CONTINUITY OF THE MAPS $f \mapsto \bigcup_{x \in I} \omega(x,f)$ AND $f \mapsto \{\omega(x,f) : x \in I\}$

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We study the behavior of two maps in an effort to better understand the stability of ω-limit sets $\omega(x,f)$ as we perturb either x or f, or both. The first map is the set-valued function Λ taking f in $C(I,I)$ to its collection of ω-limit points $\Lambda(f) = \bigcup_{x \in I} \omega(x,f)$, and the second is the map Ω taking f in $C(I,I)$ to its collection of ω-limit sets $\Omega(f) = \{\omega(x,f) : x \in I\}$. We characterize those functions f in $C(I,I)$ at which each of our maps Λ and Ω is continuous, and then go on to show that both Λ and Ω are continuous on a residual subset of $C(I,I)$. We then investigate the relationship between the continuity of Λ and Ω at some function f in $C(I,I)$ with the chaotic nature of that function.

1. Introduction

We begin with a brief historical overview in an effort to place this paper's results in context. Let f be a continuous self-map of the unit interval $I = [0,1]$, and take as our starting point the iterates of this map. In particular, we begin by considering the trajectories generated by our function $f \in C(I,I)$ for particular initial conditions x in I, and let $\tau(x,f) = \{f^k(x)\}_{k=0}^{\infty} = \{x, f(x), f(f(x)), \ldots, f^n(x), \ldots\}$ be the trajectory of $x \in I$ generated by f. Central to the study of dynamical and chaotic systems is the desire to understand how trajectories $\tau(x,f) = \{f^k(x)\}_{k=0}^{\infty}$ are affected by slight changes in the initial condition x. In order to somewhat simplify this study, one oftentimes considers the ω-limit set generated by x rather than the trajectory. We take $\omega(x,f)$—the ω-limit set generated by x—to be the collection of subsequential limit points of the trajectory $\tau(x,f)$. Whenever f is a continuous self-map of a compact interval, the ω-limit set $\omega(x,f)$ enjoys several nice properties. In particular, $\omega(x,f)$ is always nonempty, closed, and strongly invariant under the generating function f so that $f(\omega(x,f)) = \omega(x,f)$. Nonetheless, as [1] shows, ω-limit sets and the trajectories that give rise to them can be wildly complicated (see Theorem 2.2). One thread found in the literature, then, is to investigate when and how these complicated ω-limit sets can be approximated by simpler structures. If we let

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$\Lambda(f) = \bigcup_{x\in f} \omega(x, f)$ represent the closed set of *ω-limit points* of f, one natural question to consider is what conditions on f insure that $\overline{P(f)} = \Lambda(f)$, where $P(f)$ is the collection of the periodic points of f. Our first theorem is from [13].

Theorem 1.1. *If $f \in C(I,I)$ is piecewise monotonic, then $\overline{P(f)} = \Lambda(f)$.*

Examples constructed in [10, 18], however, show that $\overline{P(f)} \neq \Lambda(f)$ is possible when we restrict our attention to either the class of Lipschitz or continuously differentiable functions.

We find another approach in [9] where Coven and D'Aniello study the relationship between the sets $\overline{P(f)}$ and $\Lambda(f)$ and the chaotic nature of the generating function $f \in C(I,I)$.

Theorem 1.2. *Let $A = \{f \in C(I,I) : \overline{P(f)} \neq \Lambda(f)\}$. Then A is dense in $C(I,I)$ and any map f in A is Li-Yorke chaotic.*

In [3] the relationship between periodic orbits and ω-limit sets is studied directly, and a very interesting result dealing with wandering intervals and the prevalence of periodic orbits is established.

Theorem 1.3. *If $f \in C(I,I)$ and each wandering interval of f converges to a periodic orbit, then the family of periodic orbits of f is dense with respect to the Hausdorff metric in the collection of ω-limit sets of f.*

We note that $\overline{P(f)} = \Lambda(f)$ whenever the family of periodic orbits of f is dense with respect to the Hausdorff metric in the collection of ω-limit sets.

Another recurring thread in the literature is to investigate the stability of certain structures when the generating function is perturbed. Consider the following result from [14].

Theorem 1.4. *Suppose $f \in C(I,I)$, f has zero topological entropy, $P(f)$ is nowhere dense, and any simple system of f has nonempty interior. Then, for any $\varepsilon > 0$, there exist $n(\varepsilon)$ a natural number and $\delta(\varepsilon) > 0$ so that the following condition holds: if $\|f - g\| < \delta(\varepsilon)$, then for any ω-limit set ω_0 of g there exists a 2^k-cycle p of g such that $k \leq n(\varepsilon)$ and the Hausdorff distance between ω and p is less than ε.*

What this theorem tells us is that every ω-limit set of a function g can be ε-approximated in the Hausdorff metric space by one of its 2^k-cycles whenever g is sufficiently close to a particularly well-behaved function f. Recalling Theorem 1.3, it is worth noting that if $f \in C(I,I)$, f has zero topological entropy, and any simple system of f has nonempty interior, then the 2^k-cycles of f are dense with respect to the Hausdorff metric in the collection of ω-limit sets of f.

The primary purpose of this paper is to provide complete answers to several stability queries posed by Bruckner at the Twentieth Summer Symposium in Real Analysis [5]. In particular, how are the set of ω-limit points and the collection of ω-limit sets of a function affected by slight changes in that function? As Bruckner points out, we also may want to ask these questions when restricting our attention to particular subsets of $C(I,I)$, such as those functions that are in some way nonchaotic, or those functions that satisfy a particular smoothness condition. We will see that in answering Bruckner's queries we are able to develop insight into, or extend, the previously mentioned results. Here is how we proceed.

We work in four metric spaces. We use the regular, Euclidean metric d on $I = [0,1]$, and make occasional use of neighborhoods of closed sets F of the form $B_\varepsilon(F) = \{x \in I : d(x,y) < \varepsilon, y \in F\}$. Within $C(I,I)$ we use the supremum metric given by $\|f - g\| = \sup\{|f(x) - g(x)| : x \in I\}$. Our third metric space (K,H) is composed of the class of nonempty closed sets K in I endowed with the Hausdorff metric H given by $H(E,F) = \inf\{\delta > 0 : E \subset B_\delta(F), F \subset B_\delta(E)\}$. This space is compact [6]. Our final metric space (K^*,H^*) consists of the nonempty closed subsets of K. Thus, $K \in K^*$ if K is a nonempty family of nonempty closed sets in I such that K is closed in K with respect to H. We endow K^* with the metric H^*, so that K_1 and K_2 are close with respect to H^* if each member of K_1 is close to some member of K_2 with respect to H, and vice versa.

Our interest in, and the utility of, the metric spaces (K,H) and (K^*,H^*) stems at least in part from the following two theorems from [4, 15], respectively.

Theorem 1.5. *For any f in $C(I,I)$, the set $\Lambda(f)$ is closed in I.*

Theorem 1.6. *For any f in $C(I,I)$, the set $\Omega(f)$ is closed in (K,H).*

These theorems allow us to formulate earlier stability queries via the maps $\Lambda : (C(I,I), \|\cdot\|) \to (K,H)$ given by $f \to \Lambda(f)$ and $\Omega : (C(I,I),\|\cdot\|) \to (K^*,H^*)$ given by $f \to \Omega(f)$. Specifically, Bruckner asked how one could characterize those functions f at which each of the maps $\Lambda : (C(I,I),\|\cdot\|) \to (K,H)$ and $\Omega : (C(I,I),\|\cdot\|) \to (K^*,H^*)$ is continuous. In order to make these ideas explicit, three examples are developed in some detail. These examples will provide some insight into the behavior of the functions Λ and Ω as well as focus efforts in the ensuing sections.

