PARABOLIC INEQUALITIES IN L^1 AS LIMITS OF RENORMALIZED EQUATIONS

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The paper deals with the existence of solutions of some parabolic bilateral problems approximated by the renormalized solutions of some parabolic equations.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ and $T > 0$. We denote by Q the cylinder $\Omega \times (0,T)$ and $\Gamma = \partial Q$.

Let

$$A(u) = -\operatorname{div}\left(a(x,t,u,\nabla u)\right) \tag{1.1}$$

be a Leray-Lions operator acting on $L^p(0,T;W_0^{1,p}(\Omega))$, $1 < p < \infty$, into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))(1/p + 1/p' = 1)$. Consider the following parabolic problem:

$$\begin{gathered} u \in \mathcal{K} = \left\{v \in L^p\left(0,T;W_0^{1,p}(\Omega)\right) : v(t) \in K \text{ a.e.}\right\}, \\ \int_0^T \left\langle \frac{\partial v}{\partial t}, u - v\right\rangle dt + \int_Q a(x,t,u,\nabla u)(\nabla u - \nabla v)\,dx\,dt \le \int_0^T \langle f, u - v\rangle\,dt, \\ \forall v \in \mathcal{K} \cap \left\{v \in L^p\left(0,T;W_0^{1,p}(\Omega)\right) : \frac{\partial v}{\partial t} \in L^{p'}\left(0,T;W^{-1,p'}(\Omega)\right);\ v(0) = 0\right\}, \end{gathered} \tag{P}$$

where K is a given convex in $W_0^{1,p}(\Omega)$ and $f \in L^{p'}(0,T;W^{-1,p'}(\Omega))$.

It is well known that (P) admits at least one solution via a classical penalty method (see Lions [5] for $p \ge 2$ and Landes-Mustonen [4] for $1 < p < 2$). Recently in [6], the authors

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approximated (P) by the following sequence of parabolic equations:

$$\frac{\partial u_n}{\partial t} + A(u_n) + |h(x,u_n)|^{n-1} h(x,u_n)\,|G(x,t,u_n,\nabla u_n)| = f \quad \text{in } Q,$$
$$u_n(x,t) = 0 \quad \text{on } \partial Q, \tag{P_n}$$
$$u_n(x,0) = 0 \quad \text{in } \Omega,$$

where h and G are two Carathéodory functions satisfying some natural growth conditions. The obtained convex K depends on two obstacles constructed from h.

In the L^1 case, that is, $f \in L^1(\Omega\times]0,T[)$, the formulations (P) and (P_n) are not appropriate. So, we introduce the renormalized problem (R_n) associated to (P_n) (see the definition below). The study of the asymptotic behavior of (R_n) as $n \to \infty$ leads to some bilateral parabolic problem. Our approach allows us also to prove the existence of solutions for general parabolic inequalities of type

$$T_k(u) \in \mathcal{K},$$
$$\int_0^T \left\langle \frac{\partial v}{\partial t}, T_k(u-v) \right\rangle dt + \int_Q a(x,t,u,\nabla u)\nabla T_k(u-v)\,dx\,dt$$
$$+ \int_Q H(x,t,u,\nabla u) T_k(u-v)\,dx\,dt \le \int_Q f\,T_k(u-v)\,dx\,dt, \quad \forall v \in \mathcal{K} \cap D \cap L^\infty(Q), \tag{1.2}$$

where $D = \{v \in L^p(0,T;W_0^{1,p}(\Omega)),\ \partial v/\partial t \in L^{p'}(0,T,W_0^{-1,p'}(\Omega)) + L^1(Q),\ v(0) = 0\}$ and where H is a given Carathéodory function satisfying some natural growth assumption.

For some recent and classical results for some parabolic inequalities problems, the reader can refer to [2, 7, 9, 10].

2. Main result

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \ge 2$ and $1 < p < +\infty$.

We denote by Q the cylinder $\Omega \times (0,T)$ and $\Gamma = \partial Q$.

