INVERSION FORMULAS FOR RIEMANN-LIOUVILLE TRANSFORM AND ITS DUAL ASSOCIATED WITH SINGULAR PARTIAL DIFFERENTIAL OPERATORS

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We define Riemann-Liouville transform $\mathcal{R}_\alpha$ and its dual ${}^t\mathcal{R}_\alpha$ associated with two singular partial differential operators. We establish some results of harmonic analysis for the Fourier transform connected with $\mathcal{R}_\alpha$. Next, we prove inversion formulas for the operators $\mathcal{R}_\alpha$, ${}^t\mathcal{R}_\alpha$ and a Plancherel theorem for ${}^t\mathcal{R}_\alpha$.

1. Introduction

The mean operator is defined for a continuous function f on $\mathbb{R}^2$, even with respect to the first variable by

$$\mathfrak{R}_0(f)(r,x) = \frac{1}{2\pi}\int_0^{2\pi} f(r\sin\theta, x + r\cos\theta)\,d\theta, \tag{1.1}$$

which means that $\mathfrak{R}_0(f)(r,x)$ is the mean value of f on the circle centered at $(0,x)$ and radius r. The dual of the mean operator ${}^t\mathfrak{R}_0$ is defined by

$${}^t\mathfrak{R}_0(f)(r,x) = \frac{1}{\pi}\int_{\mathbb{R}} f\left(\sqrt{r^2+(x-y)^2}, y\right)dy. \tag{1.2}$$

The mean operator $\mathfrak{R}_0$ and its dual ${}^t\mathfrak{R}_0$ play an important role and have many applications, for example, in image processing of the so-called synthetic aperture radar (SAR) data [11, 12] or in the linearized inverse scattering problem in acoustics [6].

Our purpose in this work is to define and study integral transforms which generalize the operators $\mathfrak{R}_0$ and ${}^t\mathfrak{R}_0$. More precisely, we consider the following singular partial differential operators:

$$\begin{gathered}\Delta_1 = \frac{\partial}{\partial x},\\ \Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial x^2}, \quad (r,x)\in]0,+\infty[\times\mathbb{R},\ \alpha \geqslant 0.\end{gathered} \tag{1.3}$$

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We associate to Δ_1 and Δ_2 the Riemann-Liouville transform $\mathfrak{R}_\alpha$, defined on $\mathscr{C}_*(\mathbb{R}^2)$ (the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable) by

$$\mathfrak{R}_\alpha(f)(r,x)=\begin{cases}\dfrac{\alpha}{\pi}\displaystyle\iint_{-1}^{1} f\left(rs\sqrt{1-t^2},x+rt\right) \\ \qquad\times(1-t^2)^{\alpha-1/2}(1-s^2)^{\alpha-1}\,dt\,ds, & \text{if } \alpha>0,\\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} f\left(r\sqrt{1-t^2},x+rt\right)\frac{dt}{\sqrt{1-t^2}}, & \text{if } \alpha=0.\end{cases} \tag{1.4}$$

The dual operator ${}^t\mathfrak{R}_\alpha$ is defined on the space $\mathscr{S}_*(\mathbb{R}^2)$ (the space of infinitely differentiable functions on $\mathbb{R}^2$, rapidly decreasing together with all their derivatives, even with respect to the first variable) by

$${}^t\mathfrak{R}_\alpha(f)(r,x)=\begin{cases}\dfrac{2\alpha}{\pi}\displaystyle\int_r^{+\infty}\int_{-\sqrt{u^2-r^2}}^{\sqrt{u^2-r^2}} f(u,x+v)(u^2-v^2-r^2)^{\alpha-1}u\,du\,dv, & \text{if } \alpha>0,\\ \dfrac{1}{\pi}\displaystyle\int_{\mathbb{R}} f\left(\sqrt{r^2+(x-y)^2},y\right)dy, & \text{if } \alpha=0.\end{cases} \tag{1.5}$$

For more general fractional integrals and fractional differential equations, we can see the works of Debnath [3, 4] and Debnath with Bhatta [5].

We establish for the operators $\mathfrak{R}_\alpha$ and ${}^t\mathfrak{R}_\alpha$ the same results given by Helgason, Ludwig, and Solmon for the classical Radon transform on $\mathbb{R}^2$ [10, 14, 17] and we find the results given in [15] for the spherical mean operator. Especially

(i) we give some harmonic analysis results related to the Fourier transform associated with the Riemann-Liouville transform $\mathfrak{R}_\alpha$;

(ii) we define and characterize some spaces of the functions on which $\mathfrak{R}_\alpha$ and ${}^t\mathfrak{R}_\alpha$ are isomorphisms;

(iii) we give the following inversion formulas for $\mathfrak{R}_\alpha$ and ${}^t\mathfrak{R}_\alpha$:

$$\begin{aligned} f&=\mathfrak{R}_\alpha K_\alpha^1\,{}^t\mathfrak{R}_\alpha(f), & f&=K_\alpha^1\,{}^t\mathfrak{R}_\alpha\mathfrak{R}_\alpha(f),\\ f&={}^t\mathfrak{R}_\alpha\,K_\alpha^2\mathfrak{R}_\alpha(f), & f&=K_\alpha^2\,\mathfrak{R}_\alpha\,{}^t\mathfrak{R}_\alpha(f),\end{aligned} \tag{1.6}$$

where K_α^1 and K_α^2 are integro-differential operators;

(iv) we establish a Plancherel theorem for ${}^t\mathfrak{R}_\alpha$;

(v) we show that $\mathfrak{R}_\alpha$ and ${}^t\mathfrak{R}_\alpha$ are transmutation operators.

This paper is organized as follows. In Section 2, we show that for $(\mu,\lambda)\in\mathbb{C}^2$, the differential system

$$\begin{aligned} \Delta_1 u(r,x)&=-i\lambda u(r,x),\\ \Delta_2 u(r,x)&=-\mu^2 u(r,x),\\ u(0,0)=1,\qquad &\frac{\partial u}{\partial r}(0,x)=0,\quad \forall x\in\mathbb{R},\end{aligned} \tag{1.7}$$

admits a unique solution $\varphi_{\mu,\lambda}$ given by

$$\varphi_{\mu,\lambda}(r,x) = j_\alpha\left(r\sqrt{\mu^2+\lambda^2}\right)\exp(-i\lambda x), \tag{1.8}$$

where j_α is the modified Bessel function defined by

$$j_\alpha(s) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(s)}{s^\alpha}, \tag{1.9}$$

and J_α is the Bessel function of first kind and index α. Next, we prove a Mehler integral representation of $\varphi_{\mu,\lambda}$ and give some properties of $\mathfrak{R}_\alpha$.

In Section 3, we define the Fourier transform $\mathfrak{F}_\alpha$ connected with $\mathfrak{R}_\alpha$, and we establish some harmonic analysis results (inversion formula, Plancherel theorem, Paley-Wiener theorem) which lead to new properties of the operator $\mathfrak{R}_\alpha$ and its dual ${}^t\mathfrak{R}_\alpha$.

In Section 4, we characterize some subspaces of $\mathcal{S}_*(\mathbb{R}^2)$ on which $\mathfrak{R}_\alpha$ and ${}^t\mathfrak{R}_\alpha$ are isomorphisms, and we prove the inversion formulas cited below where the operators K_α^1 and K_α^2 are given in terms of Fourier transforms. Next, we introduce fractional powers of the Bessel operator,

$$\ell_\alpha = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r}, \tag{1.10}$$

and the Laplacian operator,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial x^2}, \tag{1.11}$$

that we use to simplify K_α^1 and K_α^2.

Finally, we prove the following Plancherel theorem for ${}^t\mathfrak{R}_\alpha$:

$$\int_{\mathbb{R}}\int_0^{+\infty} |f(r,x)|^2 r^{2\alpha+1}dr\,dx = \int_{\mathbb{R}}\int_0^{+\infty} |K_\alpha^3({}^t\mathfrak{R}_\alpha(f))(r,x)|^2 dr\,dx, \tag{1.12}$$

where K_α^3 is an integro-differential operator.

In Section 5, we show that $\mathfrak{R}_\alpha$ and ${}^t\mathfrak{R}_\alpha$ satisfy the following relations of permutation:

$$\begin{aligned} {}^t\mathfrak{R}_\alpha(\Delta_2 f) &= \frac{\partial^2}{\partial r^2}{}^t\mathfrak{R}_\alpha(f), \qquad {}^t\mathfrak{R}_\alpha(\Delta_1 f) = \Delta_1{}^t\mathfrak{R}_\alpha(f), \\ \Delta_2\mathfrak{R}_\alpha(f) &= \mathfrak{R}_\alpha\left(\frac{\partial^2 f}{\partial r^2}\right), \qquad \Delta_1\mathfrak{R}_\alpha(f) = \mathfrak{R}_\alpha(\Delta_1 f). \end{aligned} \tag{1.13}$$

2. Riemann-Liouville transform and its dual associated with the operators Δ_1 and Δ_2

In this section, we define the Riemann-Liouville transform $\mathfrak{R}_\alpha$ and its dual ${}^t\mathfrak{R}_\alpha$, and we give some properties of these operators. It is well known [21] that for every $\lambda \in \mathbb{C}$, the system

$$\begin{aligned} \ell_\alpha v(r) &= -\lambda^2 v(r); \\ v(0) = 1; &\qquad v'(0) = 0, \end{aligned} \tag{2.1}$$

where ℓ_α is the Bessel operator, admits a unique solution, that is, the modified Bessel function $r \mapsto j_\alpha(r\lambda)$. Thus, for all $(\mu,\lambda) \in \mathbb{C}\times\mathbb{C}$, the system

$$\begin{gathered}\Delta_1 u(r,x) = -i\lambda u(r,x),\\ \Delta_2 u(r,x) = -\mu^2 u(r,x),\\ u(0,0) = 1, \qquad \frac{\partial u}{\partial r}(0,x) = 0, \quad \forall x \in \mathbb{R},\end{gathered} \tag{2.2}$$

admits the unique solution given by

$$\varphi_{\mu,\lambda}(r,x) = j_\alpha\left(r\sqrt{\mu^2+\lambda^2}\right)\exp(-i\lambda x). \tag{2.3}$$

The modified Bessel function j_α has the Mehler integral representation, (we refer to [13, 21])

$$j_\alpha(s) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}\int_{-1}^{1}\left(1-t^2\right)^{\alpha-1/2}\exp(-ist)dt. \tag{2.4}$$

In particular,

$$\forall k \in \mathbb{N},\ \forall s \in \mathbb{R}, \quad \left| j_\alpha^{(k)}(s)\right| \leqslant 1. \tag{2.5}$$

On the other hand,

$$\sup_{r\in\mathbb{R}} \left| j_\alpha(r\lambda)\right| = 1 \quad \text{iff } \lambda \in \mathbb{R}. \tag{2.6}$$

