \beta_t\leq l<\infty} \left|a_{m_s,n_t,k,l} x_{k,l}\right| \\
& + \sum_{k,l=\alpha_s,\beta_t}^{\infty,\infty} \left|a_{m_s,n_t,k,l}\right| \left|y_{k,l}\right| + \sum_{p=s}^{\infty} \sum_{j=1}^{\beta_p} \left|a_{m_s,n_t,\alpha_p,j}\right| \left|z_{\alpha_p,j} - y_{\alpha_p,j}\right| \\
& + \sum_{q=t}^{\infty} \sum_{i=1}^{\alpha_q-1} \left|a_{m_s,n_t,i,\beta_q}\right| \left|z_{i,\beta_q} - y_{i,\beta_q}\right| < \frac{7}{2^{s+t}}.
\end{aligned} \tag{3.4}$$

Therefore, each P-limit point of Ay is a P-limit of Az. This completes the proof of this theorem. □

Theorem 3.2. *If x is a double-complex sequence and A is a row pairwise-finite four-dimensional matrix satisfying conditions RH_1 through RH_5 of RH-regularity, then there exists a λ-rearrangement y of x such that every limit point of x (finite or infinite) is a limit point of (Ay).*

Proof. We will assume without loss of generality that x has a definite divergent subsequence and at least one finite P-limit point. Let us consider the double sequence ν defined in the proof of Theorem 3.1. Let (α_1,β_1), (m_1,n_1), and $(\overline{m}_1,\overline{n}_1)$ be selected index pairs and let $\{z_{i,j}\}_{i,j=1,1}^{\alpha_1,\beta_1}$ be the λ-permutation of the terms in the $\alpha_1\beta_1$-block of x. Let us select (m_2,n_2) such that $m_2 > \overline{m}_1$ and $n_2 > \overline{n}_1$ such that

$$\sum_{k,l=1,1}^{\alpha_1,\beta_1} \left|a_{m_2,n_2,k,l} z_{k,l}\right| < \frac{1}{2^4}. \tag{3.5}$$

Since A is pairwise finite and we have RH_3 and RH_4, we are granted the following:

$$\begin{gathered}
\sum_{(k>\alpha_1 \cup 1\leq l\leq \beta_1)\cap(1/\lambda<k/l<\lambda)} \left|a_{m_2,n_2,k,l} z_{k,l}\right| < \frac{1}{2^4}, \\
\sum_{(k>\alpha_1 \cup 1\leq l\leq \beta_1)\cap(k/l\leq 1/\lambda \cup k/l\geq \lambda)} \left|a_{m_2,n_2,k,l} x_{k,l}\right| < \frac{1}{2^4}, \\
\sum_{(1\leq k\leq \alpha_1 \cap l>\beta_1)\cap(1/\lambda<k/l<\lambda)} \left|a_{m_2,n_2,k,l} z_{k,l}\right| < \frac{1}{2^4}, \\
\sum_{(1\leq k\leq \alpha_1 \cap l>\beta_1)\cap(k/l\leq 1/\lambda \cup k/l\geq \lambda)} \left|a_{m_2,n_2,k,l} x_{k,l}\right| < \frac{1}{2^4}.
\end{gathered} \tag{3.6}$$

The RH-regularity conditions RH_1 through RH_5 imply that

$$\left|\sum_{k>\alpha_1,l>\beta_1} a_{m_2,n_2,k,l}-1\right|<\frac{1}{2^4(|\nu_{2,2}|+1)}. \tag{3.7}$$

Let $k_2=\sup\{k:|a_{m_2,n_2,k,l}|>0\}$ and $l_2=\sup\{l:|a_{m_2,n_2,k,l}|>0\}$ and choose $\{z_{i,j}\}_{i,j=\alpha_1+1,\beta_1+1}^{k_2,l_2}$ in a Pringsheim subsequence sense from $x\backslash\{z_{i,j}\}_{i,j=1,1}^{\alpha_1,\beta_1}$ with elements in the λ-wedge of x such that

$$\begin{gathered}\left|\sum_{k,l=(\alpha_1+1,\beta_1+1)\cap(1/\lambda<k/l<\lambda)}^{k_2,l_2} a_{m_2,n_2,k,l}z_{k,l}-\nu_{2,2}\right|<\frac{1}{2^4},\\ \sum_{k,l=(\alpha_1+1,\beta_1+1)\cap(k/l\le 1/\lambda\cup k/l\ge\lambda)}^{k_2,l_2}|a_{m_2,n_2,k,l}x_{k,l}|<\frac{1}{2^4}.\end{gathered} \tag{3.8}$$