Let $A(u) = -\operatorname{div}(a(x,t,\nabla u))$ be a Leray-Lions operator defined on $L^p(0,T;W_0^{1,p}(\Omega))$ into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$, where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying for a.e. $x \in \Omega$, for all $t \in \mathbb{R}$ and for all $\zeta, \zeta' \in \mathbb{R}^N$, $(\zeta \ne \zeta')$ the following hold:

$$|a(x,t,\zeta)| \le \beta(k(x,t) + |\zeta|^{p-1}),$$
$$(a(x,t,\zeta) - a(x,t,\zeta'))(\zeta - \zeta') > 0, \tag{2.1}$$
$$a(x,t,\zeta)\zeta \ge \alpha|\zeta|^p,$$

with $\alpha > 0$, $\beta > 0$, $k \in L^{p'}(Q)$.

Furthermore, let $h : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$h(x,0) = 0, \qquad h(x,s) \text{ is nondecreasing with respect to } s. \tag{2.2}$$

G is a Carathéodory function satisfying the following assumptions:

$$|G(x,t,s,\xi)| \leq b(|s|)(c(x,t)+|\xi|^p), \qquad G(x,t,s,0)=0, \tag{2.3}$$

$$\begin{aligned}&\{v \in L^p(0,T;W_0^{1,p}(\Omega)) : G(x,t,v,\nabla v)=0 \text{ a.e. in } Q\}\\ &\quad\subset \{v \in L^p(0,T;W_0^{1,p}(\Omega)) : |h(x,v)| \leq 1 \text{ a.e. in } Q\}.\end{aligned} \tag{2.4}$$

Let us suppose

$$\begin{aligned}&\text{for almost } x \in \Omega \setminus \Omega_+^\infty \text{ there exists } \epsilon=\epsilon(x)>0 \text{ such that}\\ &\qquad h(x,s)>1, \quad \forall s \in \,]q_+(x),q_+(x)+\epsilon[,\\ &\text{for almost } x \in \Omega \setminus \Omega_-^\infty \text{ there exists } \epsilon=\epsilon(x)>0 \text{ such that}\\ &\qquad h(x,s)<-1, \quad \forall s \in \,]q_-(x)-\epsilon,q_-(x)[,\end{aligned} \tag{2.5}$$

where b is a continuous nondecreasing function and $c(x,t) \in L^1(Q)$, $c \geq 0$, and

$$\begin{aligned}q_+(x)&=\inf\{s>0,\ h(x,s)\geq 1\},\\ q_-(x)&=\sup\{s>0,\ h(x,s)\leq -1\},\\ \Omega_+^\infty&=\{x\in\Omega : q_+(x)=+\infty\},\\ \Omega_-^\infty&=\{x\in\Omega : q_-(x)=-\infty\}.\end{aligned} \tag{2.6}$$

We define for all s and k in $\mathbb{R}$, $k \geq 0$, $T_k(s)=\max(-k,\min(k,s))$.

We will say that u_n is a renormalized solution of (P_n) if

$$\begin{gathered}T_k(u_n) \in L^p(0,T;W_0^{1,p}(\Omega)), \quad \forall k>0,\\ \lim_{h\to\infty}\int_{\{h\leq |u_n|\leq h+1\}} a(x,t,\nabla u_n)\nabla u_n\,dx\,dt=0,\\ u_n \text{ satisfies in the distributional sense}\\ (A(u_n))_t-\operatorname{div}(a(x,t,\nabla u_n)A'(u_n))+a(x,t,\nabla u_n)\nabla u_n A''(u_n)\\ +|h(x,u_n)|^{n-1}h(x,u_n)|G(x,t,u_n,\nabla u_n)|A'(u_n)=fA'(u_n),\\ \forall A \in C^1(\mathbb{R}),\ A',A'' \in L^\infty(\Omega), \quad A' \text{ has a compact support and } u_n \text{ satisfies}\\ \text{the initial condition in the sense that } A(u_n)\in C([0,T],L^1(\Omega)).\end{gathered} \tag{R_n}$$

Thanks to [8, Theorem 3.2, page 164], there exists at least one solution u_n of (R_n).

