EXPLICIT ISOMORPHISMS OF REAL CLIFFORD ALGEBRAS

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It is well known that the Clifford algebra $\mathrm{Cl}_{p,q}$ associated to a nondegenerate quadratic form on $\mathbb{R}^n$ $(n = p+q)$ is isomorphic to a matrix algebra $K(m)$ or direct sum $K(m) \oplus K(m)$ of matrix algebras, where $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$. On the other hand, there are no explicit expressions for these isomorphisms in literature. In this work, we give a method for the explicit construction of these isomorphisms.

1. Preliminaries

Let F be a field and let V be a finite-dimensional vector space over F and $Q : V \to F$ a quadratic form on V. The Clifford algebra $\mathrm{Cl}(V,Q)$ is an associative algebra with unit 1, which contains and is generated by V, with $v \cdot v = Q(v) \cdot 1$ for all $v \in V$. Formally, one can define the Clifford algebra $\mathrm{Cl}(V,Q)$ as follows.

Definition 1.1. The Clifford algebra $\mathrm{Cl}(V,Q)$ associated to a vector space V over F with quadratic form Q can be defined as

$$\mathrm{Cl}(V,Q) = \frac{T(V)}{I(Q)}, \tag{1.1}$$

where $T(V)$ is the tensor algebra $T(V) = F \oplus V \oplus (V \otimes V) \oplus \cdots$ and $I(Q)$ is the two-sided ideal in $T(V)$ generated by elements $v \otimes v - Q(v) \cdot 1$.

Just like the tensor algebra and the exterior algebra, the Clifford algebra has the following universal property.

Theorem 1.2. *Given an associative unital F-algebra A (with unit 1) and a linear map $f : V \to A$ with $f(v) \cdot f(v) = Q(v) \cdot 1$ for all $v \in V$, then there is a unique homomorphism*

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of algebras $f : \mathrm{Cl}(V,Q) \to A$ *such that the following diagram commutes:*

$$\begin{array}{ccc} V & \xrightarrow{i_Q} & \mathrm{Cl}(V,Q) \\ {\scriptstyle j}\big\downarrow & \swarrow{\scriptstyle \theta} & \\ A & & \end{array} \tag{1.2}$$

where i_Q *is natural inclusion. In particular, the algebra* $\mathrm{Cl}(V,Q)$ *together with the map* $i_Q : V \to \mathrm{Cl}(V,Q)$ *satisfying* $i_Q(v)\cdot i_Q(v) = Q(v)\cdot 1$ *is uniquely determined by this property up to isomorphism (see [3]).*

If $Q = 0$, one recovers precisely the definition of exterior algebra, so $\wedge(V) = \mathrm{Cl}(V,Q = 0)$.

For the realization of the Clifford algebra $\mathrm{Cl}(V,Q)$, the following lemma is useful.

Lemma 1.3. *The structure map* $i_Q : V \to \mathrm{Cl}(V,Q)$ *is injective. Thus* V *will be viewed as a subspace of* $\mathrm{Cl}(V,Q)$. *If* $e_1,e_2,\dots,e_n$ *form a basis: for* V, *then the products*

$$e_{i_1}e_{i_2}\cdots e_{i_k}, \quad 1 \le i_1 < \cdots < i_k \le n, \tag{1.3}$$

and 1 *form a basis of the real vector space* $\mathrm{Cl}(V,Q)$ *(see [2, 3]).*

We deal with the real vector spaces with nondegenerate quadratic form Q. Due to the Sylvester theorem, any nondegenerate quadratic form on $\mathbb{R}^n$ is equivalent to a quadratic form of type

$$Q(x_1,x_2,\dots,x_n) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2 \tag{1.4}$$

(see [1]). If $V = \mathbb{R}^n$ is a real vector space with the quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$, $Q(x_1,x_2,\dots,x_n) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2$, then the corresponding Clifford algebra $\mathrm{Cl}(V,Q)$ is denoted by $\mathrm{Cl}_{p,q}$ $(n = p+q)$. Let $e_1,e_2,\dots,e_p$, $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_q$ be a Sylvester basis for $\mathbb{R}^n$, then following relations hold: $e_i^2 = 1$ $(1 \le i \le p)$, $\varepsilon_i^2 = -1$ $(1 \le i \le q)$ and $e_ie_j = -e_je_i$, $\varepsilon_i\varepsilon_j = -\varepsilon_j\varepsilon_i$ for $i \ne j$ and $e_i\varepsilon_j = -\varepsilon_je_i$ for $1 \le i \le p$, $1 \le j \le q$.

1.1. Calculations for some lower dimensions. Let $\Psi_{p,q}$ denote the isomorphism from the Clifford algebra $\mathrm{Cl}_{p,q}$ to the related matrix algebra and for $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, we denote by $K(n)$ the algebra of $n\times n$-matrices with entries in K.