This involves that

$$\sup_{(r,x)\in\mathbb{R}^2} \left|\varphi_{\mu,\lambda}(r,x)\right| = 1 \quad \text{iff } (\mu,\lambda) \in \Gamma, \tag{2.7}$$

where Γ is the set defined by

$$\Gamma = \mathbb{R}^2 \cup \left\{(i\mu,\lambda);\ (\mu,\lambda) \in \mathbb{R}^2,\ |\mu| \leqslant |\lambda|\right\}. \tag{2.8}$$

Proposition 2.1. *The eigenfunction $\varphi_{\mu,\lambda}$ given by (2.3) has the following Mehler integral representation:*

$$\varphi_{\mu,\lambda}(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\iint_{-1}^{1}\cos\left(\mu r s\sqrt{1-t^2}\right)\exp\left(-i\lambda(x+rt)\right)\left(1-t^2\right)^{\alpha-1/2}\left(1-s^2\right)^{\alpha-1}dt\,ds, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \alpha > 0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1}\cos\left(r\mu\sqrt{1-t^2}\right)\exp\left(-i\lambda(x+rt)\right)\frac{dt}{\sqrt{1-t^2}}, \quad \text{if } \alpha = 0. \end{cases} \tag{2.9}$$

Proof. From the following expansion of the function j_α:

$$j_\alpha(s) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(s)}{s^\alpha} = \Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\left(\frac{s}{2}\right)^{2k}, \tag{2.10}$$

we deduce that

$$j_\alpha\left(r\sqrt{\mu^2+\lambda^2}\right) = \Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(k+\alpha+1)}\left(\frac{r\mu}{2}\right)^{2k} j_{\alpha+k}(r\lambda), \tag{2.11}$$

and from the equality (2.4), we obtain

$$j_\alpha\left(r\sqrt{\mu^2+\lambda^2}\right) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}\int_{-1}^{1} j_{\alpha-1/2}\left(r\mu\sqrt{1-t^2}\right)\exp(-ir\lambda t)\left(1-t^2\right)^{\alpha-1/2}dt. \tag{2.12}$$

Then, the results follow by using again the relation (2.4) for $\alpha > 0$, and from the fact that

$$j_{-1/2}(s) = \cos s, \quad \text{for } \alpha = 0. \tag{2.13}$$

□

Definition 2.2. The Riemann-Liouville transform $\mathfrak{R}_\alpha$ associated with the operators Δ_1 and Δ_2 is the mapping defined on $\mathscr{C}_*(\mathbb{R}^2)$ by the following. For all $(r,x) \in \mathbb{R}^2$,

$$\mathfrak{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\iint_{-1}^{1} f\left(rs\sqrt{1-t^2}, x+rt\right) \\ \qquad \times\left(1-t^2\right)^{\alpha-1/2}\left(1-s^2\right)^{\alpha-1}dt\,ds, & \text{if } \alpha > 0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} f\left(r\sqrt{1-t^2}, x+rt\right)\frac{dt}{\sqrt{1-t^2}}, & \text{if } \alpha = 0. \end{cases} \tag{2.14}$$

Remark 2.3. (i) From Proposition 2.1 and Definition 2.2, we have

$$\varphi_{\mu,\lambda}(r,x) = \mathfrak{R}_\alpha\left(\cos(\mu.)\exp(-i\lambda.)\right)(r,x). \tag{2.15}$$

(ii) We can easily see, as in [2], that the transform $\mathfrak{R}_\alpha$ is continuous and injective from $\mathscr{E}_*(\mathbb{R}^2)$ (the space of infinitely differentiable functions on $\mathbb{R}^2$, even with respect to the first variable) into itself.

Lemma 2.4. *For $f \in \mathscr{C}_*(\mathbb{R}^2)$, f bounded, and $g \in \mathscr{S}_*(\mathbb{R}^2)$,*

$$\int_{\mathbb{R}}\int_0^{+\infty} \mathfrak{R}_\alpha(f)(r,x)g(r,x)r^{2\alpha+1}dr\,dx = \int_{\mathbb{R}}\int_0^{+\infty} f(r,x)\,{}^t\mathfrak{R}_\alpha(g)(r,x)dr\,dx, \tag{2.16}$$

where ${}^t\mathfrak{R}_\alpha$ is the dual transform defined by

$${}^t\mathfrak{R}_\alpha(g)(r,x) = \begin{cases} \dfrac{2\alpha}{\pi}\displaystyle\int_r^{+\infty}\int_{-\sqrt{u^2-r^2}}^{\sqrt{u^2-r^2}} g(u,x+v)\left(u^2-v^2-r^2\right)^{\alpha-1}u\,du\,dv, & \text{if } \alpha > 0, \\ \dfrac{1}{\pi}\displaystyle\int_{\mathbb{R}} g\left(\sqrt{r^2+(x-y)^2},y\right)dy, & \text{if } \alpha = 0. \end{cases} \tag{2.17}$$

To obtain this lemma, we use Fubini's theorem and an adequate change of variables.

Remark 2.5. By a simple change of variables, we have

$$\mathfrak{R}_0(f)(r,x) = \frac{1}{2\pi}\int_0^{2\pi} f(r\sin\theta, x + r\cos\theta)d\theta. \tag{2.18}$$

3. Fourier transform associated with Riemann-Liouville operator

In this section, we define the Fourier transform associated with the operator $\mathfrak{R}_\alpha$, and we give some results of harmonic analysis that we use in the next sections.

We denote by

(i) $d\nu(r,x)$ the measure defined on $[0,+\infty[\times\mathbb{R}$ by

$$d\nu(r,x) = \frac{1}{\sqrt{2\pi}2^\alpha\Gamma(\alpha+1)}r^{2\alpha+1}dr\otimes dx, \tag{3.1}$$

(ii) $L^1(d\nu)$ the space of measurable functions f on $[0,+\infty[\times\mathbb{R}$ satisfying

$$\|f\|_{1,\nu} = \int_{\mathbb{R}}\int_0^{+\infty} |f(r,x)|\,d\nu(r,x) < +\infty. \tag{3.2}$$

Definition 3.1. (i) The translation operator associated with Riemann-Liouville transform is defined on $L^1(d\nu)$ by the following. For all $(r,x),(s,y) \in [0,+\infty[\times\mathbb{R}$,

$$\mathscr{T}_{(r,x)}f(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\int_0^\pi f\left(\sqrt{r^2+s^2+2rs\cos\theta}, x+y\right)\sin^{2\alpha}\theta d\theta. \tag{3.3}$$

(ii) The convolution product associated with the Riemann-Liouville transform of f, $g \in L^1(d\nu)$ is defined by the following. For all $(r,x) \in [0,+\infty[\times\mathbb{R}$,

$$f\#g(r,x) = \int_{\mathbb{R}}\int_0^{+\infty} \mathscr{T}_{(r,-x)}\check{f}(s,y)g(s,y)d\nu(s,y), \tag{3.4}$$

where $\check{f}(s,y) = f(s,-y)$.

We have the following properties.

(i) Since

$$\forall r,s \geqslant 0, \quad j_\alpha(r\lambda)j_\alpha(s\lambda) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\int_0^\pi j_\alpha\big(\lambda\sqrt{r^2+s^2+2rs\cos\theta}\big)\sin^{2\alpha}\theta\,d\theta, \quad (3.5)$$

(we refer to [21]) we deduce that the eigenfunction $\varphi_{\mu,\lambda}$ defined by the relation (2.3) satisfies the product formula

$$\mathcal{T}_{(r,x)}\varphi_{\mu,\lambda}(s,y) = \varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y). \quad (3.6)$$

(ii) If $f \in L^1(d\nu)$, then for all $(r,x) \in [0,+\infty[\times\mathbb{R}$, $\mathcal{T}_{(r,x)}f$ belongs to $L^1(d\nu)$, and we have

$$\big|\big|\mathcal{T}_{(r,x)}f\big|\big|_{1,\nu} \leqslant \|f\|_{1,\nu}. \quad (3.7)$$

(iii) For $f,g \in L^1(d\nu)$, $f\#g$ belongs to $L^1(d\nu)$, and the convolution product is commutative and associative.

(iv) For $f,g \in L^1(d\nu)$,

$$\|f\#g\|_{1,\nu} \leqslant \|f\|_{1,\nu}\|g\|_{1,\nu}. \quad (3.8)$$

Definition 3.2. The Fourier transform associated with the Riemann-Liouville operator is defined by

$$\forall(\mu,\lambda)\in\Gamma, \quad \mathfrak{F}_\alpha(f)(\mu,\lambda) = \int_{\mathbb{R}}\int_0^{+\infty} f(r,x)\varphi_{\mu,\lambda}(r,x)\,d\nu(r,x), \quad (3.9)$$

where Γ is the set defined by the relation (2.8).

We have the following properties.

(i) Let f be in $L^1(d\nu)$. For all $(r,x) \in [0,+\infty[\times\mathbb{R}$, we have

$$\forall(\mu,\lambda)\in\Gamma, \quad \mathfrak{F}_\alpha(\mathcal{T}_{(r,-x)}f)(\mu,\lambda) = \varphi_{\mu,\lambda}(r,x)\mathfrak{F}_\alpha(f)(\mu,\lambda). \quad (3.10)$$

(ii) For $f,g \in L^1(d\nu)$, we have

$$\forall(\mu,\lambda)\in\Gamma, \quad \mathfrak{F}_\alpha(f\#g)(\mu,\lambda) = \mathfrak{F}_\alpha(f)(\mu,\lambda)\mathfrak{F}_\alpha(g)(\mu,\lambda). \quad (3.11)$$

(iii) For $f \in L^1(d\nu)$, we have

$$\forall(\mu,\lambda)\in\Gamma, \quad \mathfrak{F}_\alpha(f)(\mu,\lambda) = B\circ\widetilde{\mathfrak{F}}_\alpha(f)(\mu,\lambda), \quad (3.12)$$

where

$$\begin{gathered}\forall(\mu,\lambda)\in\mathbb{R}^2, \quad \widetilde{\mathfrak{F}}_\alpha(f)(\mu,\lambda) = \int_{\mathbb{R}}\int_0^{+\infty} f(r,x)j_\alpha(r\mu)\exp(-i\lambda x)\,d\nu(r,x), \\ \forall(\mu,\lambda)\in\Gamma, \quad Bf(\mu,\lambda) = f\big(\sqrt{\mu^2+\lambda^2},\lambda\big).\end{gathered} \quad (3.13)$$

3.1. Inversion formula and Plancherel theorem for $\mathfrak{F}_\alpha$. We denote by (see [15])

(i) $\mathcal{S}_*(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$ rapidly decreasing together with all their derivatives, even with respect to the first variable;