Theorem 2.1. *Under the hypotheses (2.1)–(2.5), $f \in L^1(Q)$, the problem (P_n) has at least one renormalized solution (u_n) such that*

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{strongly in } L^p(0,T;W_0^{1,p}(\Omega)), \tag{2.7}$$

where u is a solution of the following obstacle problem:

$$q_-(x) \le u(x,t) \le q_+(x) \quad \text{a.e. } (x,t) \in Q,$$
$$T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega)),$$
$$\int_0^T \left\langle \frac{\partial v}{\partial t}, T_k(u-v) \right\rangle dt + \int_Q a(x,t,\nabla u)\nabla T_k(u-v)\,dx\,dt \qquad (R)$$
$$\le \int_Q f T_k(u-v)\,dx\,dt, \quad \forall v \in \mathcal{K} \cap D \cap L^\infty(Q),$$

where

$$D = \left\{ v \in L^p(0,T;W_0^{1,p}(\Omega)),\ \frac{\partial v}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q),\ v(0) = 0 \right\},$$
$$\mathcal{K} = \{ v \in L^p(0,T;W_0^{1,p}(\Omega)),\ v(t) \in K \}, \qquad K = \{ v \in W_0^{1,p}(\Omega),\ q_- \le v \le q_+ \}. \tag{2.8}$$

Moreover, if $q_-, q_+ \in L^\infty(\Omega)$, then $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$.

Remark 2.2. The same result can be obtained when dealing with general operator of Leray-Lions type depending also on u, that is, $A(u) = -\operatorname{div}(a(x,t,u,\nabla u))$.

Proof of Theorem 2.1.

Step 1. Let $A(t) = H_m(t)$, $H_m(t) = \int_0^t h_m(s)\,ds$, where

$$h_m(s) = \begin{cases} 1 & \text{if } |s| \le m, \\ \text{affine} & \text{if } m \le |s| \le m+1, \\ 0 & \text{if } m+1 \le |s|. \end{cases} \tag{2.9}$$

Taking now $T_k(H_m(u_n))$ as test function in (R_n), we obtain

$$\begin{aligned} &\int_0^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, T_k(H_m(u_n)) \right\rangle dt + \int_{|H_m(u_n)|<k} a(x,t,\nabla u_n)\nabla u_n h_m^2(u_n)\,dx\,dt \\ &\quad + \int_Q |h(x,u_n)|^{n-1} h(x,u_n)\,|G(x,t,u_n,\nabla u_n)|\,h_m(u_n) T_k(H_m(u_n))\,dx\,dt \\ &\quad + \int_Q a(\cdot,t,\nabla u_n)\nabla u_n h_m'(u_n) T_k(H_m(u_n))\,dx\,dt = \int_Q f h_m(u_n) T_k(H_m(u_n))\,dx\,dt. \end{aligned} \tag{2.10}$$

Since

$$\begin{aligned} &\int_0^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, T_k(H_m(u_n)) \right\rangle dt \\ &\quad = \int_\Omega \left(\int_0^{H_m(u_n(x,T))} T_k(s)\,ds \right) dx - \int_\Omega \left(\int_0^{H_m(u_n(x,0))} T_k(s)\,ds \right) dx \end{aligned} \tag{2.11}$$

and by using the fact that $\int_\Omega(\int_0^{H_m(u_n(x,T))} T_k(s)ds) \geq 0$, we obtain

$$\int_{\{|H_m(u_n)|<k\}} a(x,t,\nabla u_n)\nabla u_n h_m^2(u_n)\,dx\,dt \leq Ck + \int_{\{m\leq|u_n|\leq m+1\}} a(x,t,\nabla u_n)\nabla u_n\,dx\,dt,$$

$$\begin{aligned}&\int_Q |h(x,u_n)|^{n-1}h(x,u_n)\,|G(x,t,u_n,\nabla u_n)|\,h_m(u_n)T_k(H_m(u_n))\,dx\,dt\\ &\quad\leq Ck + \int_{\{m\leq|u_n|\leq m+1\}} a(x,t,\nabla u_n)\nabla u_n\,dx\,dt.\end{aligned} \tag{2.12}$$

We have $H_m(s)$ (resp., $h_m(s)$) tends to s (resp., to 1) as m goes to $+\infty$.