Example 1.7. Consider $f_n(x) = x^{(n-1)/n}$. As n goes to infinity, we see that f_n goes to the identity function f. Thus, $\Lambda(f) = [0,1]$. Since $\Lambda(f_n) = \{0,1\}$ for all n, we see that Λ is not continuous at f, so that Ω must necessarily be discontinuous there, too. While this does rule out the best possible result—that Ω, and therefore Λ, are continuous—our example does not rule out a natural generalization of the theorem found in [4].

Recall that our four authors in [4] show that if $\{\omega_n\} \subseteq \Omega(f)$, and $\omega_n \to \omega$ in K, then $\omega \in \Omega(f)$. In Example 1.7, $\{0\} \in \Omega(f_n)$ for every n, and $\{0\} \in \Omega(f)$. Perhaps, then, the following is true: if $\omega_n \in \Omega(f_n)$ for each n, $f_n \to f$ and $\omega_n \to L$, then $L \in \Omega(f)$. This conjecture simplifies to the result of [4] if we let $f_n = f$ for all n.

For our next example, we need the following definition. Let M be a nowhere dense compact set in I, with $A = \{a_0, a_1, \ldots, a_{k-1}\} \neq \emptyset$ a set of limit points of M. Suppose there is a system $\{M_n^i\}_{n=0}^{\infty}, i = 0,1,\ldots,k-1$, of nonempty pairwise disjoint compact subsets of M such that $M \setminus \bigcup_{i,n} M_n^i = A$ and $\lim_{n\to\infty} M_n^i = a_i$ for each i. Let $f : M \to M$ be a continuous map with A a k-cycle of f such that $f(a_i) = a_{i-1}$ for $i > 0$ and $f(a_0) = a_{k-1}$. If $f(M_n^i) = M_n^{i-1}$ for $i > 0$ and any n, $f(M_n^0) = M_{n-1}^{k-1}$ for $n > 0$, and $f(M_0^0) = a_{k-1}$, then M is called a homoclinic set of order k with respect to f.

Example 1.8. We will construct a sequence of homoclinic ω -limit sets ω_n for functions f_n in $C(I,I)$ so that $\omega_n \to L$, $f_n \to f$, yet L is not contained in $\Lambda(f)$. This negates our conjectured generalization of the result from [4].

We begin by constructing our ω-limit sets ω_n. For each portion M_n^i, we take a scaled copy of the middle-third Cantor set with the indicated convex closure.

For ω_1, let $a_0 = 1/2$ and $\overline{\text{conv}} M_n^0 = [1/2 + 1/2^{2+n}, 1/2 + 1/2^{2+n} + 1/2^{3+n}]$. Set $A_0 = a_0 \cup \{\bigcup_{n=0}^{\infty} M_n^0\}$. Now, let $a_1 = 0$ and $\overline{\text{conv}} M_n^1 = [1/2^{2+n}, 1/2^{2+n} + 1/2^{3+n}]$.

For ω_2, we begin with the set A_0 described above, and take $a_1 = 1/4$ and $\overline{\text{conv}} M_n^1 = [1/4 + 1/2^{3+n}, 1/4 + 1/2^{3+n} + 1/2^{4+n}]$; let $A_1 = a_1 \cup \{\bigcup_{n=0}^{\infty} M_n^1\}$. Now, let $a_2 = 0$ and $\overline{\text{conv}}$ $M_n^2 = [1/2^{3+n}, 1/2^{3+n} + 1/2^{4+n}]$.

In general, for ω_m, we begin with the sets $A_0, A_1, \dots, A_{m-2}$ and take $a_{m-1} = 1/2^m$ and $\overline{\text{conv}} M_n^{m-1} = [1/2^{m+1} + 1/2^{m+2+n}, 1/2^{m+1} + 1/2^{m+2+n} + 1/2^{m+3+n}]$; let $A_{m-1} = a_{m-1} \cup \{\bigcup_{n=0}^{\infty} M_n^{m-1}\}$. Now, let $a_m = 0$ and $\overline{\text{conv}} M_n^m = [1/2^{m+2+n}, 1/2^{m+2+n} + 1/2^{m+3+n}]$.

We see that each of our sets ω_n will be homoclinic of order $n+1$, and the sequence $\{\omega_n\}$ converges in K to the set $L = \{0\} \cup \{\bigcup_{n=0}^{\infty} A_n\}$. How our functions $f_n : \omega_n \to \omega_n$ are defined is clear from our definition of a homoclinic trajectory as well as the construction of the sets ω_n. Moreover, since each resulting f_n is continuous, we can use [8] to extend $f_n : \omega_n \to \omega_n$ to a function we will also call f_n that is in $C(I,I)$ and has the property that $\omega_n = \omega(x, f_n)$ for some $x \in I$. Since we can take $f_n \mid A_1 \cup \cdots \cup A_m = f_k \mid A_1 \cup \cdots \cup A_m$ for all n and k greater than $m+2$, and $A_n \to 0$ as $n \to \infty$, we can take our f_n so that $f = \lim_{n\to\infty} f_n$ exists, and $f(x) = 0$ for $x \in [1/2, 1]$. Thus, $\Lambda(f) \cap [1/2, 1] = \varnothing$ as $f(0) = \lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 1/2^n$.

It is worth pointing out that not only is L not an ω-limit set of f, but we lose a considerable portion of our ω-limit points as well. For each $n, A_0 \subseteq \omega_n \subseteq \Lambda(f_n)$ with $A_0 \subseteq [1/2, 1]$, yet $\Lambda(f) \cap [1/2, 1] = \varnothing$.

Example 1.9 [5]. Let $f(x) = x$ on I, and for $\varepsilon > 0$, choose $1/n < \varepsilon$. An appropriate polygonal function f_n that possesses the orbit $0 \to 1/n \to 2/n \to \cdots \to (n-1)/n \to 1 \to (n-1/2)/n \to (n-3/2)/n \to \cdots \to 1/2n \to 0$ has a periodic orbit that spans I, and the property that $\|f - f_n\| \leq 1/n$. Since $\Omega(f) = \{\{x\} : x \in I\}$, it follows that $H^*(\Omega(f_n), \Omega(f)) = 1/2$ for all n. By choosing a subsequence if necessary, one may assume that $\lim_{n\to\infty} \Omega(f_n)$ exists, since (K^*, H^*) is compact. Then $\lim_{n\to\infty} f_n = f$, and $H^*(\lim_{n\to\infty} \Omega(f_n), \Omega(f)) = 1/2$. Thus, Ω is discontinuous at the identity function, a function with zero topological entropy. Unlike Example 1.8, however, in this example we did not lose any ω-limit points in going from $\Lambda(f_n)$ to $\Lambda(f)$, as $\Lambda(f) = [0, 1]$, but we did lose all of our nontrivial ω-limit sets in the limit.

We should note that in Examples 1.8 and 1.9, we can take the sequence $\{f_n\}$ to be equicontinuous as well as bounded, so that $\{f_n\}$ has a compact closure in $C(I,I)$. We conclude, then, that $f_n \to f, \omega_n \in \Omega(f_n)$ and $\omega_n \to L$ do not imply that L is in $\Omega(f)$ even for compact sequences $\{f_n\}$. As we see in Section 4, however, whenever $f_n \to f$, $\omega_n \to L$ and $\omega_n \in \Omega(f_n)$ for every n, the limit set L does enjoy some of the properties of an ω-limit set, even though L may not be an element of $\Omega(f)$ (see Proposition 4.3). Later in Section 4 we also see that these sets L play a critical role in characterizing those $f \in C(I,I)$ at which $\Omega : C(I,I) \to K^*$ is upper semicontinuous (see Theorem 4.4).