For $n = 0$, $\mathrm{Cl}_{0,0} \cong \mathbb{R}$

For $n = 1$, $\mathrm{Cl}_{0,1} \cong \mathbb{C}$ by $\Psi_{0,1}(e) = i$ and $\mathrm{Cl}_{1,0} \cong \mathbb{R}\oplus\mathbb{R}$ by $\Psi_{1,0}(e) = (-1,1)$

For $n = 2$,

(i) the Clifford algebra $\mathrm{Cl}_{0,2}$ is isomorphic to the quaternion algebra $\mathbb{H}$ by th isomorphism $\Psi_{0,2} : \mathrm{Cl}_{0,2} \to \mathbb{H}$, $\Psi_{0,2}(e_1) = i$, $\Psi_{0,2}(e_2) = j$ and so $\Psi_{0,2}(e_1e_2) = k$;

(ii) the Clifford algebra $\mathrm{Cl}_{2,0}$ is isomorphic to the matrix algebra $\mathbb{R}(2)$ by the isomorphism $\Psi_{2,0} : \mathrm{Cl}_{2,0} \to \mathbb{R}(2)$, $\Psi_{2,0}(e_1) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\Psi_{2,0}(e_2) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and so $\Psi_{2,0}(e_1e_2) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$;

(iii) the Clifford algebra $\mathrm{Cl}_{1,1}$ is isomorphic to the matrix algebra $\mathbb{R}(2)$ by the isomorphism $\Psi_{1,1} : \mathrm{Cl}_{1,1} \to \mathbb{R}(2)$, $\Psi_{1,1}(e_1) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\Psi_{1,1}(\varepsilon_1) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and so $\Psi_{1,1}(e_1\varepsilon_1) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

To determine $\mathrm{Cl}_{p,q}$ for higher values of $n = p + q$, the following proposition is useful.

Proposition 1.4. *There are isomorphisms*

$$\begin{aligned} \mathrm{Cl}_{0,m+2} &\cong \mathrm{Cl}_{m,0} \otimes \mathrm{Cl}_{0,2}, \\ \mathrm{Cl}_{m+2,0} &\cong \mathrm{Cl}_{0,m} \otimes \mathrm{Cl}_{2,0}, \\ \mathrm{Cl}_{p+1,q+1} &\cong \mathrm{Cl}_{p,q} \otimes \mathrm{Cl}_{1,1}. \end{aligned} \tag{1.5}$$

(We note that ungraded tensor product is meant throughout the paper.)

The first isomorphism $\pi_1 : \mathrm{Cl}_{0,m+2} \to \mathrm{Cl}_{m,0} \otimes \mathrm{Cl}_{0,2}$ is given by

$$\pi_1(e_i) = \begin{cases} \varepsilon_{i-2} \otimes e_1 e_2, & \text{if } 3 \le i \le m+2, \\ 1 \otimes e_i, & \text{if } i = 1 \text{ and } i = 2, \end{cases} \tag{1.6}$$

the second isomorphism $\pi_2 : \mathrm{Cl}_{m+2,0} \to \mathrm{Cl}_{0,m} \otimes \mathrm{Cl}_{2,0}$ can be given by

$$\pi_2(\varepsilon_i) = \begin{cases} e_{i-2} \otimes \varepsilon_1 \varepsilon_2, & \text{if } 3 \le i \le m+2, \\ 1 \otimes \varepsilon_i, & \text{if } i = 1 \text{ and } i = 2 \end{cases} \tag{1.7}$$

and the third one $\pi_3 : \mathrm{Cl}_{p+1,q+1} \to \mathrm{Cl}_{p,q} \otimes \mathrm{Cl}_{1,1}$ can be given by

$$\begin{aligned} \pi_3(e_i) &= \begin{cases} e_i \otimes e_1 \varepsilon, & \text{if } 1 \le i \le p, \\ 1 \otimes e_1, & \text{if } i = p+1, \end{cases} \\ \pi_3(\varepsilon_j) &= \begin{cases} \varepsilon_j \otimes e_1 \varepsilon_1, & \text{if } 1 \le j \le q, \\ 1 \otimes \varepsilon_1, & \text{if } j = q+1. \end{cases} \end{aligned} \tag{1.8}$$

By applying the above isomorphisms recursively, it is possible to get isomorphisms of Clifford algebras, but to apply these isomorphisms, we need some further isomorphisms among the various real algebras.

Proposition 1.5. *The following isomorphisms hold.*

(i) $\mathbb{R}(m) \otimes K \cong K(m)$ *by* $[a_{ij}] \otimes k \mapsto [a_{ij}k]$ *where* $K = \mathbb{R}$, $\mathbb{C}$, *or* $\mathbb{H}$.

(ii) $\mathbb{R}(m) \otimes \mathbb{R}(n) \cong \mathbb{R}(mn)$ *by* $A \otimes B \mapsto [a_{ij}B]$ *(this operation is called the Kronecker product of A and B), where* $A = [a_{ij}]$.

(iii) $\mathbb{C} \otimes \mathbb{H} \cong \mathbb{C}(2)$. *For this isomorphism, consider* $\mathbb{H}$ *as a* $\mathbb{C}$ *module under left scalar multiplication, and define an* $\mathbb{R}$*-bilinear map* $\Psi : \mathbb{C} \times \mathbb{H} \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{H},\mathbb{H})$ *by setting* $\Psi_{z,q}(x) = zx\overline{q}$ *and this extends (by the universal property of tensor product) to an* $\mathbb{R}$*-linear map* $\Psi : \mathbb{C} \otimes \mathbb{H} \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{H},\mathbb{H}) \cong \mathbb{C}(2)$. *This is an isomorphism (see [3]). The images of the basis elements* $1 \otimes 1$, $1 \otimes i$, $1 \otimes j$, $1 \otimes k$, $i \otimes 1$, $i \otimes i$, $i \otimes j$, $i \otimes k$ *of*

$\mathbb{C} \otimes \mathbb{H}$ under this isomorphism are as follows:

$$\Psi_{1,1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Psi_{1,i} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \quad \Psi_{1,j} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \Psi_{1,k} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix},$$
$$\Psi_{i,1} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \quad \Psi_{i,i} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \Psi_{i,j} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \Psi_{i,k} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{1.9}$$