(ii) $\mathcal{S}_*(\Gamma)$ the space of functions $f:\Gamma\to\mathbb{C}$ infinitely differentiable, even with respect to the first variable and rapidly decreasing together with all their derivatives, that is, for all $k_1,k_2,k_3\in\mathbb{N}$,

$$\sup_{(\mu,\lambda)\in\Gamma}\left(1+|\mu|^2+|\lambda|^2\right)^{k_1}\left|\left(\frac{\partial}{\partial\mu}\right)^{k_2}\left(\frac{\partial}{\partial\lambda}\right)^{k_3}f(\mu,\lambda)\right|<+\infty, \tag{3.14}$$

where

$$\frac{\partial f}{\partial\mu}(\mu,\lambda)=\begin{cases}\dfrac{\partial}{\partial r}\big(f(r,\lambda)\big), & \text{if } \mu=r\in\mathbb{R},\\[2mm] \dfrac{1}{i}\dfrac{\partial}{\partial t}\big(f(it,\lambda)\big), & \text{if } \mu=it,\ |t|\leqslant|\lambda|.\end{cases} \tag{3.15}$$

Each of these spaces is equipped with its usual topology:

(i) $L^2(d\nu)$ the space of measurable functions on $[0,+\infty[\times\mathbb{R}$ such that

$$\|f\|_{2,\nu}=\left(\int_{\mathbb{R}}\int_0^{+\infty}|f(r,x)|^2d\nu(r,x)\right)^{1/2}<+\infty; \tag{3.16}$$

(ii) $d\gamma(\mu,\lambda)$ the measure defined on Γ by

$$\begin{aligned}&\iint_\Gamma f(\mu,\lambda)d\gamma(\mu,\lambda)\\&=\frac{1}{\sqrt{2\pi}2^\alpha\Gamma(\alpha+1)}\left\{\int_{\mathbb{R}}\int_0^{+\infty}f(\mu,\lambda)\left(\mu^2+\lambda^2\right)^\alpha\mu\,d\mu\,d\lambda+\int_{\mathbb{R}}\int_0^{|\lambda|}f(i\mu,\lambda)\left(\lambda^2-\mu^2\right)^\alpha\mu\,d\mu\,d\lambda\right\};\end{aligned} \tag{3.17}$$

(iii) $L^p(d\gamma)$, $p=1$, $p=2$, the space of measurable functions on Γ satisfying

$$\|f\|_{p,\gamma}=\left(\iint_\Gamma|f(\mu,\lambda)|^p d\gamma(\mu,\lambda)\right)^{1/p}<+\infty. \tag{3.18}$$

Remark 3.3. It is clear that a function f belongs to $L^1(d\nu)$ if, and only if, the function Bf belongs to $L^1(d\gamma)$, and we have

$$\iint_\Gamma Bf(\mu,\lambda)d\gamma(\mu,\lambda)=\int_{\mathbb{R}}\int_0^{+\infty}f(r,x)d\nu(r,x). \tag{3.19}$$

Proposition 3.4 (inversion formula for $\mathfrak{F}_\alpha$). *Let $f\in L^1(d\nu)$ such that $\mathfrak{F}_\alpha(f)$ belongs to $L^1(d\gamma)$, then for almost every $(r,x)\in[0,+\infty[\times\mathbb{R}$,*

$$f(r,x)=\iint_\Gamma\mathfrak{F}_\alpha(f)(\mu,\lambda)\overline{\varphi}_{\mu,\lambda}(r,x)d\gamma(\mu,\lambda). \tag{3.20}$$

Proof. From [9, 19], one can see that if $f \in L^1(d\nu)$ is such that $\tilde{\mathfrak{F}}_\alpha(f) \in L^1(d\nu)$, then for almost every $(r,x) \in [0,+\infty[\times\mathbb{R}$,

$$f(r,x) = \int_{\mathbb{R}}\int_0^{+\infty} \tilde{\mathfrak{F}}_\alpha(f)(\mu,\lambda)\, j_\alpha(r\mu)\exp(i\lambda x)\,d\nu(\mu,\lambda). \tag{3.21}$$

Then, the result follows from the relation (3.12) and Remark 3.3. □

Theorem 3.5. (i) *The Fourier transform $\mathfrak{F}_\alpha$ is an isomorphism from $\mathcal{S}_*(\mathbb{R}^2)$ onto $\mathcal{S}_*(\Gamma)$.*

(ii) *(Plancherel formula) for $f \in \mathcal{S}_*(\mathbb{R}^2)$,*

$$\|\mathfrak{F}_\alpha(f)\|_{2,\gamma} = \|f\|_{2,\nu}. \tag{3.22}$$

(iii) *(Plancherel theorem) the transform $\mathfrak{F}_\alpha$ can be extended to an isometric isomorphism from $L^2(d\nu)$ onto $L^2(d\gamma)$.*

Proof. This theorem follows from the relation (3.12), Remark 3.3, and the fact that $\tilde{\mathfrak{F}}_\alpha$ is an isomorphism from $\mathcal{S}_*(\mathbb{R}^2)$ onto itself, satisfying that for all $f \in \mathcal{S}_*(\mathbb{R}^2)$,

$$\|\tilde{\mathfrak{F}}_\alpha(f)\|_{2,\nu} = \|f\|_{2,\nu}. \tag{3.23}$$

□

Lemma 3.6. *For $f \in \mathcal{S}_*(\mathbb{R}^2)$,*

$$\forall(\mu,\lambda) \in \mathbb{R}^2, \quad \mathfrak{F}_\alpha(f)(\mu,\lambda) = \Lambda_\alpha \circ {}^t\mathfrak{R}_\alpha(f)(\mu,\lambda), \tag{3.24}$$

where ${}^t\mathfrak{R}_\alpha$ is the dual transform of the Riemann-Liouville operator, and Λ_α is a constant multiple of the classical Fourier transform on $\mathbb{R}^2$ defined by

$$\Lambda_\alpha(f)(\mu,\lambda) = \int_{\mathbb{R}}\int_0^{+\infty} f(r,x)\cos(r\mu)\exp(-i\lambda x)\,dm(r,x), \tag{3.25}$$

where $dm(r,x)$ is the measure defined on $[0,+\infty[\times\mathbb{R}$ by

$$dm(r,x) = \frac{1}{\sqrt{2\pi}2^\alpha\Gamma(\alpha+1)}dr \otimes dx. \tag{3.26}$$

This lemma follows from the relation (2.15) and Lemma 2.4.

Using the relation (3.12) and the fact that the mapping B is continuous from $\mathcal{S}_*(\mathbb{R}^2)$ into itself, we deduce that the Fourier transform $\mathfrak{F}_\alpha$ is continuous from $\mathcal{S}_*(\mathbb{R}^2)$ into itself. On the other hand, Λ_α is an isomorphism from $\mathcal{S}_*(\mathbb{R}^2)$ onto itself. Then, Lemma 3.6 implies that the dual transform ${}^t\mathfrak{R}_\alpha$ maps continuously $\mathcal{S}_*(\mathbb{R}^2)$ into itself.

Proposition 3.7. (i) *${}^t\mathfrak{R}_\alpha$ is not injective when applied to $\mathcal{S}_*(\mathbb{R}^2)$.*

(ii) ${}^t\mathfrak{R}_\alpha(\mathcal{S}_*(\mathbb{R}^2)) = \mathcal{S}_*(\mathbb{R}^2)$.

Proof. (i) Let $g \in \mathscr{S}_*(\mathbb{R}^2)$ such that $\operatorname{supp} g \subset \{(r,x) \in \mathbb{R}^2,\ |r| \leqslant |x|\}, g \neq 0$.

Since $\widetilde{\mathfrak{F}}_\alpha$ is an isomorphism from $\mathscr{S}_*(\mathbb{R}^2)$ onto itself, there exists $f \in \mathscr{S}_*(\mathbb{R}^2)$ such that $\widetilde{\mathfrak{F}}_\alpha(f) = g$. From the relation (3.12) and Lemma 3.6, we deduce that ${}^t\mathfrak{R}_\alpha(f) = 0$.

(ii) We obtain the result by the same way as in [1]. □

3.2. Paley-Wiener theorem. We denote by

(i) $\mathscr{D}_*(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, even with respect to the first variable, and with compact support;

(ii) $\mathbb{H}_*(\mathbb{C}^2)$ the space of entire functions $f : \mathbb{C}^2 \to \mathbb{C}$, even with respect to the first variable rapidly decreasing of exponential type, that is, there exists a positive constant M, such that for all $k \in \mathbb{N}$,

$$\sup_{(\mu,\lambda)\in\mathbb{C}^2} \left(1 + |\mu|^2 + |\lambda|^2\right)^k |f(\mu,\lambda)| \exp\left(-M(|\operatorname{Im}\mu| + |\operatorname{Im}\lambda|)\right) < +\infty; \tag{3.27}$$

(iii) $\mathbb{H}_{*,0}(\mathbb{C}^2)$ the subspace of $\mathbb{H}_*(\mathbb{C}^2)$, consisting of functions $f : \mathbb{C}^2 \to \mathbb{C}$, such that for all $k \in \mathbb{N}$,

$$\sup_{\substack{(\mu,\lambda)\in\mathbb{R}^2\\ |\mu|\leqslant|\lambda|}} \left(1 - \mu^2 + 2\lambda^2\right)^k |f(i\mu,\lambda)| < +\infty; \tag{3.28}$$

(iv) $\mathscr{E}'_*(\mathbb{R}^2)$ the space of distributions on $\mathbb{R}^2$, even with respect to the first variable, and with compact support;

(v) $\mathscr{H}_*(\mathbb{C}^2)$ the space of entire functions $f : \mathbb{C}^2 \to \mathbb{C}$, even with respect to the first variable, slowly increasing of exponential type, that is, there exist a positive constant M and an integer k, such that

$$\sup_{(\mu,\lambda)\in\mathbb{C}^2} \left(1 + |\mu|^2 + |\lambda|^2\right)^{-k} |f(\mu,\lambda)| \exp\left(-M(|\operatorname{Im}\mu| + |\operatorname{Im}\lambda|)\right) < +\infty; \tag{3.29}$$

(vi) $\mathscr{H}_{*,0}(\mathbb{C}^2)$ the subspace of $\mathscr{H}_*(\mathbb{C}^2)$, consisting of functions $f : \mathbb{C}^2 \to \mathbb{C}$, such that there exists an integer k, satisfying

$$\sup_{\substack{(\mu,\lambda)\in\mathbb{R}^2\\ |\mu|\leqslant|\lambda|}} \left(1 - \mu^2 + 2\lambda^2\right)^{-k} |f(i\mu,\lambda)| < +\infty. \tag{3.30}$$

Each of these spaces is equipped with its usual topology.