We proceed through several sections. After presenting the definitions and previously known results needed in the sequel, we begin our analysis in Section 3. There we study the map $\Lambda : (C(I,I), \| \circ \|) \to (K,H)$ given by $f \to \Lambda(f)$, and characterize those functions $f \in C(I,I)$ at which Λ is continuous with Theorem 3.1. We show that Λ is continuous on a residual subset of $C(I,I)$ with Proposition 3.5 and Theorem 3.6. Section 4 is dedicated to the study of the map $\Omega : (C(I,I), \| \circ \|) \to (K^*,H^*)$ given by $f \to \Omega(f)$. Theorem 4.8 characterizes those functions f at which Ω is continuous when we restrict the domain to the set $E = \{f \in C(I,I) : f$ has zero topological entropy$\}$, and Theorem 4.7 characterizes those functions at which Ω is continuous without any domain restrictions. Section 4 concludes with Propositions 4.11 and 4.12 which establish the continuity of $\Omega : (C(I,I), \| \circ \|) \to (K^*,H^*)$ on a residual subset of $C(I,I)$. Section 5 addresses the relationship between the chaotic nature of a function and the behavior of Λ and Ω there.

2. Preliminaries

We make the following definitions in addition to those already presented in Section 1. Let $P(f)$ represent those points $x \in I$ that are *periodic* under f, and if x is a periodic point of period n for which $f^n(x) - x$ takes on both positive and negative values in any deleted neighborhood of x, then x is called a *stable periodic point*; we let $S(f)$ represent the stable periodic points of f. We let $\widetilde{S(f)} = \{\omega : \omega$ is a stable periodic orbit of f in $C(I,I)\}$ be the collection of *stable periodic orbits* of f, and set $\widetilde{\Omega(f)} = \{L : L \subset [0,1]$ is nonempty and closed, $f(L) = L$, and for any nonempty proper closed $F \subset L$ one has $F \cap \overline{f(L-F)} \neq \emptyset\}$. Now, let $\varepsilon > 0$ be given, and take x and y to be any points in $[0,1]$. An *ε-chain* from x to y with respect to a function f is a finite set of points $\{x_0,x_1,\dots,x_n\}$ in $[0,1]$ with $x = x_0, y = x_n$ and $|f(x_{k-1}) - x_k| < \varepsilon$ for $k = 0,1,\dots,n-1$. We call x a *chain recurrent point* of f if there is an ε-chain from x to itself for any $\varepsilon > 0$, and write $x \in CR(f)$. We note that for every f in $C(I,I)$, $S(f) \subseteq P(f) \subseteq \Lambda(f) \subseteq CR(f)$. Central to the ensuing study of the maps $\Lambda : (C(I,I), \| \circ \|) \to (K,H)$ and $\Omega : (C(I,I), \| \circ \|) \to (K^*,H^*)$ is the notion of semicontinuity. Consider the set-valued function $F : (C(I,I), \| \circ \|) \to (K,H)$ with $f \in C(I,I)$. We say that F is *upper semicontinuous* at f if for any $\varepsilon > 0$ there exists $\delta > 0$ so that $F(g) \subset B_\varepsilon(F(f))$ whenever $\|f - g\| < \delta$. Similarly, F is *lower semicontinuous* at f if for any $\varepsilon > 0$ there exists $\delta > 0$ so that $F(f) \subset B_\varepsilon(F(g))$ whenever $\|f - g\| < \delta$.

In part of the sequel we will restrict our attention to a closed subset E of $C(I,I)$ composed of those functions f having *zero topological entropy*, denoted by $\mathbf{h}(f) = 0$. The reader is referred to [11, Theorem A] for an extensive list of equivalent formulations of topological entropy zero. For our purposes, it suffices to note that every periodic orbit of a continuous function with zero topological entropy has cardinality of a power of two [12]. The following theorem, due to Smítal [16], sheds considerable light on the structure of infinite ω-limit sets for functions with zero topological entropy.

Theorem 2.1. *If ω is an infinite ω-limit set of $f \in C(I,I)$ possessing zero topological entropy, then there exists a sequence of closed intervals $\{J_k\}_{k=1}^{\infty}$ in $[0,1]$ such that*

(1) *for each k, $\{f^i(J_k)\}_{i=1}^{2^k}$ are pairwise disjoint, and $J_k = f^{2^k}(J_k)$;*
(2) *for each k, $J_{k+1} \cup f^{2^k}(J_{k+1}) \subset J_k$;*

(3) *for each* $k, \omega \subset \bigcup_{i=1}^{2^k} f^i(J_k)$;
(4) *for each* k *and* $i, \omega \cap f^i(J_k) \neq \emptyset$.

Let J be an interval in I so that $J, f(J), \dots, f^{n-1}(J)$ are pairwise disjoint, and $f^n(J) = J$. We call J a *periodic interval*, and refer to $\{f^i(J)\}_{i=1}^{n-1} = \operatorname{orb} J$ as a *cycle of intervals*. Now, take $J_0 \supset J_1 \supset J_2 \supset \cdots$ to be periodic intervals with periods $m_0, m_1, m_2, \dots$, so that m_i must divide m_{i+1} for any i. If $m_i \to \infty$, then the intervals $\{J_i\}_{1=0}^{\infty}$ generate a *solenoidal system* of f, and any invariant closed set $S \subset L = \bigcap_{i \geq 0} \operatorname{orb} J_i$ is called a *solenoidal set* of f. From Smítal's theorem one sees that every map f with zero topological entropy is solenoidal on each of its infinite ω-limit sets, with J_k having period 2^k for every k. When the period of J_k is 2^k for every interval J_k, we refer to the solenoidal system as a *simple system*.

Now, let us suppose that f has a cycle of intervals $M = J \cup f(J) \cup \cdots \cup f^{n-1}(J)$, and consider the set $\{x \in M : \text{for any relative neighborhood } U \text{ of } x \text{ in } M \text{ we have } \overline{\operatorname{orb} U} = M\}$. This closed set M is invariant, and we refer to M as a *basic set* of f, provided that it is infinite. A fundamental structure associated with both positive topological entropy and basic sets is a *horseshoe*. If $f \in C(I,I)$ has positive topological entropy, then there exist intervals K and L in I having at most one point in common such that $K \cup L \subset f^m(K) \cap f^m(L)$, for some natural number m. This horseshoe structure gives rise to a set $F \subset I$ such that $f(F) = F$ and $f^m \mid F$ is semiconjugate to the shift operator on two symbols. Speaking loosely, the horseshoe structure shows us that there is a considerable amount of expansion that takes place within basic sets that is not present in solenoidal systems.

The following theorem characterizes ω-limit sets for continuous functions [1].

Theorem 2.2. *Let $F \subseteq I$ be a nonempty closed set. Then F is an ω-limit set for some $f \in C(I,I)$ if and only if F is either nowhere dense, or F is the union of finitely many nondegenerate closed intervals.*

3. Continuity of $\Lambda : (C(I,I), \|\circ\|) \to (K,H)$

The main result of this section, Theorem 3.1, characterizes those functions $f \in C(I,I)$ at which the map $\Lambda : (C(I,I), \|\circ\|) \to (K,H)$ is continuous.

Theorem 3.1. *Λ is continuous at f if and only if $\overline{S(f)} = CR(f)$.*

This result follows from Lemmas 3.3 and 3.4, as Lemma 3.3 characterizes those continuous functions at which Λ is upper semicontinuous, and Lemma 3.4 characterizes those continuous functions at which Λ is lower semicontinuous. In the proof of Lemma 3.3, it is helpful to recall the following result from [2].

Lemma 3.2. *If $x \in CR(f)$, then any open neighborhood of f in $C(I,I)$ contains a function g for which $x \in P(g)$.*

Lemma 3.3. *Let $f \in C(I,I)$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ so that $\Lambda(g) \subset B_\varepsilon(\Lambda(f))$ whenever $\|f - g\| < \delta$ if and only if $\Lambda(f) = CR(f)$.*

Proof. Suppose $\Lambda(f) = CR(f)$. Since $CR : (C(I,I), \|\circ\|) \to (K,H)$ given by $g \mapsto CR(g)$ is upper semicontinuous, for any $\varepsilon > 0$ there is a $\delta > 0$ so that $CR(g) \subset B_\varepsilon(CR(f))$ whenever

$\|f-g\| < \delta$ [2]. By hypothesis, we have that $\Lambda(f) = CR(f)$, so that $\Lambda(g) \subset CR(g) \subset B_\varepsilon(\Lambda(f))$, and our conclusion follows.