(iv) $\mathbb{H} \otimes \mathbb{H} \cong \mathbb{R}(4)$. *For this isomorphism, consider the $\mathbb{R}$-bilinear map $\Psi : \mathbb{H} \times \mathbb{H} \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{H}, \mathbb{H}) \cong \mathbb{R}(4)$ given by $\Psi_{q_1,q_2}(x) = q_1 x \overline{q_2}$. This map extends (by the universal property of tensor product) to an $\mathbb{R}$-linear map $\Psi : \mathbb{H} \otimes \mathbb{H} \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{H}, \mathbb{H}) \cong \mathbb{R}(4)$. This is an isomorphism (see [3]). The images of the basis elements $1 \otimes 1$, $1 \otimes i$, $1 \otimes j$, $1 \otimes k$, $i \otimes 1$, $i \otimes i$, $i \otimes j$, $i \otimes k$, $j \otimes 1$, $j \otimes i$, $j \otimes j$, $j \otimes k$, $k \otimes 1$, $k \otimes i$, $k \otimes j$, $k \otimes k$ of $\mathbb{H} \otimes \mathbb{H}$ under this isomorphism areas are as follows:*

$$\Phi_{1,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Phi_{1,i} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \Phi_{1,j} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$
$$\Phi_{1,k} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \Phi_{i,1} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \Phi_{i,i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$
$$\Phi_{i,j} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \Phi_{i,k} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \Phi_{j,1} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$
$$\Phi_{j,i} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \Phi_{j,j} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \Phi_{j,k} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$
$$\Phi_{k,1} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \Phi_{k,i} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \Phi_{k,j} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$
$$\Phi_{k,k} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.10}$$

Now we can determine some further Clifford algebras as follows.

Recall that $\mathrm{Cl}_{0,0} \cong \mathbb{R}$, $\mathrm{Cl}_{0,1} \cong \mathbb{C}$, $\mathrm{Cl}_{1,0} \cong \mathbb{R} \oplus \mathbb{R}$, $\mathrm{Cl}_{0,2} \cong \mathbb{H}$, and $\mathrm{Cl}_{2,0} \cong \mathbb{R}(2)$.

By applying isomorphism π_1 to $\mathrm{Cl}_{0,3}$ we have $\mathrm{Cl}_{0,3} \cong \mathrm{Cl}_{1,0} \otimes \mathrm{Cl}_{0,2}$. Since $\mathrm{Cl}_{1,0} \cong \mathbb{R} \oplus \mathbb{R}$ and $\mathrm{Cl}_{0,2} \cong \mathbb{H}$, by Proposition 1.5(i) we have $\mathrm{Cl}_{0,3} \cong (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H} \cong \mathbb{R} \otimes \mathbb{H} \oplus \mathbb{R} \otimes \mathbb{H} \cong \mathbb{H} \oplus \mathbb{H}$.

Similarly by applying the isomorphism π_2 to $\mathrm{Cl}_{3,0}$ we have $\mathrm{Cl}_{3,0} \cong \mathrm{Cl}_{0,1} \otimes \mathrm{Cl}_{2,0}$. Since $\mathrm{Cl}_{0,1} \cong \mathbb{C}$ and $\mathrm{Cl}_{2,0} \cong \mathbb{R}(2)$, by Proposition 1.5(i) we have $\mathrm{Cl}_{3,0} \cong \mathbb{C} \otimes \mathbb{R}(2) \cong \mathbb{C}(2)$.

By applying π_1 to $\mathrm{Cl}_{0,4}$ we have $\mathrm{Cl}_{0,4} \cong \mathrm{Cl}_{2,0} \otimes \mathrm{Cl}_{0,2}$. Since $\mathrm{Cl}_{2,0} \cong \mathbb{R}(2)$ and $\mathrm{Cl}_{0,2} \cong \mathbb{H}$, by Proposition 1.5(i) we have $\mathrm{Cl}_{0,4} \cong \mathbb{R}(2) \otimes H \cong \mathbb{H}(2)$.

Similarly by applying π_2 to $\mathrm{Cl}_{4,0}$ we have $\mathrm{Cl}_{4,0} \cong \mathrm{Cl}_{0,2} \otimes \mathrm{Cl}_{2,0}$. Since $\mathrm{Cl}_{2,0} \cong \mathbb{R}(2)$ and $\mathrm{Cl}_{0,2} \cong \mathbb{H}$, by Proposition 1.5(i) we have $\mathrm{Cl}_{4,0} \cong \mathbb{R}(2) \otimes \mathbb{H} \cong \mathbb{H}(2)$. If we continue in similar way, we have