Definition 3.8. The Fourier transform associated with the Riemann-Liouville operator is defined on $\mathscr{E}'_*(\mathbb{R}^2)$ by

$$\forall (\mu,\lambda) \in \mathbb{C}^2, \quad \mathfrak{F}_\alpha(T)(\mu,\lambda) = \langle T, \varphi_{\mu,\lambda}\rangle. \tag{3.31}$$

Proposition 3.9. *For every* $T \in \mathscr{E}'_*(\mathbb{R}^2)$,

$$\forall(\mu,\lambda) \in \mathbb{C}^2, \quad \mathfrak{F}_\alpha(T)(\mu,\lambda) = B \circ \widetilde{\mathfrak{F}}_\alpha(T)(\mu,\lambda), \tag{3.32}$$

where

$$\forall(\mu,\lambda) \in \mathbb{C}^2, \quad \widetilde{\mathfrak{F}}_\alpha(T)(\mu,\lambda) = \langle T, j_\alpha(\mu.)\exp(-i\lambda.)\rangle, \tag{3.33}$$

and B is the transform defined by the relation (3.12).

Using [7, Lemma 2] (see also [15]) and the fact that $\widetilde{\mathfrak{F}}_\alpha$ is an isomorphism from $\mathscr{D}_*(\mathbb{R}^2)$ (resp., $\mathscr{E}'_*(\mathbb{R}^2)$) onto $\mathbb{H}_*(\mathbb{C}^2)$ (resp., $\mathscr{H}_*(\mathbb{C}^2)$), we deduce the following theorem.

Theorem 3.10 (of Paley-Wiener). *The Fourier transform* $\mathfrak{F}_\alpha$ *is an isomorphism*
(i) *from* $\mathscr{D}_*(\mathbb{R}^2)$ *onto* $\mathbb{H}_{*,0}(\mathbb{C}^2)$*;*
(ii) *from* $\mathscr{E}'_*(\mathbb{R}^2)$ *onto* $\mathscr{H}_{*,0}(\mathbb{C}^2)$.

From Lemma 3.6, Theorem 3.10, and the fact that Λ_α is an isomorphism from $\mathscr{D}_*(\mathbb{R}^2)$ onto $\mathbb{H}_*(\mathbb{C}^2)$, we have the following corollary.

Corollary 3.11. (i) ${}^t\mathfrak{R}_\alpha$ *maps injectively* $\mathscr{D}_*(\mathbb{R}^2)$ *into itself.*
(ii) ${}^t\mathfrak{R}_\alpha(\mathscr{D}_*(\mathbb{R}^2)) \neq \mathscr{D}_*(\mathbb{R}^2)$.

4. Inversion formulas for $\mathfrak{R}_\alpha$ and ${}^t\mathfrak{R}_\alpha$ and Plancherel theorem for ${}^t\mathfrak{R}_\alpha$

In this section, we will define some subspaces of $\mathscr{S}_*(\mathbb{R}^2)$ on which $\mathfrak{R}_\alpha$ and ${}^t\mathfrak{R}_\alpha$ are isomorphisms, and we give their inverse transforms in terms of integro-differential operators. Next, we establish Plancherel theorem for ${}^t\mathfrak{R}_\alpha$.

We denote by

(i) $\mathcal{N}$ the subspace of $\mathscr{S}_*(\mathbb{R}^2)$, consisting of functions f satisfying

$$\forall k \in \mathbb{N},\ \forall x \in \mathbb{R}, \quad \left(\frac{\partial}{\partial r^2}\right)^k f(0,x) = 0, \tag{4.1}$$

where

$$\frac{\partial}{\partial r^2} = \frac{1}{r}\frac{\partial}{\partial r}; \tag{4.2}$$

(ii) $\mathscr{S}_{*,0}(\mathbb{R}^2)$ the subspace of $\mathscr{S}_*(\mathbb{R}^2)$, consisting of functions f, such that

$$\forall k \in \mathbb{N},\ \forall x \in \mathbb{R}, \quad \int_0^{+\infty} f(r,x) r^{2k}\,dr = 0; \tag{4.3}$$

(iii) $\mathscr{S}^0_*(\mathbb{R}^2)$ the subspace of $\mathscr{S}_*(\mathbb{R}^2)$, consisting of functions f, such that

$$\operatorname{supp} \widetilde{\mathfrak{F}}_\alpha(f) \subset \{(\mu,\lambda) \in \mathbb{R}^2;\ |\mu| \geqslant |\lambda|\}. \tag{4.4}$$

Lemma 4.1. (i) *The mapping* Λ_α *is an isomorphism from* $\mathcal{S}_{*,0}(\mathbb{R}^2)$ *onto* $\mathcal{N}$.

(ii) *The subspace* $\mathcal{N}$ *can be written as*

$$\mathcal{N}=\left\{f\in\mathcal{S}_*(\mathbb{R}^2);\ \forall k\in\mathbb{N},\forall x\in\mathbb{R};\ \left(\frac{\partial}{\partial r}\right)^{2k}f(0,x)=0\right\}. \tag{4.5}$$

Proof. Let $f\in\mathcal{S}_{*,0}(\mathbb{R}^2)$.

(i) For $\nu>-1$, we have

$$\left(\frac{\partial}{\partial\mu^2}\right)^k\left(j_\nu(r\mu)\right)=\frac{\Gamma(\nu+1)}{2^k\Gamma(\nu+k+1)}\left(-r^2\right)^k j_{\nu+k}(r\mu), \tag{4.6}$$

thus, from the expression of Λ_α, given in Lemma 3.6, and the fact that $j_{-1/2}(s)=\cos s$, we obtain

$$\left(\frac{\partial}{\partial\mu^2}\right)^k\left(\Lambda_\alpha(f)\right)(0,\lambda)=\frac{\sqrt{\pi}}{2^k\Gamma(k+1/2)}(-1)^k\int_{\mathbb{R}}\int_0^{+\infty}f(r,x)r^{2k}\exp(-i\lambda x)dm(r,x), \tag{4.7}$$

which gives the result.

(ii) The proof of (ii) is immediate. □

Theorem 4.2. (i) *For all real numbers* γ, *the mappings*

(i) $f\mapsto(r^2+x^2)^\gamma f$

(ii) $f\mapsto|r|^\gamma f$

are isomorphisms from $\mathcal{N}$ *onto itself.*

(ii) *For* $f\in\mathcal{N}$, *the function* g *defined by*

$$g(r,x)=\begin{cases}f\left(\sqrt{r^2-x^2},x\right) & \text{if } |r|\geqslant|x|,\\ 0 & \text{otherwise,}\end{cases} \tag{4.8}$$

belongs to $\mathcal{S}_*(\mathbb{R}^2)$.

Proof. (i) Let $f\in\mathcal{N}$, by Leibnitz formula, we have

$$\begin{aligned}&\left(\frac{\partial}{\partial r}\right)^{k_1}\left(\frac{\partial}{\partial x}\right)^{k_2}\left[(r^2+x^2)^\gamma f\right](r,x)\\&\quad=\sum_{j=0}^{k_1}\sum_{i=0}^{k_2}C_{k_1}^jC_{k_2}^iP_j(r)P_i(x)(r^2+x^2)^{\gamma-i-j}\frac{\partial^{k_1+k_2-i-j}}{\partial r^{k_1-j}\partial x^{k_2-i}}f(r,x),\end{aligned} \tag{4.9}$$

where P_i and P_j are real polynomials.

Let $n \in \mathbb{N}$ such that $\gamma - k_1 - k_2 + n > 0$. By Taylor formula and the fact that $f \in \mathcal{N}$, we have

$$\begin{aligned}\left(\frac{\partial}{\partial r}\right)^{k_1-j}(f)(r,x) &= \frac{r^{2n}}{(2n-1)!}\int_0^1 (1-t)^{2n-1}\left(\frac{\partial}{\partial r}\right)^{k_1-j+2n}(f)(rt,x)dt \\ &= -\frac{r^{2n}}{(2n-1)!}\int_1^{+\infty}(1-t)^{2n-1}\left(\frac{\partial}{\partial r}\right)^{k_1-j+2n}(f)(rt,x)dt,\end{aligned} \tag{4.10}$$

$$\begin{aligned}\left(\frac{\partial}{\partial x}\right)^{k_2-i}\left(\frac{\partial}{\partial r}\right)^{k_1-j}f(r,x) &= \frac{r^{2n}}{(2n-1)!}\int_0^1 (1-t)^{2n-1}\left(\frac{\partial}{\partial x}\right)^{k_2-i}\left(\frac{\partial}{\partial r}\right)^{k_1-j+2n}f(rt,x)dt \\ &= -\frac{r^{2n}}{(2n-1)!}\int_1^{+\infty}(1-t)^{2n-1}\left(\frac{\partial}{\partial x}\right)^{k_2-i}\left(\frac{\partial}{\partial r}\right)^{k_1-j+2n}f(rt,x)dt.\end{aligned} \tag{4.11}$$

The relations (4.9) and (4.11) imply that the function

$$(r,x) \longmapsto (r^2+x^2)^{\gamma} f(r,x) \tag{4.12}$$

belongs to $\mathcal{N}$ and that the mapping

$$f \longmapsto (r^2+x^2)^{\gamma} f \tag{4.13}$$

is continuous from $\mathcal{N}$ onto itself. The inverse mapping is given by

$$f \longmapsto (r^2+x^2)^{-\gamma} f. \tag{4.14}$$

By the same way, we show that the mapping

$$f \longmapsto |r|^{\gamma} f \tag{4.15}$$

is an isomorphism from $\mathcal{N}$ onto itself.