Now, let us suppose that $x \in CR(f) - \Lambda(f)$. Then there exists $\{f_n\} \subset C(I,I)$ so that $f_n \to f$ and $x \in P(f_n)$ for each n, so that $x \in \lim \Lambda(f_n) - \Lambda(f)$. □

Lemma 3.4. *Let $f \in C(I,I)$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ so that $\Lambda(f) \subset B_\varepsilon(\Lambda(g))$ whenever $\|f-g\| < \delta$ if and only if $\overline{S(f)} = \Lambda(f)$.*

Proof. The sufficiency of our lemma follows immediately from the definition of a stable periodic orbit, and the compactness of $\Lambda(f)$. As for the necessity, let us suppose $\overline{S(f)}$ is a proper subset of $\Lambda(f)$, and let J be an open interval in $[0,1]$ for which $\Lambda(f) \cap J \neq \varnothing$, but $\overline{S(f)} \cap J = \varnothing$. If $P(f) \cap J \neq \varnothing$, then there exists K an open interval contained in J for which $P(f) \cap K \subseteq P_n(f)$, for some natural number n. If $P(f) \cap J = \varnothing$, set $K = J$. In either case, then, there exists $\{f_n\} \subset C(I,I)$ and an open interval K for which $K \cap \Lambda(f) \neq \varnothing$, but $P(f_n) \cap K = \varnothing$ for all natural numbers n, and $f_n \to f$. Since we may take f_n to be piecewise monotonic on I, $\Lambda(f_n) = \overline{P(f_n)}$ for each n [2], so that by taking a subsequence of $\{f_n\}$ if necessary, we have $\Lambda(f_n) = \overline{P(f_n)} \to F$ in (K,H), with $F \cap K = \varnothing$. Thus, $\Lambda(f)$ is not contained in $\lim \Lambda(f_n)$. □

Theorem 3.6 shows that the map $\Lambda : (C(I,I), \| \circ \|) \to (K,H)$ is continuous on a residual subset of $C(I,I)$. This will follow immediately from Proposition 3.5, where we show that $\overline{S(g)} = CR(g)$ for the typical f in $C(I,I)$.

Proposition 3.5. *The set $S = \{f \in C(I,I) : \overline{S(f)} = CR(f)\}$ is residual in $(C(I,I), \| \circ \|)$.*

Proof. Since $S(f) \subseteq CR(f)$ and $CR(f)$ is closed in I it follows that $\overline{S(f)} \subseteq CR(f)$. To show that $H(\overline{S(f)}, CR(f)) < \varepsilon$, then, it suffices to show that for any $x \in CR(f)$ there exists $y \in S(f)$ so that $| x - y | < \varepsilon$. Set $S_n = \{f \in C(I,I) : H(\overline{S(f)}, CR(f)) < 1/n\}$. Since $S = \bigcap_{n=1}^{\infty} S_n$, we need to show that S_n is both dense and open in $C(I,I)$.

We first verify that S_n is a dense subset of $C(I,I)$. Let $f \in C(I,I) - S_n$ with $\varepsilon > 0$. Since $CR : C(I,I) \to K$ is upper semicontinuous, there exists $\delta > 0$ so that $\|f-g\| < \delta$ implies $CR(g) \subset B_\varepsilon(CR(f))$. Take $\delta > 0$ so that $CR(g) \subset B_{1/2n}(CR(f))$ whenever $\|f-g\| < \delta$, and let $\{x_1, x_2, \ldots, x_m\} \subseteq CR(f)$ be a $1/2n$-net of $CR(f)$. Now, choose $g \in C(I,I)$ so that $x_i \in S(g)$ for $1 \le i \le m$ and $\|f-g\| < \min\{\delta, \varepsilon\}$. Then $CR(f) \subset B_{1/2n}(S(g))$ since $\{x_1, x_2, \ldots, x_m\} \subseteq S(g)$ and $CR(g) \subset B_{1/2n}(CR(f))$, so that $CR(g) \subset B_{1/n}(S(g))$. We conclude that $H(\overline{S(g)}, CR(g)) < 1/n$.

We now show that S_n is an open subset of $C(I,I)$. Let $f \in S_n$ with $n \ge 4$. Say $H(\overline{S(f)}, CR(f)) = \alpha < 1/n$, and set $\gamma = 1/n - \alpha$. Let $\delta_1 > 0$ so that $\|f-g\| < \delta_1$ implies $CR(g) \subset B_{\gamma/n}(CR(f))$. Take $\{x_1, x_2, \ldots, x_m\} \subseteq S(f)$ to be an $(\alpha + \gamma/n)$-net of $CR(f)$. Now, there exists $\delta_2 > 0$ so that $\|f-g\| < \delta_2$ implies $S(g) \cap B_{\gamma/n}(x_i) \neq \varnothing$ for $i = 1, 2, \ldots, m$. If $g \in C(I,I)$ for which $\|f-g\| < \min\{\delta_1, \delta_2\}$, then $\bigcup_{i=1}^m x_i \subset B_{\gamma/n}(S(g))$, $CR(f) \subset B_{\alpha+\gamma/n}(\bigcup_{i=1}^m x_i)$ and $CR(g) \subset B_{\gamma/n}(CR(f))$. It follows that $CR(g) \subset B_{1/n}(S(g))$ so that $H(\overline{S(g)}, CR(g)) < 1/n$, and $g \in S_n$. □

Theorem 3.6. *The map $\Lambda : (C(I,I), \| \circ \|) \to (K,H)$ given by $f \mapsto \Lambda(f)$ is continuous on a residual subset of $C(I,I)$.*

Theorem 1.2 shows that the set $A = \{f \in C(I,I) : \overline{P(f)} \neq \Lambda(f)\}$ is dense in $C(I,I)$; Proposition 3.5 shows that A cannot be too large, however, since it is contained in a set of the first category. Moreover, not only is $\overline{P(f)} = \Lambda(f)$ for the class of "nice" piecewise monotonic maps (Theorem 1.1), but $\overline{P(f)} = \Lambda(f)$ also is true for the typical $f \in C(I,I)$. We note that the typical element of $C(I,I)$ has positive topological entropy, so while Li-Yorke chaos is necessary for $\overline{P(f)} \neq \Lambda(f)$, it is not sufficient.

4. Continuity of $\Omega : (C(I,I), \| \circ \|) \to (K^*, H^*)$

As with $\Lambda : (C(I,I), \| \circ \|) \to (K,H)$, semicontinuity is central to our analysis of $\Omega : (C(I,I), \| \circ \|) \to (K^*, H^*)$. Since the upper semicontinuity of $\Lambda : (C(I,I), \| \circ \|) \to (K,H)$ is a necessary condition for the upper semicontinuity of $\Omega : (C(I,I), \| \circ \|) \to (K^*, H^*)$, Lemma 3.3 provides a necessary condition for the upper semicontinuity of Ω.

Proposition 4.1. *A necessary condition for the function f to be a point of upper semicontinuity of the map $\Omega : (C(I,I), \| \circ \|) \to (K^*, H^*)$ is that $\Lambda(f) = CR(f)$.*

That the condition of Proposition 4.1 is not sufficient to insure that $f \in C(I,I)$ is a point of upper semicontinuity of $\Omega : (C(I,I), \| \circ \|) \to (K^*, H^*)$ follows from consideration of the hat map h where $h(x) = 2x$ for $x \in [0,1/2]$ and $h(x) = 2(1-x)$ for $x \in (1/2,1]$. Since $\overline{S(h)} = CR(h) = [0,1]$, Λ is continuous at h. Now, let us consider $g \in C(I,I)$ so that $g(0) = \varepsilon/2, g(1/2) = 1, g(1) = 0$, and g is linear on both $(0,1/2)$ and $(1/2,1)$. Then $\|g - h\| < \varepsilon$, yet $\omega(x,g) \cap [1/2,1] \neq \varnothing$ for all x in I. Since $\{0\} = \omega(0,h) \in \Omega(h)$, we see that $H^*(\Omega(g), \Omega(f)) \geq 1/2$, so that Ω is discontinuous at h.