$$\begin{aligned}
\mathrm{Cl}_{0,5} &\cong \mathrm{Cl}_{0,1} \otimes \mathrm{Cl}_{2,0} \otimes \mathrm{Cl}_{0,2} \cong \mathbb{C} \otimes \mathbb{R}(2) \otimes \mathbb{H} \cong \mathbb{C} \otimes (\mathbb{H} \otimes \mathbb{R}(2)) \\
&\cong (\mathbb{C} \otimes \mathbb{H}) \otimes \mathbb{R}(2) \cong \mathbb{C}(2) \otimes \mathbb{R}(2) \cong (\mathbb{C} \otimes \mathbb{R}(2)) \otimes \mathbb{R}(2) \\
&\cong \mathbb{C} \otimes (\mathbb{R}(2) \otimes \mathbb{R}(2)) \cong \mathbb{C} \otimes \mathbb{R}(4) \cong \mathbb{C}(4), \\
\mathrm{Cl}_{0,6} &\cong \mathrm{Cl}_{0,2} \otimes \mathrm{Cl}_{2,0} \otimes \mathrm{Cl}_{0,2} \cong \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{H} \\
&\cong \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{R}(2) \cong \mathbb{R}(4) \otimes \mathbb{R}(2) \cong \mathbb{R}(8), \\
\mathrm{Cl}_{0,7} &\cong \mathrm{Cl}_{0,3} \otimes \mathrm{Cl}_{2,0} \otimes \mathrm{Cl}_{0,2} \cong (\mathbb{H} \oplus \mathbb{H}) \otimes \mathbb{R}(2) \otimes \mathbb{H} \\
&\cong (\mathbb{H} \oplus \mathbb{H}) \otimes \mathbb{H} \otimes \mathbb{R}(2) \cong (\mathbb{H} \otimes \mathbb{H} \oplus \mathbb{H} \otimes \mathbb{H}) \otimes \mathbb{R}(2) \\
&\cong (\mathbb{R}(4) \oplus \mathbb{R}(4)) \otimes \mathbb{R}(2) \cong \mathbb{R}(8) \oplus \mathbb{R}(8), \\
\mathrm{Cl}_{0,8} &\cong \mathrm{Cl}_{0,4} \otimes \mathrm{Cl}_{2,0} \otimes \mathrm{Cl}_{0,2} \cong \mathrm{Cl}_{2,0} \otimes \mathrm{Cl}_{0,2} \otimes \mathrm{Cl}_{2,0} \otimes \mathrm{Cl}_{0,2} \\
&\cong \mathbb{R}(2) \otimes \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{H} \cong \mathbb{R}(2) \otimes \mathbb{R}(2) \otimes \mathbb{H} \otimes \mathbb{H} \cong \mathbb{R}(4) \otimes \mathbb{R}(4) \cong \mathbb{R}(16), \\
\mathrm{Cl}_{5,0} &\cong \mathrm{Cl}_{1,0} \otimes \mathrm{Cl}_{0,2} \otimes \mathrm{Cl}_{2,0} \cong (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H} \otimes \mathbb{R}(2) \\
&\cong (\mathbb{R} \oplus \mathbb{R}) \otimes (\mathbb{R}(2) \otimes \mathbb{H}) \cong (\mathbb{R}(2) \oplus \mathbb{R}(2)) \otimes \mathbb{H} \\
&\cong (\mathbb{R}(2) \otimes \mathbb{H}) \otimes (\mathbb{R}(2) \otimes \mathbb{H}) \cong (\mathbb{H} \otimes \mathbb{R}(2)) \otimes (\mathbb{H} \otimes \mathbb{R}(2)) \\
&\cong \mathbb{H}(2) \oplus \mathbb{H}(2), \\
\mathrm{Cl}_{6,0} &\cong \mathrm{Cl}_{2,0} \otimes \mathrm{Cl}_{0,2} \otimes \mathrm{Cl}_{2,0} \cong \mathbb{R}(2) \otimes \mathbb{H} \otimes \mathbb{R}(2) \cong \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{R}(2) \\
&\cong \mathbb{H} \otimes \mathbb{R}(4) \cong \mathbb{H}(4), \\
\mathrm{Cl}_{7,0} &\cong \mathrm{Cl}_{3,0} \otimes \mathrm{Cl}_{0,2} \otimes \mathrm{Cl}_{2,0} \cong \mathbb{C}(2) \otimes \mathbb{H} \otimes \mathbb{R}(2) \cong \mathbb{C} \otimes \mathbb{R}(2) \otimes \mathbb{H} \otimes \mathbb{R}(2) \\
&\cong \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{R}(2) \cong \mathbb{C}(2) \otimes \mathbb{R}(4) \cong \mathbb{C} \otimes \mathbb{R}(2) \otimes \mathbb{R}(4) \\
&\cong \mathbb{C} \otimes \mathbb{R}(8) \cong \mathbb{C}(8), \\
\mathrm{Cl}_{8,0} &\cong \mathrm{Cl}_{4,0} \otimes \mathrm{Cl}_{0,2} \otimes \mathrm{Cl}_{2,0} \cong \mathbb{H}(2) \otimes \mathbb{H} \otimes \mathbb{R}(2) \cong \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{H} \otimes \mathbb{R}(2) \\
&\cong \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{R}(2) \cong \mathbb{R}(4) \otimes \mathbb{R}(4) \cong \mathbb{R}(16).
\end{aligned} \tag{1.11}$$

Table 1.1

m	0	1	2	3	4	5	6	7	8
$\mathrm{Cl}_{0,m}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$
$\mathrm{Cl}_{m,0}$	$\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$

Table 1.2

$-q\ (\mathrm{mod}\,8)$	$\mathrm{Cl}_{0,q}$
0, 2	$\mathbb{R}(2^{q/2})$
1	$\mathbb{R}(2^{(q-1)/2})\oplus\mathbb{R}(2^{(q-1)/2})$
3, 7	$\mathbb{C}(2^{(q-1)/2})$
4, 6	$\mathbb{H}(2^{(q-2)/2})$
5	$\mathbb{H}(2^{(q-1)/2})\oplus\mathbb{H}(2^{(q-1)/2})$

All of these calculations yields Table 1.1.