Using Fatou's lemma and the definition of the renormalized solution leads to

$$\int_Q |\nabla T_k(u_n)|^p\,dx\,dt \leq Ck, \tag{2.13}$$

$$\int_Q |h(x,u_n)|^{n-1}h(x,u_n)\,|G(x,t,u_n,\nabla u_n)|\,T_k(u_n)\,dx\,dt \leq Ck, \tag{2.14}$$

which gives

$$\int_Q |h(x,u_n)|^n\,|G(x,t,u_n,\nabla u_n)|\,\frac{|T_k(u_n)|}{k}\,dx\,dt \leq C, \tag{2.15}$$

and as $k \to 0$ we obtain

$$\int_Q |h(x,u_n)|^n\,|G(x,t,u_n,\nabla u_n)|\,dx\,dt \leq C. \tag{2.16}$$

Choosing now a C^2 function ρ_k, such that $\rho_k(s) = s$ for $|s| \leq k$ and $2k\operatorname{sign}(s)$ for $|s| > 2k$, we get

$$\begin{aligned}&(\rho_k(u_n))_t - \operatorname{div}(a(x,t,\nabla u_n)\rho_k'(u_n)) + a(x,t,\nabla u_n)\nabla u_n\rho_k''(u_n)\\ &\quad + |h(x,u_n)|^{n-1}h(x,u_n)\,|G(x,t,u_n,\nabla u_n)|\,\rho_k'(u_n) = f\rho_k(u_n).\end{aligned} \tag{2.17}$$

We deduce that $(\rho_k(u_n))_t$ is bounded in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))$.

Now thanks to the following result.

Lemma 2.3 [11]. *Let $p > 1$. If (u_n) is a bounded sequence of $L^p(0,T;W_0^{1,p}(\Omega))$ such that $\partial u_n/\partial t$ is bounded in $L^1 + L^{p'}(0,T;W^{-1,p'}(\Omega))$, then u_n is relatively compact in $L^p(Q)$.*

We deduce that $\rho_k(u_n)$ is relatively compact in $L^p(Q)$ and so there exists a measurable function u such that $u_n \to u$ a.e. in Q.

Finally, we deduce from (2.13) that $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $L^p(0,T;W_0^{1,p}(\Omega))$, and strongly in $L^p(Q)$.

Step 2. We are dealing now with the almost convergence of the gradient.

We have to prove that, for $0<\theta<1$,

$$\lim_{n\to\infty}\int_Q \left(\left[a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u))\right]\left[\nabla T_k(u_n) - \nabla T_k(u)\right]\right)^\theta dx\,dt = 0. \tag{2.18}$$

Let $\omega \in L^p(0,T;W_0^{1,p}(\Omega))$, we define for any $\mu>0, \omega_\mu$ the time regularization of ω,

$$\omega_\mu(x,t) = \mu\int_{-\infty}^{t} \overline{\omega}(x,s)\exp\left(\mu(s-t)\right)ds, \tag{2.19}$$

where $\overline{\omega}$ is the zero extension of ω for $s>T$. Furthermore, ω_μ satisfies the following properties (see [3]):

$$\begin{gathered}\omega_\mu \longrightarrow \omega \quad \text{strongly in } L^p\left(0,T;W_0^{1,p}(\Omega)\right),\\ \frac{\partial\omega_\mu}{\partial t} = \mu(\omega-\omega_\mu) \quad \text{in the distributional sense.}\end{gathered} \tag{2.20}$$

Letting $\eta>0$, we obtain

$$\begin{aligned}&\int_Q \left(\left[a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u))\right]\left[\nabla T_k(u_n) - \nabla T_k(u)\right]\right)^\theta\\ &\quad\le C\,\text{meas}\left\{\left|T_k(u_n) - T_k(u)_\mu\right| \ge \eta\right\}^{1-\theta}\\ &\qquad + C\Bigg(\int_{\{|T_k(u_n)-T_k(u)_\mu|<\eta\}}\left[a(x,t,\nabla T_k(u_n))\right.\\ &\qquad\qquad \left. - a(x,t,\nabla T_k(u))\right]\left[\nabla T_k(u_n) - \nabla T_k(u)\right]\Bigg)^\theta.\end{aligned} \tag{2.21}$$