Turning now to our next results, Proposition 4.2 recalls a couple of basic properties of ω-limit sets [2], and Proposition 4.3 shows that these properties are shared by closed sets L whenever $f_n \to f, \omega_n \to L$ and $\omega_n \in \Omega(f_n)$ for each n.

Proposition 4.2. *Suppose $f \in C(I,I)$ with $\omega \in \Omega(f)$. Then*

(1) $f(\omega) = \omega$,

(2) *if F is any nonempty proper closed subset of ω, then $F \cap \overline{f(\omega \setminus F)} \neq \varnothing$.*

Proposition 4.3. *Suppose $f_n \to f, \omega_n \to L$ and $\omega_n \in \Omega(f_n)$ for each n. Then*

(1) $f(L) = L$,

(2) *if F is any nonempty proper closed subset of L, then $F \cap \overline{f(L \setminus F)} \neq \varnothing$.*

Proof. We first show that $f(L) = L$.

$f(L) \subseteq L$: let $y \in L$, and take $\{y_n\}$ so that $y_n \in \omega_n$ for each n, and $y_n \to y$. Then $f_n(y_n) \to f(y)$, and since $f_n(y_n) \in \omega_n$, it follows that $f(y) \in L$.

$L \subseteq f(L)$: let $y \in L$, and take $\{y_n\}$ so that $y_n \in \omega_n$ for each n, and $y_n \to y$. Suppose $x_n \in f_n^{-1}(y_n) \cap \omega_n$, with $\{x_{n_k}\} \subseteq \{x_n\}$ a convergent subsequence; say $x_{n_k} \to x$. Then $x \in L$, and $f(x) = y$, as $|f(x) - y| \leq |f(x) - f(x_{n_k})| + |f(x_{n_k}) - f_{n_k}(x_{n_k})| + |f_{n_k}(x_{n_k}) - y|$.

In order to prove the second part of our proposition, suppose, to the contrary, that F and $\overline{f(L \setminus F)}$ are disjoint. Then there exist open sets G_1, G_2 such that $L \setminus F \subseteq G_1, F \subseteq G_2$ and $\overline{G_2}$ is disjoint from $\overline{f(G_1)}$. Say $\sigma = \min\{|x-y| : x \in \overline{G_2}, y \in \overline{f(G_1)}\}$. Since $\omega_n \to L$, there exists M a natural number such that $\omega_n \subseteq G_1 \cup G_2$ and $\omega_n \cap G_1 \neq \varnothing, \omega_n \cap G_2 \neq \varnothing$ for all $n \geq M$. Also, since $f_n \to f$, there is a natural number N so that $|f_m(x) - f(x)| < \sigma/2$

for all $m \geq N$ and $x \in I$. Let us take n, then, so that $n > \max\{M,N\}$, and set $F_n = \omega_n \cap \overline{G_2}$. Then F_n is a closed, nonempty, proper subset of ω_n, and $\overline{G_2}$ is disjoint from $\overline{f_n(G_1)}$. Let $x_n \in I$ so that $\omega_n = \omega(x_n, f_n)$. For all large k, $f_n^k(x_n)$ belongs to either G_1 or G_2, and it belongs to each of them infinitely often. Thus there is an infinite sequence $k_1 < k_2 < k_3 < \cdots$ so that $f_n^{k_i}(x_n) \in G_1$, and $f_n^{k_i+1}(x_n) \in G_2$. If y is a limit point of the sequence $f_n^{k_i}(x_n)$, then $y \in \overline{G_1}$, and $f(y) \in \overline{G_2}$, which is a contradiction. □

Proposition 4.3 holds the key to characterizing those functions $f \in C(I,I)$ at which $\Omega : (C(I,I), \|\circ\|) \to (K^*, H^*)$ is upper semicontinuous. Any ω-limit set of a continuous function f satisfies both parts of the conclusion of Proposition 4.3; however, a function f will be a point of upper semicontinuity of $\Omega : (C(I,I), \|\circ\|) \to (K^*, H^*)$ only when these conditions characterize its ω-limit sets. This is the content of Theorem 4.4.

Theorem 4.4. *The map $\Omega : (C(I,I), \|\circ\|) \to (K^*, H^*)$ is upper semicontinuous at the function f if and only if $L \in \Omega(f)$ whenever $L \in K$ for which $f(L) = L$ and $F \cap \overline{f(L \setminus F)} \neq \varnothing$ for any nonempty proper closed subset F of L.*

Significant progress in proving Theorem 4.4 is made with the development of the following proposition.

Proposition 4.5. *Let $f \in C(I,I)$ with $L \in K$ for which $f(L) = L$ and $F \cap \overline{f(L \setminus F)} \neq \varnothing$ for any nonempty proper closed subset F of L. For any $\varepsilon > 0$ there exist $g \in C(I,I)$ and $\omega \in \Omega(g)$ so that $\|f - g\| < \varepsilon$ and $H(\omega, L) < \varepsilon$.*

Proof. Let $\varepsilon > 0$. Since $f \in C(I,I)$, f is uniformly continuous on I, so that there exists $\delta_1 > 0$ with the property that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta_1$. Choose δ so that $0 < \delta < \min\{\delta_1, \varepsilon\}$, and take $\{x_1, x_2, \ldots, x_n\} \subseteq L$ to be a δ-net for L. It suffices to perturb f to get a function $g \in C(I,I)$ possessing a periodic attractor ω so that $\|f - g\| < \varepsilon$ and $\{x_1, x_2, \ldots, x_n\} \subseteq \omega \subseteq \bigcup_{i=1}^n B_\delta(x_i)$, as this implies $H(\omega, L) < \varepsilon$. That this is possible follows from our hypothesis that $F \cap \overline{f(L \setminus F)} \neq \varnothing$ for any nonempty proper closed subset F contained in L. In particular,

(1) let $F = L \backslash B_\delta(x_i)$ to see that there exists $x \in B_\delta(x_i)$ such that $f(x) \in B_\delta(x_j)$ for some $j \neq i$, for any $i = 1, 2, \ldots, n$;

(2) let $F = L \backslash \bigcup_{j \neq i} B_\delta(x_j)$ to see that there exists $x \in B_\delta(x_j)$ for some $j \neq i$ so that $f(x) \in B_\delta(x_i)$, for any $i = 1, 2, \ldots, n$;

(3) let $S \subseteq \{1, 2, \ldots, n\}$ with $F = L \backslash \bigcup_{i \in S} B_\delta(x_i)$ to see that there exists $x \in \bigcup_{i \in S} B_\delta(x_i)$ so that $f(x) \in B_\delta(x_j)$ for some $j \in \{1, 2, \ldots, n\} \backslash S$. □

With Proposition 4.5, a proof of Theorem 4.4 follows easily.

Proof of Theorem 4.4. Suppose $\Omega : (C(I,I), \|\circ\|) \to (K^*, H^*)$ is upper semicontinuous at f, and $L \in K$ for which $f(L) = L$ and $F \cap \overline{f(L \setminus F)} \neq \varnothing$ for any nonempty proper closed subset F of L. By Proposition 4.5, there exists $\{f_n\} \subseteq C(I,I)$ with $\omega_n \in \Omega(f_n)$ for any n so that $\lim_{n \to \infty} f_n = f$ and $\lim_{n \to \infty} \omega_n = L$. Since Ω is upper semicontinuous at f, it follows that $L \in \Omega(f)$.