By composing the isomorphisms π_1 and π_2 we get an isomorphism from the Clifford algebra $\mathrm{Cl}_{0,m+4}$ to $\mathrm{Cl}_{0,m}\otimes\mathrm{Cl}_{2,0}\otimes\mathrm{Cl}_{0,2}$ as follows:

$$\begin{aligned}
\pi:\mathrm{Cl}_{0,m+4} &\longrightarrow \mathrm{Cl}_{m+2,0}\otimes\mathrm{Cl}_{0,2}\longrightarrow \mathrm{Cl}_{0,m}\otimes\mathrm{Cl}_{2,0}\otimes\mathrm{Cl}_{0,2},\\
\varepsilon_1 &\longmapsto 1\otimes\varepsilon_1\longmapsto 1\otimes 1\otimes\varepsilon_1,\\
\varepsilon_2 &\longmapsto 1\otimes\varepsilon_2\longmapsto 1\otimes 1\otimes\varepsilon_2,\\
\varepsilon_3 &\longmapsto e_1\otimes\varepsilon_1\varepsilon_2\longmapsto 1\otimes e_1\otimes\varepsilon_1\varepsilon_2,\\
\varepsilon_4 &\longmapsto e_2\otimes\varepsilon_1\varepsilon_2\longmapsto 1\otimes e_2\otimes\varepsilon_1\varepsilon_2,\\
\varepsilon_5 &\longmapsto e_3\otimes\varepsilon_1\varepsilon_2\longmapsto \varepsilon_1\otimes e_1e_2\otimes\varepsilon_1\varepsilon_2,\\
\varepsilon_6 &\longmapsto e_4\otimes\varepsilon_1\varepsilon_2\longmapsto \varepsilon_2\otimes e_1e_2\otimes\varepsilon_1\varepsilon_2,\\
&\vdots\longmapsto\vdots\longmapsto\vdots\\
\varepsilon_{n+4} &\longmapsto e_{n+2}\otimes\varepsilon_1\varepsilon_2\longmapsto \varepsilon_n\otimes e_1e_2\otimes\varepsilon_1\varepsilon_2.
\end{aligned}\tag{1.12}$$

In particular, if we take $m=8$ and use the isomorphism π two times, then we can write

$$\mathrm{Cl}_{0,8}\cong\mathrm{Cl}_{2,0}\otimes\mathrm{Cl}_{0,2}\otimes\mathrm{Cl}_{2,0}\otimes\mathrm{Cl}_{0,2}.\tag{1.13}$$

On the other hand, if we start with the Clifford algebra $\mathrm{Cl}_{0,m+8}$ and apply the isomorphism π two times, then we get the isomorphism $\mathrm{Cl}_{0,m+8}\cong\mathrm{Cl}_{0,m}\otimes\mathrm{Cl}_{2,0}\otimes\mathrm{Cl}_{0,2}\otimes\mathrm{Cl}_{2,0}\otimes\mathrm{Cl}_{0,2}$. If use (1.13) in the last expression, then we get the periodicity relation

$$\mathrm{Cl}_{0,m+8}\cong\mathrm{Cl}_{0,m}\otimes\mathrm{Cl}_{0,8}\cong\mathrm{Cl}_{0,m}\otimes\mathbb{R}(16).\tag{1.14}$$

The periodicity $\mathrm{Cl}_{m+8,0}\cong\mathrm{Cl}_{m,0}\otimes\mathrm{Cl}_{8,0}\cong\mathrm{Cl}_{m,0}\otimes\mathbb{R}(16)$ can be obtained similarly.

By using the above periodicity relations we can easily determine the Clifford algebras $\mathrm{Cl}_{0,m}$ and $\mathrm{Cl}_{m,0}$ recursively for the higher values of m and we get Tables 1.2 and 1.3.

Table 1.3

$p \pmod 8$	$\mathrm{Cl}_{p,0}$
0, 2	$\mathbb{R}(2^{p/2})$
1	$\mathbb{R}(2^{(p-1)/2}) \oplus \mathbb{R}(2^{(p-1)/2})$
3, 7	$\mathbb{C}(2^{(p-1)/2})$
4, 6	$\mathbb{H}(2^{(p-2)/2})$
5	$\mathbb{H}(2^{(p-1)/2}) \oplus \mathbb{H}(2^{(p-1)/2})$

Table 1.4

$(p-q) \pmod 8$	$p+q$	$\mathrm{Cl}_{p,q}$
0, 2	$2m$	$\mathbb{R}(2^m)$
1	$2m+1$	$\mathbb{R}(2^m) \oplus \mathbb{R}(2^m)$
3, 7	$2m+1$	$\mathbb{C}(2^m)$
4, 6	$2m+2$	$\mathbb{H}(2^m)$
5	$2m+3$	$\mathbb{H}(2^m) \oplus \mathbb{H}(2^m)$

To determine Clifford algebras of type $\mathrm{Cl}_{p,q}$ $(p,q>0)$, the isomorphism π_3 and Tables 1.2 and 1.3 are enough. For example, if we start by applying isomorphism π_3 to $\mathrm{Cl}_{2,2}$, we have $\mathrm{Cl}_{2,2} \cong \mathrm{Cl}_{1,1} \otimes \mathrm{Cl}_{1,1}$. Since $\mathrm{Cl}_{1,1} \cong \mathbb{R}(2)$, by Proposition 1.5(ii) we have $\mathrm{Cl}_{2,2} \cong \mathbb{R}(2) \otimes \mathbb{R}(2) \cong \mathbb{R}(4)$. Similarly $\mathrm{Cl}_{3,3} \cong \mathrm{Cl}_{2,2} \otimes \mathrm{Cl}_{1,1} \cong \mathbb{R}(4) \otimes \mathbb{R}(2) \cong \mathbb{R}(8)$. Similarly for $p = q$ we have $\mathrm{Cl}_{p,p} \cong \mathbb{R}(2^p)$. If we apply π_3 to $\mathrm{Cl}_{1,2}$, then we get $\mathrm{Cl}_{1,2} \cong \mathrm{Cl}_{0,1} \otimes \mathrm{Cl}_{1,1}$. Since $\mathrm{Cl}_{0,1} \cong \mathbb{C}$ and $\mathrm{Cl}_{1,1} \cong \mathbb{R}(2)$, by Proposition 1.5(i) we can write $\mathrm{Cl}_{1,2} \cong \mathbb{C} \otimes \mathbb{R}(2) \cong \mathbb{C}(2)$. Similarly $\mathrm{Cl}_{2,3} \cong \mathrm{Cl}_{1,2} \otimes \mathrm{Cl}_{1,1} \cong \mathbb{C}(2) \otimes \mathbb{R}(2) \cong \mathbb{C}(4)$. Similarly for $q = p+1$ we have $\mathrm{Cl}_{p,p+1} \cong \mathbb{C}(2^p)$. Therefore, by continuing in a completely similar fashion for other values of p, q (p, $q > 0$), we obtain Table 1.4.