Also let us select the following z's: $(\{z_{i,k_2+1}:1\le i\le l_2+1\}\cap\{1/\lambda<i/(k_2+1)<\lambda\})\cup(\{z_{l_2+1,i}:1\le i<k_2\}\cap\{1/\lambda<(l_2+1)/i<\lambda\})$ and denote these z's by $\{z_{\zeta,\eta}\}$. In addition, $\{z_{\zeta,\eta}\}$ are selected such that $\{\zeta,\eta\}$ corresponds to the first index of x in $x\backslash\{z_{i,j}\}_{i,j=1,1}^{\alpha_2-1,\beta_2-1}$. By the RH-regularity conditions, there exist $\overline{m}_2>m_2$, $\overline{n}_2>n_2$, $\alpha_2>k_2+1$, and $\beta_2>l_2+1$ such that $|a_{\overline{m}_2,\overline{n}_2,\alpha_2,\beta_2}|>0$ and $|a_{\overline{m}_2,\overline{n}_2,k,l}|=0$, where $k>\alpha_2$ or $l>\beta_2$. Choose $\{z_{i,j}\}_{i,j=k_2+2,l_2+2}^{\alpha_2-1,\beta_2-1}$ in a Pringsheim subsequence sense from $x\backslash\{z_{i,j}\}_{i,j=1,1}^{k_2+1,l_2+1}$ in the λ-wedge of x. Let us denote $(\{z_{i,\beta_2}:1\le i\le\alpha_2\}\cap\{1/\lambda<i/\beta_2<\lambda\})\cup(\{z_{\alpha_2,i}:1\le i<\beta_2\}\cap\{1/\lambda<\alpha_2/i<\lambda\})$ by $\{z_{\zeta,\eta}\}$, where $\{z_{\zeta,\eta}\}$ are selected such that $\{\zeta,\eta\}$ corresponds to the first index of x in $x\backslash\{z_{i,j}\}_{i,j=1,1}^{\alpha_2-1,\beta_2-1}$ such that

$$\left|\sum_{k,l=1,1}^{\alpha_2,\beta_2} a_{\overline{m}_2,\overline{n}_2,k,l}z_{k,l}\right|>2^4. \tag{3.9}$$

Thus, in general we select two double sequences (m_r,n_s) and $(\overline{m}_r,\overline{n}_s)$ as follows: let $(\alpha_{r-1},\beta_{s-1})$, (m_{r-1},n_{s-1}), and $(\overline{m}_{r-1},\overline{n}_{s-1})$ be selected index pairs and let $\{z_{i,j}\}_{i,j=1,1}^{\alpha_{r-1},\beta_{s-1}}$ be the λ-permutation of the terms in the $\alpha_{r-1}\beta_{s-1}$-block of x. Let us select (m_r,n_s) such that $m_r>\overline{m}_{r-1}$ and $n_s>\overline{n}_{s-1}$ such that

$$\sum_{k,l=1,1}^{\alpha_{r-1},\beta_{s-1}}|a_{m_r,n_s,k,l}z_{k,l}|<\frac{1}{2^{r+s}}. \tag{3.10}$$

Since A is pairwise finite and we have RH_3 and RH_4, we are granted the following:

$$\sum_{(k>\alpha_{r-1}\cup 1\le l\le\beta_{s-1})\cap(1/\lambda<k/l<\lambda)}|a_{m_r,n_s,k,l}z_{k,l}|<\frac{1}{2^{r+s}},$$

$$\sum_{(k>\alpha_{r-1}\cup 1\le l\le\beta_{s-1})\cap(k/l\le 1/\lambda\cup k/l\ge\lambda)}|a_{m_r,n_s,k,l}x_{k,l}|<\frac{1}{2^{r+s}},$$

$$\sum_{(1\le k\le \alpha_{r-1}\cap l>\beta_{s-1})\cap(1/\lambda<k/l<\lambda)} \left| a_{m_r,n_s,k,l} z_{k,l} \right| < \frac{1}{2^{r+s}},$$

$$\sum_{(1\le k\le \alpha_{r-1}\cap l>\beta_{s-1})\cap(k/l\le 1/\lambda \cup k/l\ge\lambda)} \left| a_{m_r,n_s,k,l} x_{k,l} \right| < \frac{1}{2^{r+s}}. \tag{3.11}$$