Now, suppose that $L \in \Omega(f)$ whenever $L \in K$ for which $f(L) = L$ and $F \cap \overline{f(L \setminus F)} \neq \varnothing$ for any nonempty proper closed subset F of L. Let $\{f_n\} \subseteq C(I,I)$ with $\omega_n \in \Omega(f_n)$ for any n so that $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} \omega_n = L^*$. Since $f(L^*) = L^*$ and $F \cap \overline{f(L^* \setminus F)} \neq \emptyset$ for any nonempty proper closed subset F of L^* by Proposition 4.3, it follows that $L^* \in \Omega(f)$, so that $\Omega : (C(I,I), \| \circ \|) \to (K^*, H^*)$ is upper semicontinuous at f. □

The next task is to develop some insight into those functions f in $C(I,I)$ at which $\Omega : (C(I,I), \| \circ \|) \to (K^*, H^*)$ is lower semicontinuous.

Theorem 4.6. *If $f \in C(I,I)$ for which $P(f) - S(f) \neq \emptyset$, then $\Omega : (C(I,I), \| \circ \|) \to (K^*, H^*)$ is not lower semicontinuous at f.*

Proof. Let $x_0 \in P(f) - S(f)$, say of period n. Since $x_0 \in P(f) - S(f)$, for each $0 \le i \le n-1$ there exists a neighborhood N_i containing $x_i = f^i(x)$ so that $f^n(x_i) - x_i$ is unisigned on $N_i - \{x_i\}$. By taking sufficiently small neighborhoods we may assume that $N_i \cap N_j = \emptyset$ whenever $i \neq j$. Now, let $\varepsilon > 0$ so that $B_\varepsilon(x_i) \subseteq N_i$ for all i and $f(B_\varepsilon(x_i)) \subsetneq N_{i+1}$ for $0 \le i \le n-2$ with $f(B_\varepsilon(x_{n-1})) \subsetneq N_0$. We now take $\delta > 0$ so that $g(B_\varepsilon(x_i)) \subsetneq N_{i+1}$, too, for $0 \le i \le n-2$ with $g(B_\varepsilon(x_{n-1})) \subsetneq N_0$ whenever $\|f - g\| < \delta$. Fix $g \in C(I,I)$ so that $P_n(g) \cap N_i = \emptyset$ for $0 \le i \le n-1$ and $\|f - g\| < \delta$. Should $f^n(y) \ge y$ for all $y \in N_0 - \{x_0\}$, it suffices to take $g(x) = f(x) + \delta/2$ for $x \in B_\varepsilon(x_{n-1})$, $g(x) = f(x)$ for $x \notin N_{n-1}$, and extend g appropriately to the remainder of N_{n-1}. We show that $H(\omega(x_0, f), \omega(y, g)) > \varepsilon$ for all y in I. Suppose, to the contrary, that there exists y^* in I so that $H(\omega(x_0, f), \omega^*) < \varepsilon$, where $\omega^* = \omega(y^*, g)$. Then $\omega^* \subseteq B_\varepsilon(\omega(x_0, f)) \subseteq \bigcup_{i=0}^{n-1} N_i$, and by choosing g as we did we know that $g(\omega^* \cap N_i) = \omega^* \cap N_{i+1}$ for $0 \le i \le n-2$ with $g(\omega^* \cap N_{n-1}) = \omega^* \cap N_0$. Thus, $g^n(\omega^* \cap N_i) = \omega^* \cap N_i$ for all i, so that the convex closure $\overline{\operatorname{conv}}(\omega^* \cap N_i)$ contains a periodic point of period n. This, however, contradicts our earlier choice of g. □

We are now in a position to state and prove a main result of the section. We note that condition (1) implies the continuity of the map $\Lambda : (C(I,I), \| \circ \|) \to (K, H)$ at f and condition (3) insures the upper semicontinuity of the map $\Omega : (C(I,I), \| \circ \|) \to (K^*, H^*)$ there. Both of these are clearly necessary to the continuity of Ω at f.

Theorem 4.7. *Let $f \in C(I,I)$. The map $\Omega : (C(I,I), \| \circ \|) \to (K^*, H^*)$ is continuous at f if and only if*

(1) $\overline{S(f)} = CR(f)$,
(2) *all the periodic points of f are stable,*
(3) *$L \in \Omega(f)$ whenever $L \in K$ for which $f(L) = L$ and $F \cap \overline{f(L \setminus F)} \neq \emptyset$ for every nonempty proper closed subset F of L.*

Proof. Suppose the map $\Omega : (C(I,I), \| \circ \|) \to (K^*, H^*)$ is continuous at f. It follows immediately from the definition of the map $\Lambda : (C(I,I), \| \circ \|) \to (K, H)$ that it, too, must be continuous there. Moreover, if $P(f) - S(f) \neq \emptyset$, then Ω would not be lower semicontinuous at f, so that Ω could not be continuous there. Finally, as Ω is continuous at f, Ω must be upper semicontinuous there, so that (3) holds as it characterizes those functions at which Ω is upper semicontinuous.

Now, let us suppose that conditions (1) through (3) hold for some $f \in C(I,I)$.

Since condition (3) characterizes those continuous functions at which Ω is upper semicontinuous, we see that Ω is necessarily upper semicontinuous at f. We must show that Ω is lower semicontinuous at f.

We begin by showing that any infinite ω-limit set of f is contained in the Hausdorff closure of its periodic orbits. Let ω^* be a basic set of f. From Theorem 1.3, any ω-limit set contained in ω^* is in the closure of the periodic orbits of f whenever there is no wandering interval J converging to an infinite ω-limit set in ω^*. To see that this is not possible, assume that J is such an interval and set $A = \bigcup_{n=0}^{\infty} f^{-n}(J)$. Then $\omega^* \subset \overline{A}$ so that $A \cap P(f) = \emptyset$ and $A \cap CR(f) \neq \emptyset$. But this contradicts (1). It remains to show that any simple set W is contained in the closure of the periodic orbits. Let $\{J_k\}$ be a nested family of compact intervals such that J_k has period 2^k and $W \subset \bigcap_{k=1}^{\infty} \bigcup_{i=0}^{2^k-1} f^i(J_k) = L$. Since every interval $U \subset L$ consists of nonperiodic chain recurrent points, condition (1) implies that L has empty interior so that W can be approximated by periodic orbits, too.

We now show that Ω is lower semicontinuous at f. If $\omega \in \Omega(f)$ is finite, then ω is stable by (2), so that for any $\varepsilon > 0$ there exists $\delta > 0$ such that any $g \in C(I,I)$ for which $\|f-g\| < \delta$ possesses a periodic orbit α with the property that $H(\alpha,\omega) < \varepsilon$. If $\omega \in \Omega(f)$ is infinite, then ω can be Hausdorff approximated by stable periodic orbits; this provides the lower semicontinuity of Ω at f. $\square$

Now, suppose that $f \in E$ and $\overline{S(f)} = CR(f)$. Then $intL = \emptyset$ for any simple system of f, and $P(f)$ has nonempty interior, so that such a function f satisfies the hypotheses of Price and Smítal's Theorem 1.4.

Theorem 4.8. *If $f \in E$, then $\Omega : C(I,I) \to K^*$ is continuous at f if and only if $\overline{S(f)} = CR(f)$.*

This theorem from [19] sharpens the result of [14] and indicates that whenever $f \in E$, the map Ω is continuous at f if and only if the map Λ is continuous there. From the definition of a solenoidal system, f must possess a wandering interval that converges to an infinite ω-limit set of f whenever a solenoidal system has nonempty interior. As [3] indicates, this wandering interval may generate an infinite ω-limit set that is not found in the Hausdorff closure of the periodic orbits so that $\Omega : C(I,I) \to K^*$ may not be lower semicontinuous there. Theorem 4.4 shows that $\Omega : C(I,I) \to K^*$ is never upper semicontinuous at such an f.

Our next objective is to establish the continuity of $\Omega : C(I,I) \to K^*$ on a residual subset of $C(I,I)$. Proposition 4.9 shows that $\widetilde{\Omega(f)}$ is closed in (K,H). This allows $\widetilde{\Omega}$ to play a role analogous to that of the chain recurrent set in the analysis of $\Lambda : C(I,I) \to K$ in Section 3.