We also point out that there are periodicity isomorphisms $\mathrm{Cl}_{p+8,q} \cong \mathrm{Cl}_{p,q} \otimes \mathrm{Cl}_{8,0}$ and $\mathrm{Cl}_{p,q+8} \cong \mathrm{Cl}_{p,q} \otimes \mathrm{Cl}_{0,8}$ for Clifford algebras (see [4]).

Our goal is give a method for the explicit expressions of the Clifford algebra isomorphisms. To do this firstly we obtain isomorphisms for the Clifford algebras of type $\mathrm{Cl}_{0,m}$.

2. Isomorphisms of nondegenerate Clifford algebras

2.1. Isomorphisms for the Clifford algebra $\mathrm{Cl}_{0,m}$. First we obtain isomorphisms of $\mathrm{Cl}_{0,m}$ for $1 \le m \le 8$, then by using the periodicity isomorphism $\mathrm{Cl}_{0,m+8} \cong \mathrm{Cl}_{0,m} \otimes \mathrm{Cl}_{0,8} \cong \mathrm{Cl}_{0,m} \otimes \mathbb{R}(16)$ we achieve the other isomorphisms.

2.1.1. Isomorphisms of $\mathrm{Cl}_{0,m}$ for $1 \le m \le 8$. Above we have given the isomorphisms $\Psi_{1,0} : \mathrm{Cl}_{0,1} \to \mathbb{C}$ and $\Psi_{0,2} : \mathrm{Cl}_{0,2} \to \mathbb{H}$ and the others are as follows.

(1) $\Psi_{0,3} : \mathrm{Cl}_{0,3} \to \mathbb{H} \oplus \mathbb{H}$,

$$\begin{gathered} \mathrm{Cl}_{0,3} \longrightarrow \mathrm{Cl}_{1,0} \otimes \mathrm{Cl}_{0,2} \longrightarrow (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H} \longrightarrow \mathbb{H} \oplus \mathbb{H}, \\ \varepsilon_1 \longmapsto 1 \otimes \varepsilon_1 \longmapsto (1,1) \otimes i \longmapsto (i,i), \\ \varepsilon_2 \longmapsto 1 \otimes \varepsilon_2 \longmapsto (1,1) \otimes j \longmapsto (j,j), \\ \varepsilon_3 \longmapsto e_1 \otimes \varepsilon_1\varepsilon_2 \longmapsto (1,-1) \otimes k \longmapsto (k,-k). \end{gathered} \tag{2.1}$$

(2) $\Psi_{0,4} : \mathrm{Cl}_{0,4} \to \mathbb{H}(2)$,

$$\begin{gathered} \mathrm{Cl}_{0,4} \longrightarrow \mathrm{Cl}_{2,0} \otimes \mathrm{Cl}_{0,2} \longrightarrow \mathbb{R}(2) \otimes \mathbb{H} \longrightarrow \mathbb{H}(2), \\ \varepsilon_1 \longmapsto 1 \otimes \varepsilon_1 \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes i \longmapsto \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \\ \varepsilon_2 \longmapsto 1 \otimes \varepsilon_2 \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes j \longmapsto \begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix}, \\ \varepsilon_3 \longmapsto e_1 \otimes \varepsilon_1\varepsilon_2 \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes k \longmapsto \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix}, \\ \varepsilon_4 \longmapsto e_2 \otimes \varepsilon_1\varepsilon_2 \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes k \longmapsto \begin{bmatrix} k & 0 \\ 0 & -k \end{bmatrix}. \end{gathered} \tag{2.2}$$

(3) $\Psi_{0,5} : \mathrm{Cl}_{0,5} \to \mathbb{C}(4)$,

$$\begin{gathered} \mathrm{Cl}_{0,5} \longrightarrow \mathbb{C} \otimes \mathbb{R}(4) \longrightarrow \mathbb{C}(4), \\ \varepsilon_1 \longmapsto i \otimes \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{bmatrix}, \\ \varepsilon_2 \longmapsto 1 \otimes \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \end{gathered}$$

$$\varepsilon_3 \longmapsto i \otimes \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix},$$

$$\varepsilon_4 \longmapsto i \otimes \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix},$$

$$\varepsilon_5 \longmapsto 1 \otimes \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{2.3}$$

(4) $\Psi_{0,6} : \mathrm{Cl}_{0,6} \to \mathbb{R}(8)$,

$$\begin{aligned} \Psi_{0,6}(\varepsilon_1) &= -\sigma_2 \otimes \sigma_1\sigma_2 \otimes I, \\ \Psi_{0,6}(\varepsilon_2) &= -\sigma_1\sigma_2 \otimes I \otimes I, \\ \Psi_{0,6}(\varepsilon_3) &= -\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1, \\ \Psi_{0,6}(\varepsilon_4) &= -\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2, \\ \Psi_{0,6}(\varepsilon_5) &= \sigma_1 \otimes I \otimes \sigma_1\sigma_2, \\ \Psi_{0,6}(\varepsilon_6) &= -\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2, \end{aligned} \tag{2.4}$$

where $I = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $\sigma_1 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, and $\sigma_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$.