(ii) Let $f \in \mathcal{N}$, and

$$g(r,x) = \begin{cases} f(\sqrt{r^2-x^2},x) & \text{if } |r| \geqslant |x|, \\ 0 & \text{if } |r| \leqslant |x|, \end{cases} \tag{4.16}$$

we have

$$\left(\frac{\partial}{\partial x}\right)^{k_2}\left(\frac{\partial}{\partial r}\right)^{k_1}(g)(r,x) = \sum_{j=0}^{k_1} P_j(r)\left(\sum_{p,q=0}^{k_2} Q_{p,q}(x)\left(\frac{\partial}{\partial x}\right)^{p}\left(\frac{\partial}{\partial r^2}\right)^{q+j}(f)(\sqrt{r^2-x^2},x)\right), \tag{4.17}$$

where P_j and $Q_{p,q}$ are real polynomials. This equality, together with the fact that f belongs to $\mathcal{N}$, implies that g belongs to $\mathcal{S}_*(\mathbb{R}^2)$. □

Theorem 4.3. *The Fourier transform $\mathfrak{F}_\alpha$ associated with Riemann-Liouville transform is an isomorphism from $\mathcal{S}^0_*(\mathbb{R}^2)$ onto $\mathcal{N}$.*

Proof. Let $f \in \mathcal{S}^0_*(\mathbb{R}^2)$. From the relation (3.12), we get

$$\begin{aligned}\left(\frac{\partial}{\partial\mu^2}\right)^k \mathfrak{F}_\alpha(f)(0,\lambda) &= \left(\frac{\partial}{\partial\mu^2}\right)^k (B\circ\widetilde{\mathfrak{F}}_\alpha(f))(0,\lambda)\\ &= B\left(\left(\frac{\partial}{\partial\mu^2}\right)^k \widetilde{\mathfrak{F}}_\alpha(f)\right)(0,\lambda)\\ &= \left(\frac{\partial}{\partial\mu^2}\right)^k \widetilde{\mathfrak{F}}_\alpha(f)(\lambda,\lambda) = 0,\end{aligned} \tag{4.18}$$

because $\operatorname{supp}\widetilde{\mathfrak{F}}_\alpha(f) \subset \{\ (\mu,\lambda) \in \mathbb{R}^2,\ |\mu| \geqslant |\lambda|\}$, this shows that $\mathfrak{F}_\alpha$ maps injectively $\mathcal{S}^0_*(\mathbb{R}^2)$ into $\mathcal{N}$. On the other hand, let $h \in \mathcal{N}$ and

$$g(r,x) = \begin{cases} h(\sqrt{r^2-x^2},x) & \text{if } |r| \geqslant |x|,\\ 0 & \text{if } |r| \leqslant |x|.\end{cases} \tag{4.19}$$

From Theorem 4.2(ii), g belongs to $\mathcal{S}_*(\mathbb{R}^2)$, so there exists $f \in \mathcal{S}_*(\mathbb{R}^2)$ satisfying $\widetilde{\mathfrak{F}}_\alpha(f) = g$. Consequently, $f \in \mathcal{S}^0_*(\mathbb{R}^2)$ and $\mathfrak{F}_\alpha(f) = h$. □

From Lemmas 3.6, 4.1, and Theorem 4.3, we deduce the following result.

Corollary 4.4. *The dual transform ${}^t\mathfrak{R}_\alpha$ is an isomorphism from $\mathcal{S}^0_*(\mathbb{R}^2)$ onto $\mathcal{S}_{*,0}(\mathbb{R}^2)$.*

4.1. Inversion formula for $\mathfrak{R}_\alpha$ and ${}^t\mathfrak{R}_\alpha$

Theorem 4.5. (i) *The operator K^1_α defined by*

$$K^1_\alpha(f)(r,x) = \Lambda_\alpha^{-1}\left(\frac{\pi}{2^{2\alpha+1}\Gamma^2(\alpha+1)}(\mu^2+\lambda^2)^\alpha|\mu|\Lambda_\alpha(f)\right)(r,x) \tag{4.20}$$

is an isomorphism from $\mathcal{S}_{*,0}(\mathbb{R}^2)$ onto itself.

(ii) *The operator K^2_α defined by*

$$K^2_\alpha(g)(r,x) = \mathfrak{F}_\alpha^{-1}\left(\frac{\pi}{2^{2\alpha+1}\Gamma^2(\alpha+1)}(\mu^2+\lambda^2)^\alpha|\mu|\mathfrak{F}_\alpha(g)\right)(r,x) \tag{4.21}$$

is an isomorphism from $\mathcal{S}^0_*(\mathbb{R}^2)$ onto itself.

This theorem follows from Lemma 4.1, Theorems 4.2 and 4.3.

Theorem 4.6. (i) *For $f \in \mathcal{S}_{*,0}(\mathbb{R}^2)$ and $g \in \mathcal{S}^0_*(\mathbb{R}^2)$, there exists the inversion formula for $\mathfrak{R}_\alpha$:*

$$g = \mathfrak{R}_\alpha K^1_\alpha\,{}^t\mathfrak{R}_\alpha(g), \qquad f = K^1_\alpha\,{}^t\mathfrak{R}_\alpha\mathfrak{R}_\alpha(f). \tag{4.22}$$

(ii) *For $f \in \mathcal{S}_{*,0}(\mathbb{R}^2)$ and $g \in \mathcal{S}^0_*(\mathbb{R}^2)$, there exists the inversion formula for ${}^t\mathfrak{R}_\alpha$:*

$$f = {}^t\mathfrak{R}_\alpha K^2_\alpha\mathfrak{R}_\alpha(f), \qquad g = K^2_\alpha\mathfrak{R}_\alpha\,{}^t\mathfrak{R}_\alpha(g). \tag{4.23}$$

Proof. (i) Let $g \in \mathcal{S}^0_*(\mathbb{R}^2)$. From the relation (2.15), Proposition 3.4, Lemma 3.6, and Theorem 4.3, we have

$$\begin{aligned} g(r,x) &= \int_{\mathbb{R}}\int_0^{+\infty} (\mu^2+\lambda^2)^\alpha \mu \Lambda_\alpha \circ {}^t\mathfrak{R}_\alpha(g)(\mu,\lambda)\mathfrak{R}_\alpha(\cos(\mu.)\exp(i\lambda.))(r,x)dm(\mu,\lambda) \\ &= \mathfrak{R}_\alpha\Big(\int_{\mathbb{R}}\int_0^{+\infty} (\mu^2+\lambda^2)^\alpha \mu \Lambda_\alpha \circ {}^t\mathfrak{R}_\alpha(g)(\mu,\lambda)\cos(\mu.)\exp(i\lambda.)dm(\mu,\lambda)\Big)(r,x) \\ &= \mathfrak{R}_\alpha\Big(\Lambda_\alpha^{-1}\Big(\frac{\pi}{2^{2\alpha+1}\Gamma^2(\alpha+1)}(\mu^2+\lambda^2)^\alpha|\mu|\Lambda_\alpha \circ {}^t\mathfrak{R}_\alpha(g)\Big)\Big)(r,x) \\ &= \mathfrak{R}_\alpha K^1_\alpha\, {}^t\mathfrak{R}_\alpha(g)(r,x). \end{aligned} \tag{4.24}$$

This relation, together with Corollary 4.4 and Theorem 4.5(i), implies that $\mathfrak{R}_\alpha$ is an isomorphism from $\mathcal{S}_{*,0}(\mathbb{R}^2)$ onto $\mathcal{S}^0_*(\mathbb{R}^2)$, and that $K^1_\alpha {}^t\mathfrak{R}_\alpha$ is its inverse; in particular for $f \in \mathcal{S}_{*,0}(\mathbb{R}^2)$, we have

$$K^1_\alpha {}^t\mathfrak{R}_\alpha\mathfrak{R}_\alpha(f) = f. \tag{4.25}$$

(ii) Let $f \in \mathcal{S}_{*,0}(\mathbb{R}^2)$. From (i), we have

$$K^1_\alpha {}^t\mathfrak{R}_\alpha\mathfrak{R}_\alpha(f) = f. \tag{4.26}$$

Let us put $g = \mathfrak{R}_\alpha(f)$, then $g \in \mathcal{S}^0_*(\mathbb{R}^2)$, and we have

$$\mathfrak{R}_\alpha^{-1}(g) = K^1_\alpha {}^t\mathfrak{R}_\alpha(g), \tag{4.27}$$

and from Lemma 3.6, it follows that

$$\begin{aligned} \mathfrak{R}_\alpha^{-1}(g) &= \Lambda_\alpha^{-1}\Big(\frac{\pi}{2^{2\alpha+1}\Gamma^2(\alpha+1)}(\mu^2+\lambda^2)^\alpha|\mu|\mathfrak{F}_\alpha(g)\Big), \\ {}^t\mathfrak{R}_\alpha^{-1}\mathfrak{R}_\alpha^{-1}(g) &= \mathfrak{F}_\alpha^{-1}\Big(\frac{\pi}{2^{2\alpha+1}\Gamma^2(\alpha+1)}(\mu^2+\lambda^2)^\alpha|\mu|\mathfrak{F}_\alpha(g)\Big) = K^2_\alpha(g), \end{aligned} \tag{4.28}$$

which gives

$$f = {}^t\mathfrak{R}_\alpha K^2_\alpha \mathfrak{R}_\alpha(f). \tag{4.29}$$

□

4.2. The expressions of the operators K^1_α and K^2_α. In the previous subsection, we have defined the operators K^1_α and K^2_α in terms of Fourier transforms Λ_α and $\mathfrak{F}_\alpha$. Here, we will give nice expressions of these operators using fractional powers of partial differential operators. For this, we need the following inevitable notations.

(i) $\mathscr{E}_*(\mathbb{R})$ is the space of even infinitely differentiable functions on $\mathbb{R}$.

(ii) $\mathcal{S}_*(\mathbb{R})$ is the subspace of $\mathscr{E}_*(\mathbb{R})$, consisting of functions rapidly decreasing together with all their derivatives.

(iii) $\mathcal{S}'_*(\mathbb{R})$ is the space of even tempered distributions on $\mathbb{R}$.

(iv) $\mathcal{S}'_*(\mathbb{R}^2)$ is the space of tempered distributions on $\mathbb{R}^2$, even with respect to the first variable.

Each of these spaces is equipped with its usual topology.

(i) For $a \in \mathbb{R}$, $a \geqslant -1/2$, $d\omega_a(r)$ is the measure defined on $[0,+\infty[$ by

$$d\omega_a(r) = \frac{1}{2^a\Gamma(a+1)} r^{2a+1} dr. \tag{4.30}$$

(ii) ℓ_a is the Bessel operator defined on $]0,+\infty[$ by

$$\ell_a = \frac{d^2}{dr^2} + \frac{2a+1}{r}\frac{d}{dr}, \quad a \geqslant -\frac{1}{2}. \tag{4.31}$$

(iii) For an even measurable function f on $\mathbb{R}$, $T_f^{\omega_a}$ is the element of $\mathcal{S}'_*(\mathbb{R})$, defined by

$$\langle T_f^{\omega_a}, \varphi\rangle = \int_0^{+\infty} f(r)\varphi(r)d\omega_a(r), \quad \varphi \in \mathcal{S}_*(\mathbb{R}). \tag{4.32}$$

(iv) For a measurable function g on $\mathbb{R}^2$, even with respect to the first variable, T_g^{ν} (resp., T_g^m) is the element of $\mathcal{S}'_*(\mathbb{R}^2)$, defined by

$$\begin{gathered}\langle T_g^{\nu}, \varphi\rangle = \int_{\mathbb{R}}\int_0^{+\infty} g(r,x)\varphi(r,x)d\nu(r,x), \\ \left(\text{resp.}, \langle T_g^{m}, \varphi\rangle = \int_{\mathbb{R}}\int_0^{+\infty} g(r,x)\varphi(r,x)dm(r,x)\right), \quad \varphi \in \mathcal{S}_*(\mathbb{R}^2),\end{gathered} \tag{4.33}$$

where $d\nu$ and dm are the measures defined by the relations (3.1) and (3.26).