Proposition 4.9. *If $f \in C(I,I)$, then $\widetilde{\Omega(f)}$ is closed in (K,H).*

Proof. Let $\{L_k\}_{k=1}^{\infty} \subset K$ with $f \in C(I,I)$ so that $L_k = f(L_k)$ for any k, and $L_k \to L$ in K. Since f is continuous, it follows that $f(L_k) \to f(L)$ in K, too. We conclude that $L = f(L)$. Now, suppose that for each k, the following holds for L_k: If $F \neq \emptyset$ is closed such that $F \subsetneq L_k$, then $F \cap \overline{f(L_k - F)} \neq \emptyset$. We show that for any $F \neq \emptyset$ closed, $F \subsetneq L$, it follows that $F \cap \overline{f(L-F)} \neq \emptyset$. Suppose, to the contrary, that there exists such an F so that $F \cap \overline{f(L-F)} = \emptyset$; say $H(F, \overline{f(L-F)}) = \sigma$. Let $\delta > 0$ so that $|x-y| < \delta$ implies $|f(x) - f(y)| < \sigma/4$ and choose n sufficiently large so that $H(L_n, L) < \gamma$, $L_n \cap B_\gamma(F) \neq \emptyset$, and $L_n \cap B_\gamma(L-F) \neq \emptyset$, where $\gamma < \min(\delta, \sigma/8)$. Set $\widetilde{F} = \overline{L_n \cap B_\gamma(F)}$. Then $\widetilde{F} \cap \overline{f(L_n - \widetilde{F})} = \emptyset$ since $\widetilde{F} \subset B_{\sigma/4}(F)$ and $\overline{f(L_n - \widetilde{F})} \subset B_{\sigma/4}(\overline{f(L-F)})$. $\square$

Proposition 4.10. *The map* $\widetilde{\Omega} : (C(I,I), \|\circ\|) \to (K^*, H^*)$ *given by* $f \mapsto \widetilde{\Omega(f)}$ *is upper semicontinuous.*

Proof. Let $f_n \to f$ in $(C(I,I), \|\circ\|)$ with $L_n \in \widetilde{\Omega(f_n)}$ for each n, and $L_n \to L$ in (K,H). We show that $L \in \widetilde{\Omega(f)}$.

We first show that $L = f(L)$. Since $f \in C(I,I)$, $f_n \to f$ uniformly and $L_n \to L$ in (K,H), we have $H(L, f(L)) = H(L, L_n) + H(L_n, f_n(L_n)) + H(f_n(L_n), f(L_n)) + H(f(L_n), f(L))$ where each of the terms on the right-hand side goes to zero as $n \to \infty$. It follows that $L = f(L)$.

Now, let us suppose to the contrary that there exists an appropriate F for which our transport property does not hold for F, $L - F$ and f. In particular, $F \neq \emptyset$ is closed, $F \subsetneq L$ and $F \cap \overline{f(L-F)} = \emptyset$. Say $H(F, \overline{f(L-F)}) = \sigma$. Since $f_n \to f$ uniformly, there is N_1 a natural number so that $n > N_1$ implies $|f(x) - f_n(x)| < \sigma/8$ for all $x \in I$. Since f is uniformly continuous on I there is a $\delta > 0$ so that $|f(x) - f(y)| < \sigma/8$ whenever $|x - y| < \delta$. Since $L_n \to L$ in (K,H) there is N_2 a natural number so that $n > N_2$ implies $H(L_k, L) < \gamma$, $L_k \cap B_\gamma(F) \neq \emptyset$, and $L_k \cap B_\gamma(L-F) \neq \emptyset$, where $\gamma < \min\{\delta, \sigma/8\}$. Now, set $\widetilde{F} = \overline{L_k \cap B_\gamma(F)}$, so that $\widetilde{F} \subset B_{\sigma/4}(F)$. Then $\overline{f_k(L_k - \widetilde{F})} \subset B_{\sigma/8}(\overline{f(L_k - \widetilde{F})})$ and $\overline{f(L_k - \widetilde{F})} \subset B_{\sigma/8}(\overline{f(L-F)})$ so that $\overline{f_k(L_k - \widetilde{F})} \subset B_{\sigma/4}(\overline{f(L-F)})$ whenever $k > max\{N_1, N_2\}$. This implies $H(\widetilde{F}, \overline{f_k(L_k - \widetilde{F})}) > \sigma/2$, a contradiction. □

The next result ties the behavior of $\Omega : (C(I,I), \|\circ\|) \to (K^*, H^*)$ to the upper semicontinuity of the map $\widetilde{\Omega} : (C(I,I), \|\circ\|) \to (K^*, H^*)$.

Proposition 4.11. *If* $f \in C(I,I)$ *for which* $\widetilde{S(f)} = \widetilde{\Omega(f)}$ *in* (K^*, H^*), *then* $\Omega : (C(I,I), \|\circ\|) \to (K^*, H^*)$ *is continuous at* f.

Proof. Recall that for any $f \in C(I,I)$, $\widetilde{S(f)} \subset \Omega(f) \subset \widetilde{\Omega(f)}$. Let us fix f and $\varepsilon > 0$. Since $\widetilde{\Omega} : (C(I,I), \|\circ\|) \to (K^*, H^*)$ is upper semicontinuous at f, there exists $\delta_1 > 0$ so that $\widetilde{\Omega(g)} \subset B_{\varepsilon/4}(\widetilde{\Omega(f)})$ whenever $\|f - g\| < \delta_1$. Since $\widetilde{S(f)}$ is dense in $\widetilde{\Omega(f)}$, there exists $\delta_2 > 0$ so that $\widetilde{\Omega(f)} \subset B_{\varepsilon/4}(\widetilde{S(g)})$ whenever $\|f - g\| < \delta_2$. If $\|f - g\| < \min\{\delta_1, \delta_2\}$, then $\Omega(g) \subset \widetilde{\Omega(g)} \subset B_{\varepsilon/4}(\widetilde{\Omega(f)}) \subset B_{\varepsilon/2}(\widetilde{S(g)}) \subset B_{\varepsilon/2}(\Omega(g))$ so that $\Omega(g) \subset B_{\varepsilon/2}(\Omega(f))$ and $\Omega(f) \subset \widetilde{\Omega(f)} \subset B_{\varepsilon/4}(\Omega(g))$. It follows that $H(\Omega(g), \Omega(f)) < \varepsilon/2$, and $\Omega : (C(I,I), \|\circ\|) \to (K^*, H^*)$ is continuous at f. □

It remains to show that $\widetilde{S(f)} = \widetilde{\Omega(f)}$ for the typical f in $C(I,I)$.

Proposition 4.12. *The set* $G = \{ f \in C(I,I) : \widetilde{S(f)} = \widetilde{\Omega(f)}\}$ *is residual in* $(C(I,I), \|\circ\|)$.

Proof. Let $B_n = \{ f \in C(I,I) : H^*(\widetilde{S(f)}, \widetilde{\Omega(f)}) > 1/n\}$. It suffices to show that B_n is nowhere dense for any n.

We first show that $C(I,I) - B_n$ is dense. Let $f \in B_n$. Since $\widetilde{\Omega} : (C(I,I), \|\circ\|) \to (K^*, H^*)$ is upper semicontinuous at f, there exists $\delta > 0$ so that $\widetilde{\Omega(g)} \subset B_{4n}(\widetilde{\Omega(f)})$ whenever $\|f - g\| < \delta$. Since $\widetilde{\Omega(f)}$ is closed in K, there exist $\{L_i\}_{i=1}^m \subset \widetilde{\Omega(f)}$ so that $\{L_i\}_{i=1}^m$ is a $4n$-net of $\widetilde{\Omega(f)}$. Choose $g \in C(I,I)$ so that $\|f-g\| < \delta$ and there is a stable periodic orbit $K_i \in \widetilde{S(g)}$ so that $H(K_i,L_i) < 1/4n$ for $i = 1,2,3,\dots,m$. It follows that $\widetilde{\Omega(g)} \subset B_{1/4n}(\widetilde{\Omega(f)}) \subset B_{1/2n}(\{L_i\}_{i=1}^m) \subset B_{1/2n}(\widetilde{S(g)})$, so that $H^*(\widetilde{S(g)},\widetilde{\Omega(g)}) < 1/2n$.