(5) $\Psi_{0,7} : \mathrm{Cl}_{0,6} \to \mathbb{R}(8) \oplus \mathbb{R}(8)$,

$$\begin{aligned} \varepsilon_1 &\longmapsto (-\sigma_2 \otimes \sigma_1\sigma_2 \otimes I, -\sigma_2 \otimes \sigma_1\sigma_2 \otimes I), \\ \varepsilon_2 &\longmapsto (-\sigma_1\sigma_2 \otimes I \otimes I, -\sigma_1\sigma_2 \otimes I \otimes I), \\ \varepsilon_3 &\longmapsto (-\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1, -\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1), \\ \varepsilon_4 &\longmapsto (-\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2, -\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2), \\ \varepsilon_5 &\longmapsto (\sigma_1 \otimes I \otimes \sigma_1\sigma_2, \sigma_1 \otimes I \otimes \sigma_1\sigma_2), \\ \varepsilon_6 &\longmapsto (-\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2, -\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2), \\ \varepsilon_7 &\longmapsto (\sigma_2 \otimes \sigma_2 \otimes \sigma_1\sigma_2, \sigma_2 \otimes \sigma_2 \otimes \sigma_1\sigma_2). \end{aligned} \tag{2.5}$$

(6) $\Psi_{0,8} : \mathrm{Cl}_{0,8} \to \mathbb{R}(16)$,

$$\begin{aligned}
\varepsilon_1 &\longmapsto -I \otimes I \otimes \sigma_2 \otimes \sigma_1\sigma_2, \\
\varepsilon_2 &\longmapsto -I \otimes I \otimes \sigma_1\sigma_2 \otimes I, \\
\varepsilon_3 &\longmapsto -I \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1\sigma_2, \\
\varepsilon_4 &\longmapsto -I \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2, \\
\varepsilon_5 &\longmapsto I \otimes \sigma_1\sigma_2 \otimes \sigma_1 \otimes I, \\
\varepsilon_6 &\longmapsto -I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_1, \\
\varepsilon_7 &\longmapsto \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2, \\
\varepsilon_8 &\longmapsto \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2.
\end{aligned} \tag{2.6}$$

2.1.2. Isomorphisms of $\mathrm{Cl}_{0,n+8}$ *for* $n \geq 1$. Now we want to obtain explicit form of the isomorphism (2):

$$\begin{gathered}
\mathrm{Cl}_{0,n+8} \longrightarrow \mathrm{Cl}_{0,n+4} \otimes \mathrm{Cl}_{2,0} \otimes \mathrm{Cl}_{0,2} \longrightarrow \mathrm{Cl}_{0,n} \otimes \mathrm{Cl}_{2,0} \otimes \mathrm{Cl}_{0,2} \otimes \mathrm{Cl}_{2,0} \otimes \mathrm{Cl}_{0,2}, \\
\varepsilon_1 \longmapsto 1 \otimes 1 \otimes \varepsilon_1 \longmapsto 1 \otimes 1 \otimes 1 \otimes 1 \otimes \varepsilon_1, \\
\varepsilon_2 \longmapsto 1 \otimes 1 \otimes \varepsilon_2 \longmapsto 1 \otimes 1 \otimes 1 \otimes 1 \otimes \varepsilon_2, \\
\varepsilon_3 \longmapsto 1 \otimes e_1 \otimes \varepsilon_1\varepsilon_2 \longmapsto 1 \otimes 1 \otimes 1 \otimes e_1 \otimes \varepsilon_1\varepsilon_2, \\
\varepsilon_4 \longmapsto 1 \otimes e_2 \otimes \varepsilon_1\varepsilon_2 \longmapsto 1 \otimes 1 \otimes 1 \otimes e_2 \otimes \varepsilon_1\varepsilon_2, \\
\varepsilon_5 \longmapsto \varepsilon_1 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2 \longmapsto 1 \otimes 1 \otimes \varepsilon_1 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2, \\
\varepsilon_6 \longmapsto \varepsilon_2 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2 \longmapsto 1 \otimes 1 \otimes \varepsilon_2 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2, \\
\varepsilon_7 \longmapsto \varepsilon_3 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2 \longmapsto 1 \otimes e_1 \otimes \varepsilon_1\varepsilon_2 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2, \\
\varepsilon_8 \longmapsto \varepsilon_4 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2 \longmapsto 1 \otimes e_2 \otimes \varepsilon_1\varepsilon_2 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2, \\
\varepsilon_9 \longmapsto \varepsilon_5 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2 \longmapsto \varepsilon_1 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2, \\
\varepsilon_{10} \longmapsto \varepsilon_6 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2 \longmapsto \varepsilon_2 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2, \\
\vdots \longmapsto \vdots \longmapsto \vdots \\
\varepsilon_{n+8} \longmapsto \varepsilon_{n+4} \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2 \longmapsto \varepsilon_n \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2.
\end{gathered} \tag{2.7}$$