Definition 4.7. (i) The translation operator τ_r^a, $r \in \mathbb{R}$, associated with Bessel operator ℓ_a is defined on $\mathcal{S}_*(\mathbb{R})$ by the following. For all $s \in \mathbb{R}$,

$$\tau_r^a f(s) = \begin{cases} \dfrac{\Gamma(a+1)}{\sqrt{\pi}\Gamma(a+1/2)} \displaystyle\int_0^{\pi} f\left(\sqrt{r^2+s^2+2rs\cos\theta}\right)\sin^{2a}\theta\, d\theta & \text{if } a > -\dfrac{1}{2}, \\ \dfrac{f(r+s)+f(|r-s|)}{2} & \text{if } a = -\dfrac{1}{2}. \end{cases} \tag{4.34}$$

(ii) The convolution product of $f \in \mathcal{S}_*(\mathbb{R})$ and $T \in \mathcal{S}'_*(\mathbb{R})$ is defined by

$$\forall r \in \mathbb{R}, \quad T *_a f(r) = \langle T, \tau_r^a f\rangle. \tag{4.35}$$

(iii) The Fourier Bessel transform is defined on $\mathcal{S}_*(\mathbb{R})$ by

$$\forall \mu \in \mathbb{R}, \quad F_a(f)(\mu) = \int_0^{+\infty} f(r) j_a(r\mu) d\omega_a(r), \tag{4.36}$$

and on $\mathcal{S}'_*(\mathbb{R})$ by

$$\forall \varphi \in \mathcal{S}_*(\mathbb{R}), \quad \langle F_a(T), \varphi\rangle = \langle T, F_a(\varphi)\rangle. \tag{4.37}$$

We have the following properties (we refer to [19]).

(i) F_a is an isomorphism from $\mathscr{S}_*(\mathbb{R})$ (resp., $\mathscr{S}'_*(\mathbb{R})$) onto itself, and we have

$$F_a^{-1} = F_a. \tag{4.38}$$

(ii) For $f \in \mathscr{S}_*(\mathbb{R})$, and $r \in \mathbb{R}$, $\tau_r^a f$ belongs to $\mathscr{S}_*(\mathbb{R})$, and we have

$$F_a(\tau_r^a f)(\mu) = j_a(r\mu)F_a(f)(\mu). \tag{4.39}$$

(iii) For $f \in \mathscr{S}_*(\mathbb{R})$ and $T \in \mathscr{S}'_*(\mathbb{R})$, the function $T *_a f$ belongs to $\mathscr{E}_*(\mathbb{R})$, and is slowly increasing, moreover

$$F_a\left(T^{\omega_a}_{T *_a f}\right) = F_a(f)F_a(T). \tag{4.40}$$

In the following, we will define the fractional powers of Bessel operator and the Laplacian operator defined on $\mathbb{R}^2$ by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial x^2} \tag{4.41}$$

that we use to give simple expressions of K_α^1 and K_α^2.

In [16], the author has proved that the mappings

$$z \longmapsto T^{\omega_a}_{|r|^z}, \qquad z \longmapsto T^{\omega_a}_{(2^{z+a+1}\Gamma(z/2+a+1)/\Gamma(-z/2))|r|^{-z-2a-2}}, \tag{4.42}$$

defined initially for $-2(a+1) < \Re e(z) < 0$, can be extended to a valued functions on $\mathscr{S}'_*(\mathbb{R})$, analytic on $\mathbb{C}\backslash\{-2(k+a), k \in \mathbb{N}^*\}$, and we have

$$T^{\omega_a}_{|r|^z} = F_a\left(T^{\omega_a}_{(2^{z+a+1}\Gamma(z/2+a+1)/\Gamma(-z/2))|r|^{-z-2a-2}}\right). \tag{4.43}$$

Definition 4.8. For $z \in \mathbb{C}\backslash\{-(k+a),\ k \in \mathbb{N}^*\}$, the fractional power of Bessel operator ℓ_a is defined on $\mathscr{S}_*(\mathbb{R})$ by

$$(-\ell_a)^z f(r) = \left(T^{\omega_a}_{(2^{2z+a+1}\Gamma(z+a+1)/\Gamma(-z))|s|^{-2z-2a-2}}\right) *_a f(r). \tag{4.44}$$

From the relations (4.40) and (4.43), we deduce that for $f \in \mathscr{S}_*(\mathbb{R})$ and $z \in \mathbb{C}\backslash\{-(k+a),\ k \in \mathbb{N}^*\}$, we have

$$F_a\left(T^{\omega_a}_{(-\ell_a)^z f}\right) = F_a(f)T^{\omega_a}_{|r|^{2z}}. \tag{4.45}$$

On the other hand, from [8, 10], we deduce that the mappings

$$z \longmapsto T^m_{(r^2+x^2)^z}, \qquad T^m_{\sqrt{2/\pi}(2^{2z+\alpha+1}\Gamma(z+1)\Gamma(\alpha+1)/\Gamma(-z))(r^2+x^2)^{-z-1}}, \tag{4.46}$$

defined initially for $-1 < \Re e(z) < 0$, can be extended to a valued functions in $\mathscr{S}'_*(\mathbb{R}^2)$, analytic on $\mathbb{C}\backslash\{-k,\ k \in \mathbb{N}^*\}$, and we have

$$T^m_{(r^2+x^2)^z} = \Lambda_\alpha\left(T^m_{\sqrt{2/\pi}(2^{2z+\alpha+1}\Gamma(z+1)\Gamma(\alpha+1)/\Gamma(-z))(r^2+x^2)^{-z-1}}\right), \tag{4.47}$$

where Λ_α is defined on $\mathcal{S}'_*(\mathbb{R}^2)$ by

$$\langle \Lambda_\alpha(T), \varphi \rangle = \langle T, \Lambda_\alpha(\varphi) \rangle, \quad \varphi \in \mathcal{S}_*(\mathbb{R}^2), \tag{4.48}$$

and $\Lambda_\alpha(\varphi)$ is given in Lemma 3.6.

Definition 4.9. For $z \in \mathbb{C}\backslash\{-k,\ k \in \mathbb{N}^*\}$, the fractional power of the Laplacian operator Δ is defined on $\mathcal{S}_*(\mathbb{R}^2)$ by

$$(-\Delta)^z f(r,x) = \left(T^m_{(1/\pi)(2^{2z+1}\Gamma(z+1)/\Gamma(-z))(s^2+y^2)^{-z-1} * f}\right)(r,x), \tag{4.49}$$

where

(i) $*$ is the usual convolution product defined by

$$T * f(r,x) = \langle T, \sigma_{(r,x)} \check{f} \rangle, \quad T \in \mathcal{S}'_*(\mathbb{R}^2),\ f \in \mathcal{S}_*(\mathbb{R}^2); \tag{4.50}$$

(ii)

$$\sigma_{(r,x)} f(s,y) = \frac{1}{2}\left[f(r+s, y-x) + f(r-s, y-x)\right], \quad f \in \mathcal{S}_*(\mathbb{R}^2). \tag{4.51}$$

It is well known that for $f \in \mathcal{S}_*(\mathbb{R}^2)$ and $T \in \mathcal{S}'_*(\mathbb{R}^2)$, the function $T * f$ belongs to $\mathcal{E}_*(\mathbb{R}^2)$ and is slowly increasing, and we have

$$\Lambda_\alpha(T^m_{T*f}) = \Lambda_\alpha(f)\Lambda_\alpha(T), \tag{4.52}$$

thus from the relations (4.47) and (4.52), we deduce that for $f \in \mathcal{S}_*(\mathbb{R}^2)$ and $z \in \mathbb{C}\backslash \{-k,\ k \in \mathbb{N}^*\}$,

$$\Lambda_\alpha\left(T^m_{\sqrt{2\pi}2^\alpha\Gamma(\alpha+1)(-\Delta)^z f}\right) = \Lambda_\alpha(f) T^m_{(r^2+x^2)^z}. \tag{4.53}$$

Theorem 4.10. *The operator K^1_α defined in Theorem 4.5 can be written as*

$$K^1_\alpha(f) = \frac{\pi}{2^{2\alpha+1}\Gamma^2(\alpha+1)}\left(-\frac{\partial^2}{\partial r^2}\right)^{1/2}(-\Delta)^\alpha f, \tag{4.54}$$

where

$$\left(-\frac{\partial^2}{\partial r^2}\right)^{1/2} f(r,x) = \left(-\ell_{-1/2}\right)^{1/2}\left(f(\cdot,x)\right)(r). \tag{4.55}$$

Proof. Let $f \in \mathcal{S}_{*,0}(\mathbb{R}^2)$. Using Fubini's theorem, we get for every $\varphi \in \mathcal{S}_*(\mathbb{R}^2)$ the following:

$$\begin{aligned}&\left\langle \Lambda_\alpha\left(T^m_{(-\partial^2/\partial r^2)^{1/2} f}\right), \varphi \right\rangle \\ &\quad = \frac{1}{2^{2\alpha+2}\Gamma^2(\alpha+1)} \int_{\mathbb{R}} \int_0^{+\infty} \left\langle T^{\omega_{-1/2}}_{(-\ell_{-1/2})^{1/2}(f(\cdot,x))}, F_{-1/2}(\varphi(\cdot,y)) \right\rangle \times \exp(-ixy)\,dx\,dy\end{aligned} \tag{4.56}$$

and by the relation (4.45), we obtain

$$\begin{aligned}&\left\langle \Lambda_\alpha\left(T^m_{(-\partial^2/\partial r^2)^{1/2}f}\right),\varphi\right\rangle\\&\quad=\frac{1}{2^{2\alpha+2}\Gamma^2(\alpha+1)}\int_{\mathbb{R}}\int_0^{+\infty}\left\langle F_{-1/2}(f(\cdot,x))T^{\omega_{-1/2}}_{|r|},\varphi(\cdot,y)\right\rangle\times\exp(-ixy)dx\,dy,\end{aligned}\tag{4.57}$$

which involves that

$$\left\langle \Lambda_\alpha\left(T^m_{(-\partial^2/\partial r^2)^{1/2}f}\right),\varphi\right\rangle=\int_{\mathbb{R}}\int_0^{+\infty}r\Lambda_\alpha(f)(r,y)\varphi(r,y)dm(r,y),\tag{4.58}$$

this shows that

$$\Lambda_\alpha\left(T^m_{(-\partial^2/\partial r^2)^{1/2}f}\right)=T^m_{|r|\Lambda_\alpha(f)}.\tag{4.59}$$

Now, from Lemma 4.1, we deduce that the function

$$(\mu,\lambda)\longmapsto|\mu|\Lambda_\alpha(f)(\mu,\lambda)\tag{4.60}$$

belongs to the subspace $\mathcal{N}$. Then, from the relation (4.59), it follows that the function $(-\partial^2/\partial r^2)^{1/2}f$ belongs to the subspace $\mathcal{S}_{*,0}(\mathbb{R}^2)$, and we have

$$\forall(\mu,\lambda)\in\mathbb{R}^2,\quad\Lambda_\alpha\left(\left(-\frac{\partial^2}{\partial r^2}\right)^{1/2}f\right)(\mu,\lambda)=|\mu|\Lambda_\alpha(\mu,\lambda).\tag{4.61}$$

By the same way, and using the relation (4.53), we deduce that for every $f\in\mathcal{S}_{*,0}(\mathbb{R}^2)$, the function $(-\Delta)^\alpha f$ belongs to the subspace $\mathcal{S}_{*,0}(\mathbb{R}^2)$, and we have that for all $(\mu,\lambda)\in\mathbb{R}^2$,

$$\Lambda_\alpha(\sqrt{2\pi}2^\alpha\Gamma(\alpha+1)(-\Delta)^\alpha f)(\mu,\lambda)=(\mu^2+\lambda^2)^\alpha\Lambda_\alpha(f)(\mu,\lambda).\tag{4.62}$$

Hence, the theorem follows from the relations (4.61) and (4.62). □

Definition 4.11. Let $a,b\in\mathbb{R}$, $b\geqslant a\geqslant-1/2$.