The RH-regularity conditions RH_1 through RH_5 imply that

$$\left| \sum_{k>\alpha_{r-1},l>\beta_{s-1}} a_{m_r,n_s,k,l} - 1 \right| < \frac{1}{2^{r+s}\left(\left| \nu_{r,s} \right| + 1 \right)}. \tag{3.12}$$

Let $k_r = \sup\{k : |a_{m_r,n_s,k,l}| > 0\}$ and $l_s = \sup\{l : |a_{m_r,n_s,k,l}| > 0\}$ and choose $\{z_{i,j}\}_{i,j=\alpha_{r-1}+1,\beta_{s-1}+1}^{k_r,l_s}$ in a Pringsheim subsequence sense from $x\backslash\{z_{i,j}\}_{i,j=1,1}^{\alpha_{r-1},\beta_{s-1}}$ with elements in the λ-wedge of x such that

$$\left| \sum_{k,l=(\alpha_{r-1}+1,\beta_{s-1}+1)\cap(1/\lambda<k/l<\lambda)}^{k_r,l_s} a_{m_r,n_s,k,l} z_{k,l} - \nu_{r,s} \right| < \frac{1}{2^{r+s}},$$

$$\sum_{k,l=(\alpha_{r-1}+1,\beta_{s-1}+1)\cap(k/l\le 1/\lambda\cup k/l\ge\lambda)}^{k_r,l_s} \left| a_{m_r,n_s,k,l} x_{k,l} \right| < \frac{1}{2^{r+s}}. \tag{3.13}$$

Also let us select the following z's : $(\{z_{i,k_r+1} : 1 \le i \le l_s + 1\} \cap \{1/\lambda < i/(k_r + 1) < \lambda\}) \cup (\{z_{l_s+1,i} : 1 \le i < k_r + 1\} \cap \{1/\lambda < (l_s + 1)/i < \lambda\})$ and denote these z's by $\{z_{\zeta,\eta}\}$. In addition, $\{z_{\zeta,\eta}\}$ are selected such that $\{\zeta, \eta\}$ corresponds to the first index of x in $x\backslash\{z_{i,j}\}_{i,j=1,1}^{\alpha_r-1,\beta_s-1}$. By the RH-regularity conditions, there exist $\overline{m}_r > m_r$, $\overline{n}_s > n_s$, $\alpha_r > k_r + 1$, and $\beta_s > l_s + 1$ such that $|a_{\overline{m}_r,\overline{n}_s,\alpha_r,\beta_s}| > 0$ and $|a_{\overline{m}_r,\overline{n}_s,k,l}| = 0$, where $k > \alpha_r$ or $l > \beta_s$. Choose $\{z_{i,j}\}_{i,j=k_r+2,l_s+2}^{\alpha_r-1,\beta_s-1}$ in a Pringsheim subsequence sense from $x\backslash\{z_{i,j}\}_{i,j=1,1}^{k_{2r}+1,l_s+1}$ in the λ-wedge of x. Let us denote $(\{z_{i,\beta_s} : 1 \le i \le \alpha_r\} \cap \{1/\lambda < i/\beta_s < \lambda\}) \cup (\{z_{\alpha_r,i} : 1 \le i < \beta_s\} \cap \{1/\lambda < \alpha_r/i < \lambda\})$ by $\{z_{\zeta,\eta}\}$, where $\{z_{\zeta,\eta}\}$ are selected such that $\{\zeta, \eta\}$ corresponds to the first index of x in $x\backslash\{z_{i,j}\}_{i,j=1,1}^{\alpha_r-1,\beta_s-1}$ such that

$$\left| \sum_{k,l=1,1}^{\alpha_r,\beta_s} a_{\overline{m}_r,\overline{n}_s,k,l} z_{k,l} \right| > 2^{r+s}. \tag{3.14}$$

This process grants us two positive integer double sequences and a λ-rearrangement z of x having the following properties: $|(Az)_{m_r,n_s} - \nu_{r,s}| = o(1)$ and $\{|(Az)_{\overline{m}_r,\overline{n}_s}|\}$ which is definite divergent. This completes the proof. □

Corollary 3.3. *If there exists a four-dimensional matrix A satisfying RH_1 through RH_5 such that Az is definite divergent for every λ-rearrangement z of x, then x is definite divergent.*

Corollary 3.4. *If there exists a four-dimensional matrix A satisfying RH_1 through RH_5 such that Az is bounded for every λ-rearrangement z of x, then x has only bounded subsequence.*

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Richard F. Patterson: Department of Mathematics and Statistics, University of North Florida, Building 14, Jacksonville, FL 32224, USA
Email address: rpatters@unf.edu