Then we get the isomorphism $\Psi_{0,n+8}$ as

$$\begin{aligned}
\mathrm{Cl}_{0,n+8} &\longrightarrow \mathrm{Cl}_{0,n} \otimes \mathbb{R}(16),\\
\varepsilon_1 &\longmapsto 1 \otimes A_1,\\
\varepsilon_2 &\longmapsto 1 \otimes A_2,\\
\varepsilon_3 &\longmapsto 1 \otimes A_3,\\
\varepsilon_4 &\longmapsto 1 \otimes A_4,\\
\varepsilon_5 &\longmapsto 1 \otimes A_5,\\
\varepsilon_6 &\longmapsto 1 \otimes A_6,\\
\varepsilon_7 &\longmapsto 1 \otimes A_7,\\
\varepsilon_8 &\longmapsto 1 \otimes A_8,\\
\varepsilon_9 &\longmapsto \varepsilon_1 \otimes B,\\
\varepsilon_{10} &\longmapsto \varepsilon_2 \otimes B,\\
\vdots &\longmapsto \vdots\\
\varepsilon_{n+8} &\longmapsto \varepsilon_n \otimes B,
\end{aligned} \tag{2.8}$$

where $A_1 = \Psi_{0,8}(\varepsilon_1)$, $A_2 = \Psi_{0,8}(\varepsilon_2)$, $A_3 = \Psi_{0,8}(\varepsilon_3)$, $A_4 = \Psi_{0,8}(\varepsilon_4)$, $A_5 = \Psi_{0,8}(\varepsilon_5)$, $A_6 = \Psi_{0,8}(\varepsilon_6)$, $A_7 = \Psi_{0,8}(\varepsilon_7)$, $A_8 = \Psi_{0,8}(\varepsilon_8)$ and $B = \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2$. Note that B is symmetric with $B^2 = I$ and it anticommutes with the matrices $A_1, A_2, \ldots, A_8$. The explicit form B is as follows:

$$B = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \tag{2.9}$$

2.2. Isomorphisms for the Clifford algebra $\mathrm{Cl}_{m,0}$**.** Now by using the above isomorphism we determine isomorphisms for Clifford algebras $\mathrm{Cl}_{m,0}$. Above we have given the isomorphisms $\Psi_{1,0} : \mathrm{Cl}_{1,0} \to \mathbb{R} \oplus \mathbb{R}$ and $\Psi_{2,0} : \mathrm{Cl}_{2,0} \to \mathbb{R}(2)$ and the others can be obtained easily. For example, we know that $\mathrm{Cl}_{3,0} \cong \mathrm{Cl}_{0,1} \otimes \mathrm{Cl}_{2,0}$,

$$\begin{gathered}
\mathrm{Cl}_{3,0} \longrightarrow \mathrm{Cl}_{0,1} \otimes \mathrm{Cl}_{2,0} \longrightarrow \mathbb{C} \otimes \mathbb{R}(2) \longrightarrow \mathbb{C}(2), \\
e_1 \longmapsto 1 \otimes e_1 \longmapsto 1 \otimes \sigma_1 \longmapsto \sigma_1, \\
e_2 \longmapsto 1 \otimes e_2 \longmapsto 1 \otimes \sigma_2 \longmapsto \sigma_2, \\
e_3 \longmapsto \varepsilon_1 \otimes e_1 e_2 \longmapsto i \otimes \sigma_1 \sigma_2 \longmapsto i \sigma_1 \sigma_2,
\end{gathered} \tag{2.10}$$

that is, $\Psi_{3,0} = \Psi_{0,1} \otimes \Psi_{2,0}$. Generally the isomorphism $\Psi_{n+2,0}$ of $\mathrm{Cl}_{n+2,0}$ can be expressed as $\Psi_{n+2,0} = \Psi_{0,n} \otimes \Psi_{2,0}$ since $\mathrm{Cl}_{n+2,0} \cong \mathrm{Cl}_{0,n} \otimes \mathrm{Cl}_{2,0}$.

2.3. Isomorphisms for the Clifford algebra $\mathrm{Cl}_{p,q}$ $(p, q > 0)$**.** Now by using the above isomorphisms we determine isomorphisms for Clifford algebras $\mathrm{Cl}_{p,q}$. Above we have given the isomorphism $\Psi_{1,1} : \mathrm{Cl}_{1,1} \to \mathbb{R}(2)$ and the others can be obtained easily. For example, we know that $\mathrm{Cl}_{2,2} \cong \mathrm{Cl}_{1,1} \otimes \mathrm{Cl}_{1,1}$,

$$\begin{gathered}
\mathrm{Cl}_{2,2} \longrightarrow \mathrm{Cl}_{1,1} \otimes \mathrm{Cl}_{1,1} \longrightarrow \mathbb{R}(2) \otimes \mathbb{R}(2) \longrightarrow \mathbb{R}(4), \\
e_1 \longmapsto 1 \otimes e_1 \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \longmapsto I \otimes \sigma_1, \\
e_2 \longmapsto e_1 \otimes e_1 \varepsilon_1 \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \longmapsto \sigma_1 \otimes \sigma_2, \\
\varepsilon_1 \longmapsto 1 \otimes \varepsilon_1 \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \longmapsto I \otimes \sigma_1 \sigma_2, \\
\varepsilon_2 \longmapsto \varepsilon_1 \otimes e_1 \varepsilon_1 \longmapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \longmapsto \sigma_1 \sigma_2 \otimes \sigma_2,
\end{gathered} \tag{2.11}$$

that is, $\Psi_{2,2} = \Psi_{1,1} \otimes \Psi_{1,1}$. Generally the isomorphism $\Psi_{p+1,q+1}$ of $\mathrm{Cl}_{p+1,q+1}$ can be expressed as $\Psi_{p+1,q+1} = \Psi_{p,q} \otimes \Psi_{1,1}$ since $\mathrm{Cl}_{p+1,q+1} \cong \mathrm{Cl}_{p,q} \otimes \mathrm{Cl}_{1,1}$.

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