(i) The Sonine transform is the mapping defined on $\mathcal{E}_*(\mathbb{R})$ by the following. For all $r\in\mathbb{R}$,

$$S_{b,a}(f)(r)=\begin{cases}\dfrac{2\Gamma(b+1)}{\Gamma(b-a)\Gamma(a+1)}\displaystyle\int_0^1\left(1-t^2\right)^{b-a-1}f(rt)t^{2a+1}dt & \text{if } b>a,\\ f(r) & \text{if } b=a.\end{cases}\tag{4.63}$$

(ii) The dual transform ${}^tS_{b,a}$ is the mapping defined on $\mathcal{S}_*(\mathbb{R})$ by the following. For all $r\in\mathbb{R}$,

$${}^tS_{b,a}(f)(r)=\begin{cases}\dfrac{2\Gamma(b+1)}{\Gamma(b-a)\Gamma(a+1)}\displaystyle\int_r^{+\infty}\left(t^2-r^2\right)^{b-a-1}f(t)t\,dt & \text{if } b>a,\\ f(r) & \text{if } b=a.\end{cases}\tag{4.64}$$

Then, we have the following.

(i) The Sonine transform is an isomorphism from $\mathscr{E}_*(\mathbb{R})$ onto itself.

(ii) The dual Sonine transform is an isomorphism from $\mathscr{S}_*(\mathbb{R})$ onto itself.

(iii) For $f \in \mathscr{E}_*(\mathbb{R})$, f bounded, and $g \in \mathscr{S}_*(\mathbb{R})$, we have

$$\int_0^{+\infty} S_{b,a}(f)(r)g(r)r^{2b+1}dr = \int_o^{+\infty} f(r)^t S_{b,a}(g)(r)r^{2a+1}dr. \tag{4.65}$$

(iv) $j_b = S_{b,a}(j_a)$.

(v)

$$F_b = \frac{\Gamma(a+1)}{2^{b-a}\Gamma(b+1)} F_a \circ {}^t S_{b,a}. \tag{4.66}$$

For more details, we refer to [18, 20, 21].

We denote the following.

(i) For $T \in \mathscr{S}'_*(\mathbb{R}^2)$, $\varphi \in \mathscr{S}_*(\mathbb{R}^2)$,

$$\langle S_{a,0}(T), \varphi \rangle = \langle T, \psi \rangle, \tag{4.67}$$

with $\psi(r,x) = {}^t S_{a,0}(\varphi(\cdot,x))(r)$.

(ii) For all $(r,x) \in \mathbb{R}^2$,

$$T \# \varphi(r,x) = \langle T, \mathscr{T}_{(r,-x)} \check{\varphi} \rangle, \tag{4.68}$$

where $\mathscr{T}_{(r,x)}$ is the translation operator given by Definition 3.1.

(iii) $\tilde{\mathfrak{F}}_\alpha$ is the mapping defined on $\mathscr{S}'_*(\mathbb{R}^2)$ by

$$\forall \varphi \in \mathscr{S}_*(\mathbb{R}^2), \quad \langle \tilde{\mathfrak{F}}_\alpha(T), \varphi \rangle = \langle T, \tilde{\mathfrak{F}}_\alpha(\varphi) \rangle. \tag{4.69}$$

(iv) L_α is the operator defined on $\mathscr{S}_*(\mathbb{R}^2)$ by

$$L_\alpha f(r,x) = (-\ell_\alpha)^{2\alpha}(f(\cdot,x))(r), \tag{4.70}$$

where $(-\ell_\alpha)^z$ is the fractional power of Bessel given by Definition 4.8.

Theorem 4.12. *The operator K_α^2, defined in Theorem 4.5, is given by*

$$K_\alpha^2(f)(r,x) = \frac{\pi}{2^{4\alpha+2}\Gamma^4(\alpha+1)} S_{\alpha,0}(T) \# (-\Delta_2) L_\alpha(\check{f})(r,-x), \quad f \in \mathscr{S}_*^0(\mathbb{R}^2), \tag{4.71}$$

where

(i) *T is the distribution defined by*

$$\langle T, \varphi \rangle = \int_{\mathbb{R}} \varphi(y,y)dy; \tag{4.72}$$

(ii) *Δ_2 is the operator defined in Section 2.*

Proof. By the definition of K_α^2, and the relation (3.12), we have that for $f \in \mathcal{S}_*^0(\mathbb{R}^2)$,

$$K_\alpha^2(f)(r,x) = \frac{\sqrt{\pi/2}}{2^{3\alpha+1}\Gamma^3(\alpha+1)} \int_{\mathbb{R}} \int_0^{+\infty} \mu^2(\mu^2+\lambda^2)^{2\alpha} \tilde{\mathfrak{F}}_\alpha(f)(\sqrt{\mu^2+\lambda^2},\lambda)\, j_\alpha(r\sqrt{\mu^2+\lambda^2}) \exp(i\lambda x)d\mu\, d\lambda. \tag{4.73}$$

By a change of variables, and using Fubini's theorem, we get

$$K_\alpha^2(f)(r,x) = \frac{\sqrt{\pi/2}}{2^{3\alpha+1}\Gamma^3(\alpha+1)} \int_0^{+\infty} \int_{-\nu}^{\nu} \nu^{4\alpha}(\nu^2-\lambda^2)\tilde{\mathfrak{F}}_\alpha(f)(\nu,\lambda)\frac{\exp(i\lambda x)}{\sqrt{\nu^2-\lambda^2}} j_\alpha(r\nu)\nu\, d\nu\, d\lambda. \tag{4.74}$$

On the other hand, for $f \in \mathcal{S}_*^0(\mathbb{R}^2)$, the function $L_\alpha f$ belongs to $\mathcal{E}_*(\mathbb{R}^2)$, and is slowly increasing. Moreover, we have

$$\tilde{\mathfrak{F}}_\alpha\left(T_{L_\alpha f}^\nu\right) = T^\nu_{|\mu|^{4\alpha}\tilde{\mathfrak{F}}_\alpha(f)}. \tag{4.75}$$

But, for $f \in \mathcal{S}_*^0(\mathbb{R}^2)$, the function $\tilde{\mathfrak{F}}_\alpha(f)$ belongs to the subspace $\mathcal{N}$; according to Theorem 4.2, we deduce that the function $L_\alpha f$ belongs to $\mathcal{S}_*(\mathbb{R}^2)$, and we have

$$\forall(\mu,\lambda) \in \mathbb{R}^2, \quad \tilde{\mathfrak{F}}_\alpha(L_\alpha f)(\mu,\lambda) = |\mu|^{4\alpha}\tilde{\mathfrak{F}}_\alpha(f)(\mu,\lambda). \tag{4.76}$$

This involves that

$$\begin{aligned} K_\alpha^2(f)(r,x) &= \frac{\sqrt{\pi/2}}{2^{3\alpha+1}\Gamma^3(\alpha+1)} \int_0^{+\infty} \int_{-\nu}^{\nu} (\nu^2-\lambda^2)\tilde{\mathfrak{F}}_\alpha(L_\alpha f)(\nu,\lambda)\frac{\exp(i\lambda x)}{\sqrt{\nu^2-\lambda^2}} j_\alpha(r\nu)\nu\, d\nu\, d\lambda \\ &= \frac{\sqrt{\pi/2}}{2^{3\alpha+1}\Gamma^3(\alpha+1)} \int_0^{+\infty} \int_{-\nu}^{\nu} \tilde{\mathfrak{F}}_\alpha((-\Delta_2)L_\alpha f)(\nu,\lambda)\frac{\exp(i\lambda x)}{\sqrt{\nu^2-\lambda^2}} j_\alpha(r\nu)\nu\, d\nu\, d\lambda. \end{aligned} \tag{4.77}$$

Since for every $f \in \mathcal{S}_*(\mathbb{R}^2)$, we have that

$$\forall(r,x),(\mu,\lambda) \in \mathbb{R}^2, \quad \tilde{\mathfrak{F}}_\alpha(\mathcal{T}_{(r,x)}f)(\nu,\lambda) = j_\alpha(r\nu)\exp(i\lambda x)\tilde{\mathfrak{F}}_\alpha(f)(\nu,\lambda), \tag{4.78}$$

we get

$$K_\alpha^2(f)(r,x) = \frac{\sqrt{\pi/2}}{2^{3\alpha+1}\Gamma^3(\alpha+1)} \int_0^{+\infty} \int_{-\nu}^{\nu} \tilde{\mathfrak{F}}_\alpha(\mathcal{T}_{(r,x)}(-\Delta_2)L_\alpha f)(\nu,\lambda)\frac{\nu\, d\nu\, d\lambda}{\sqrt{\nu^2-\lambda^2}}. \tag{4.79}$$

Using the expression of $\tilde{\mathfrak{F}}_\alpha$, we obtain

$$\begin{aligned} K_\alpha^2(f)(r,x) = \frac{1}{2^{4\alpha+2}\Gamma^4(\alpha+1)} \int_0^{+\infty} \int_{-\nu}^{\nu} \bigg[\int_{\mathbb{R}} \int_0^{+\infty} & (\mathcal{T}_{(r,x)}(-\Delta_2)L_\alpha f)(s,y) \\ & \times j_\alpha(s\nu)\exp(-i\lambda y)s^{2\alpha+1}ds\, dy \bigg] \frac{d\lambda}{\sqrt{\nu^2-\lambda^2}}\nu\, d\nu. \end{aligned} \tag{4.80}$$