We now show that $C(I,I) - B_n$ is open. Let $f \in C(I,I)$ such that $H^*(\widetilde{S(f)},\widetilde{\Omega(f)}) = \sigma < 1/n$; say $1/n - \sigma = \varepsilon$. Choose $\delta_1 > 0$ so that $\widetilde{\Omega(g)} \subset B_{\varepsilon/4}(\widetilde{\Omega(f)})$ whenever $\|f-g\| < \delta_1$, and take $\{L_i\}_{i=1}^m \subset \widetilde{S(f)}$ with the property that $H^*(\{L_i\}_{i=1}^m,\widetilde{\Omega(f)}) < \sigma + \varepsilon/4$. Since $\{L_i\}_{i=1}^m \subset \widetilde{S(f)}$, there exists $\delta_2 > 0$ so that $\|f-g\| < \delta_2$ implies the existence, for any $i = 1,2,\dots,m$, of $K_i \in \widetilde{S(g)}$ so that $H(K_i,L_i) < \varepsilon/4$. Let $g \in C(I,I)$ with $\|f-g\| < \min\{\delta_1,\delta_2\}$. Then $\widetilde{\Omega(g)} \subset B_{\varepsilon/4}(\widetilde{\Omega(f)}) \subset B_{\sigma+\varepsilon/2}(\{L_i\}_{i=1}^m) \subset B_{\sigma+3\varepsilon/4}(\{K_i\}_{i=1}^m) \subset B_{\sigma+3\varepsilon/4}(\widetilde{S(g)})$, so that $H^*(\widetilde{S(g)},\widetilde{\Omega(g)}) < \sigma + 3\varepsilon/4 < 1/n$. □

From Propositions 4.11 and 4.12 it now follows immediately that $\Omega : (C(I,I),\|\circ\|) \to (K^*,H^*)$ is continuous on a residual subset of $C(I,I)$.

Theorem 4.13. *The map $\Omega : (C(I,I),\|\circ\|) \to (K^*,H^*)$ given by $f \mapsto \Omega(f)$ is continuous at a residual set of functions f in $C(I,I)$.*

5. The relationship between stability and chaos

The goal of this section is to determine the relationship between the chaotic nature of a function f in $C(I,I)$ and the behavior of the maps $\Lambda : (C(I,I),\|\circ\|) \to (K,H)$ and $\Omega : (C(I,I),\|\circ\|) \to (K^*,H^*)$ at that function. We begin by considering functions f that are not chaotic in the sense of Li-Yorke, and then consider the evolving behavior of $\Lambda : (C(I,I),\|\circ\|) \to (K,H)$ and $\Omega : (C(I,I),\|\circ\|) \to (K^*,H^*)$ as we make the function f progressively more chaotic.

Lemma 5.1. *Suppose $f \in C(I,I)$ is not chaotic in the sense of Li-Yorke. Then one of the following possibilities must hold:*

(1) *the maps $\Lambda : (C(I,I),\|\circ\|) \to (K,H)$ and $\Omega : (C(I,I),\|\circ\|) \to (K^*,H^*)$ are both continuous at f;*

(2) *the maps $\Lambda : (C(I,I),\|\circ\|) \to (K,H)$ and $\Omega : (C(I,I),\|\circ\|) \to (K^*,H^*)$ are both discontinuous at f.*

Proof. If f is not chaotic in the sense of Li-Yorke, then f has zero topological entropy, so that Λ and Ω are either both continuous or discontinuous together at f. This follows from Theorems 3.1 and 4.8. □

As the next pair of examples shows, each of the situations described in Lemma 5.1 is possible. Suppose $f(x) = 0$ for all $x \in I$. Then f is not Li-Yorke chaotic and both Λ and Ω are continuous there. This follows from the observation that $S(f) = CR(f) = \{0\}$. Now, let $f(x) = x$ for all $x \in I$. Then f is not Li-Yorke chaotic and both Λ and Ω are discontinuous there. We note that $S(f) = \varnothing$ whereas $CR(f) = [0,1]$.

We now consider the behavior of $\Lambda : (C(I,I),\|\circ\|) \to (K,H)$ and $\Omega : (C(I,I),\|\circ\|) \to (K^*,H^*)$ at functions f that are chaotic in the sense of Li-Yorke but still have zero topological entropy.

Proposition 5.2. *Let $E = \{f \in C(I,I) : f$ has zero topological entropy$\}$. If f is an element of E chaotic in the sense of Li-Yorke, then the maps $\Lambda : (C(I,I), \| \circ \|) \to (K,H)$ and $\Omega : (C(I,I), \| \circ \|) \to (K^*,H^*)$ are both discontinuous at f.*

Proof. Let $f \in E$ be chaotic in the sense of Li-Yorke. Since $f \in E$, Λ and Ω will either be continuous or discontinuous together at f. From [7] we know that f must possess a simple system L with nonempty interior. Since $int(L) \cap S(f) = \emptyset$ and $int(L) \subset CR(f)$, we see that $\overline{S(f)} \subsetneq CR(f)$, and our conclusion follows from Theorem 3.10. □

We now apply Proposition 5.2 to functions f for which the map $\omega_f : I \to K$ is not in the first class of Baire but do still possess zero topological entropy.

Corollary 5.3. *Suppose f is an element of E and the map $\omega_f : I \to K$ is not in the first class of Baire. Then $\Lambda : (C(I,I), \| \circ \|) \to (K,H)$ and $\Omega : (C(I,I), \| \circ \|) \to (K^*,H^*)$ are both discontinuous at f.*

With Proposition 5.4 we consider the behavior of Λ and Ω at a function f possessing positive topological entropy.

Proposition 5.4. *Let $T = \{f \in C(I,I) : f$ has positive topological entropy$\}$, with $f \in T$. Then one of the following possibilities must hold:*

(1) *Λ and Ω are both continuous at f;*
(2) *Λ is continuous at f, but Ω is discontinuous there;*
(3) *Λ and Ω are both discontinuous at f.*

Proof. This proposition follows readily from Theorems 3.1 and 4.7. □

We provide examples illustrating each of the three possibilities found in Proposition 5.4. From Theorem 4.13 we know that Ω is continuous on a residual subset of $C(I,I)$. Since T is also residual in $C(I,I)$, it follows that the set $\{f \in T$: Ω is continuous at $f\}$ is also residual in $C(I,I)$. Thus, our first possibility holds on a residual subset of $C(I,I)$.

As for the second possibility, consider the hat map $h(x)$ given by $x \mapsto 2x$ for $x \in [0,1/2]$ and $x \mapsto 2(1-x)$ for $x \in (1/2,1]$. Then $\overline{S(h)} = CR(h) = [0,1]$, so that $\Lambda : (C(I,I), \| \circ \|) \to (K,H)$ is continuous at h. Since $\{0\} \in P(h) - S(h)$, by Theorem 4.7 we see that $\Omega : (C(I,I), \| \circ \|) \to (K^*,H^*)$ is discontinuous at f.

We turn our attention to the third possibility. Consider a function $f \in T$ possessing a wandering interval L such that the closure of the orbit of L contains a basic set ω_0. Since $S(f) \cap int(\text{orb}(L)) = \emptyset$ and $int(\text{orb}(L)) \cap CR(f) \neq \emptyset$, we see that $\overline{S(f)} \subsetneq CR(f)$, so that $\Lambda : (C(I,I), \| \circ \|) \to (K,H)$, and hence $\Omega : (C(I,I), \| \circ \|) \to (K^*,H^*)$, must be discontinuous there.

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