From the fact that

$$\int_{-\nu}^{\nu} \frac{\exp(-i\lambda y)}{\sqrt{\nu^2-\lambda^2}}\,d\lambda = \pi j_0(\nu y), \tag{4.81}$$

and using Fubini's theorem, we deduce that

$$\begin{aligned} K_\alpha^2(f)(r,x) &= \frac{\pi}{2^{4\alpha+2}\Gamma^4(\alpha+1)}\int_{\mathbb{R}}\left\{\iint_0^{+\infty}(\mathcal{T}_{(r,x)}(-\Delta_2)L_\alpha f)(s,y)j_\alpha(s\nu)\times j_0(\nu y)s^{2\alpha+1}ds\nu\,d\nu\right\}dy \\ &= \frac{\pi}{2^{3\alpha+2}\Gamma^3(\alpha+1)}\int_{\mathbb{R}}\left\{\int_0^{+\infty}F_\alpha((\mathcal{T}_{(r,x)}(-\Delta_2)L_\alpha f)(\cdot,y))(\nu)j_0(\nu y)\nu\,d\nu\right\}dy, \end{aligned} \tag{4.82}$$

and from the relation (4.66), we have

$$\begin{aligned} K_\alpha^2(f)(r,x) &= \frac{\pi}{2^{4\alpha+2}\Gamma^4(\alpha+1)}\int_{\mathbb{R}}\left\{\int_0^{+\infty}F_0\circ {}^tS_{\alpha,0}((\mathcal{T}_{(r,x)}(-\Delta_2)L_\alpha f)(\cdot,y))(\nu)\times j_0(\nu y)\nu\,d\nu\right\}dy, \end{aligned} \tag{4.83}$$

and the relation (4.38) implies that

$$K_\alpha^2(f)(r,x) = \frac{\pi}{2^{4\alpha+2}\Gamma^4(\alpha+1)}\int_{\mathbb{R}} {}^tS_{\alpha,0}((\mathcal{T}_{(r,x)}(-\Delta_2)L_\alpha f)(\cdot,y))(y)\,dy. \tag{4.84}$$

□

4.3. Plancherel theorem for ${}^t\mathfrak{R}_\alpha$

Proposition 4.13. *The operator K_α^3 defined by*

$$K_\alpha^3(f) = \pi\left(-\frac{\partial^2}{\partial r^2}\right)^{1/4}(-\triangle)^{\alpha/2}f \tag{4.85}$$

is an isomorphism from $\mathcal{S}_{*,0}(\mathbb{R}^2)$ onto itself, where

$$\left(-\frac{\partial^2}{\partial r^2}\right)^{1/4}f(r,x) = (-\ell_{-1/2})^{1/4}(f(\cdot,x))(r). \tag{4.86}$$

Proof. Let $f\in\mathcal{S}_{*,0}(\mathbb{R}^2)$. From the relations (4.45) and (4.53), we deduce that for all $(\mu,\lambda)\in\mathbb{R}^2$,

$$\sqrt{|\mu|}\,(\mu^2+\lambda^2)^{\alpha/2}\Lambda_\alpha(f)(\mu,\lambda) = \Lambda_\alpha\left(\sqrt{2\pi}2^\alpha\Gamma(\alpha+1)\left(-\frac{\partial^2}{\partial r^2}\right)^{1/4}(-\Delta)^{\alpha/2}f\right)(\mu,\lambda), \tag{4.87}$$

which implies that for all $(\mu,\lambda) \in \mathbb{R}^2$,

$$\Lambda_\alpha(K_\alpha^3)(f)(\mu,\lambda) = \sqrt{\frac{\pi}{2}}\frac{1}{2^\alpha\Gamma(\alpha+1)}\sqrt{|\mu|}\,(\mu^2+\lambda^2)^{\alpha/2}\Lambda_\alpha(f)(\mu,\lambda). \tag{4.88}$$

Then, the result follows from Lemma 4.1 and Theorem 4.2. □

Proposition 4.14. *For $g \in \mathcal{S}_*^0(\mathbb{R}^2)$, there exists the Plancherel formula*

$$\int_{\mathbb{R}}\int_0^{+\infty} |g(r,x)|^2 d\nu(r,x) = \int_{\mathbb{R}}\int_0^{+\infty} |K_\alpha^3({}^t\mathfrak{R}_\alpha(g))(r,x)|^2 dm(r,x). \tag{4.89}$$

Proof. Let $g \in \mathcal{S}_*^0(\mathbb{R}^2)$, from Theorem 3.5 (Plancherel formula), we have

$$\int_{\mathbb{R}}\int_0^{+\infty} |g(r,x)|^2 d\nu(r,x) = \iint_\Gamma |\mathfrak{F}_\alpha(g)(\mu,\lambda)|^2 d\gamma(\mu,\lambda). \tag{4.90}$$

From the relation (3.12), Lemma 3.6, and the fact that

$$\operatorname{supp}\widetilde{\mathfrak{F}}_\alpha(g) \subset \{(\mu,\lambda) \in \mathbb{R}^2/|\mu| \geqslant |\lambda|\}, \tag{4.91}$$

we get

$$\int_{\mathbb{R}}\int_0^{+\infty} |g(r,x)|^2 d\nu(r,x) = \int_{\mathbb{R}}\int_0^{+\infty} |\sqrt{\mu}(\mu^2+\lambda^2)^{\alpha/2}\Lambda_\alpha \circ {}^t\mathfrak{R}_\alpha(g)(\mu,\lambda)|^2 dm(\mu,\lambda). \tag{4.92}$$

We complete the proof by using the formula (4.88), and the fact that for every $f \in \mathcal{S}_*(\mathbb{R}^2)$,

$$\int_{\mathbb{R}}\int_0^{+\infty} |\Lambda_\alpha(f)(\mu,\lambda)|^2 dm(\mu,\lambda) = \frac{\pi}{2^{2\alpha+1}\Gamma^2(\alpha+1)}\int_{\mathbb{R}}\int_0^{+\infty} |f(\mu,\lambda)|^2 dm(\mu,\lambda). \tag{4.93}$$

□

We denote by
(i) $L_0^2(d\nu)$ the subspace of $L^2(d\nu)$ consisting of functions g such that

$$\operatorname{supp}\widetilde{\mathfrak{F}}_\alpha(g) \subset \{(\mu,\lambda) \in \mathbb{R}^2/|\mu| \geqslant |\lambda|\}; \tag{4.94}$$

(ii) $L^2(dm)$ the space of square integrable functions on $[0,+\infty[\times\mathbb{R}$ with respect to the measure $dm(r,x)$.

Theorem 4.15. *The operator $K_\alpha^3 \circ {}^t\mathfrak{R}_\alpha$ can be extended to an isometric isomorphism from $L_0^2(d\nu)$ onto $L^2(dm)$.*

Proof. The theorem follows from Propositions 4.13, 4.14, and the density of $\mathscr{S}_{*,0}(\mathbb{R}^2)$ (resp., $\mathscr{S}^0_*(\mathbb{R}^2)$) in $L^2(dm)$ (resp., $L^2_0(d\nu)$). □

5. Transmutation operators

Proposition 5.1. *The Riemann-Liouville transform and its dual satisfy the following permutation properties.*

(i) *For all* $f \in \mathscr{S}_*(\mathbb{R}^2)$,

$$ {}^t\mathfrak{R}_\alpha(\Delta_2 f) = \frac{\partial^2}{\partial r^2}\, {}^t\mathfrak{R}_\alpha(f), \qquad {}^t\mathfrak{R}_\alpha(\Delta_1 f) = \Delta_1\, {}^t\mathfrak{R}_\alpha(f). \tag{5.1}$$

(ii) *For all* $f \in \mathscr{E}_*(\mathbb{R}^2)$,

$$ \Delta_2 \mathfrak{R}_\alpha(f) = \mathfrak{R}_\alpha\left(\frac{\partial^2 f}{\partial r^2}\right), \qquad \Delta_1 \mathfrak{R}_\alpha(f) = \mathfrak{R}_\alpha(\Delta_1 f). \tag{5.2}$$

Proof. (i) We know that the operators Δ_1, Δ_2, $\partial^2/\partial r^2$, and ${}^t\mathfrak{R}_\alpha$ are continuous mappings from $\mathscr{S}_*(\mathbb{R}^2)$ into itself. Then, by applying the usual Fourier transform Λ_α, we have

$$ \begin{aligned} \Lambda_\alpha\left({}^t\mathfrak{R}_\alpha(\Delta_2 f)\right)(\mu,\lambda) &= -\mu^2 \Lambda_\alpha \circ {}^t\mathfrak{R}_\alpha(f)(\mu,\lambda) = \Lambda_\alpha\left(\frac{\partial^2}{\partial r^2}{}^t\mathfrak{R}_\alpha(f)\right)(\mu,\lambda), \\ \Lambda_\alpha\left(\Delta_1 {}^t\mathfrak{R}_\alpha f\right)(\mu,\lambda) &= i\lambda \Lambda_\alpha\left({}^t\mathfrak{R}_\alpha(f)\right)(\mu,\lambda) = \Lambda_\alpha\left({}^t\mathfrak{R}_\alpha(\Delta_1 f)\right)(\mu,\lambda). \end{aligned} \tag{5.3}$$

Consequently, (i) follows from the fact that Λ_α is an isomorphism from $\mathscr{S}_*(\mathbb{R}^2)$ onto itself.

(ii) We obtain the result from (i), Lemma 2.4, and the fact that for $f \in \mathscr{E}_*(\mathbb{R}^2)$, and $g \in \mathscr{D}_*(\mathbb{R}^2)$,

$$ \int_{\mathbb{R}}\int_0^{+\infty} \Delta_2 f(r,x) g(r,x)\, d\nu(r,x) = \int_{\mathbb{R}}\int_0^{+\infty} f(r,x)\Delta_2 g(r,x)\, d\nu(r,x). \tag{5.4}$$

□

Theorem 5.2. (i) *The Riemann-Liouville transform* $\mathfrak{R}_\alpha$ *is a transmutation operator of*

$$ \frac{\partial^2}{\partial r^2}, \Delta_1 \quad \text{into } \Delta_2,\ \Delta_1 \tag{5.5}$$

from

$$ \mathscr{S}_{*,0}(\mathbb{R}^2) \quad \text{onto } \mathscr{S}^0_*(\mathbb{R}^2). \tag{5.6}$$

(ii) *The dual transform* ${}^t\mathfrak{R}_\alpha$ *is a transmutation operator of*

$$\Delta_2,\ \Delta_1 \quad \text{into}\ \frac{\partial^2}{\partial r^2},\ \Delta_1 \tag{5.7}$$

from

$$\mathcal{S}^0_*(\mathbb{R}^2) \quad \text{onto}\ \mathcal{S}_{*,0}(\mathbb{R}^2). \tag{5.8}$$

This theorem follows from Proposition 5.1 and the fact that $\mathfrak{R}_\alpha$ is an isomorphism from $\mathcal{S}_{*,0}(\mathbb{R}^2)$ onto $\mathcal{S}^0_*(\mathbb{R}^2)$ and ${}^t\mathfrak{R}_\alpha$ is an isomorphism from $\mathcal{S}^0_*(\mathbb{R}^2)$ onto $\mathcal{S}_{*,0}(\mathbb{R}^2)$.

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