On the other hand, we have

$$\begin{aligned}
&\int_{\{|T_k(u_n)-T_k(u)_\mu|<\eta\}} [a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\,dx\,dt\\
&\quad\leq \int_{\{|T_k(u_n)-T_k(u)_\mu|<\eta\}} [a(x,t,\nabla T_k(u_n))\\
&\qquad\qquad - a(x,t,\nabla T_k(u)_\mu)][\nabla T_k(u_n) - \nabla T_k(u)_\mu]\,dx\,dt\\
&\qquad + \int_{\{|T_k(u_n)-T_k(u)_\mu|<\eta\}} a(x,t,\nabla T_k(u_n))(\nabla T_k(u)_\mu - \nabla T_k(u))\,dx\,dt\\
&\qquad + \int_{\{|T_k(u_n)-T_k(u)_\mu|<\eta\}} [a(x,t,\nabla T_k(u)_\mu) - a(x,t,\nabla T_k(u))]\nabla T_k(u_n)\,dx\,dt\\
&\qquad - \int_{\{|T_k(u_n)-T_k(u)_\mu|<\eta\}} a(x,t,\nabla T_k(u)_\mu)\nabla T_k(u)_\mu\,dx\,dt\\
&\qquad + \int_{\{|T_k(u_n)-T_k(u)_\mu|<\eta\}} a(x,t,\nabla T_k(u))\nabla T_k(u)\,dx\,dt\\
&\quad\leq I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{2.22}$$

Take $T_\eta(H_m(u_n) - T_k(u)_\mu)$ as test function in (R_n) with $A(t) = H_m(t)$. We obtain

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, T_\eta(H_m(u_n) - T_k(u)_\mu)\right\rangle dt\\
&\qquad + \int_{\{|H_m(u_n)-T_k(u)_\mu|<\eta\}} a(x,t,\nabla u_n)\nabla u_n h_m^2(u_n) - \nabla T_k(u)_\mu h_m(u_n)\,dx\,dt\\
&\qquad + \int_Q |h(x,u_n)|^{n-1} h(x,u_n)\,|G(x,t,u_n,\nabla u_n)|\,h_m(u_n) T_\eta(H_m(u_n) - T_k(u)_\mu)\,dx\,dt\\
&\qquad + \int_Q a(x,t,\nabla u_n)\nabla u_n h_m'(u_n) T_\eta(H_m(u_n) - T_k(u)_\mu)\,dx\,dt\\
&\quad = \int_Q f h_m(u_n) T_\eta(H_m(u_n) - T_k(u)_\mu)\,dx\,dt.
\end{aligned} \tag{2.23}$$

We have

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, T_\eta(H_m(u_n) - T_k(u)_\mu)\right\rangle dt\\
&\quad = \int_0^T \left\langle \frac{\partial H_m(u_n) - T_k(u)_\mu}{\partial t}, T_k(H_m(u_n) - T_k(u)_\mu)\right\rangle dt\\
&\qquad + \int_0^T \left\langle \frac{\partial T_k(u)_\mu}{\partial t}, T_k(H_m(u_n) - T_k(u)_\mu)\right\rangle dt.
\end{aligned} \tag{2.24}$$

Using the fact that

$$\int_0^T \left\langle \frac{\partial H_m(u_n) - T_k(u)_\mu}{\partial t}, T_\eta(H_m(u_n) - T_k(u)_\mu) \right\rangle dt \geq 0,$$

$$\begin{aligned} &\int_0^T \left\langle \frac{\partial T_k(u)_\mu}{\partial t}, T_k(H_m(u_n) - T_k(u)_\mu) \right\rangle dt \\ &\quad = \mu \int_Q (T_k(u) - T_k(u)_\mu) T_\eta(H_m(u_n) - T_k(u)_\mu)\, dx\, dt, \end{aligned} \tag{2.25}$$

consequently,

$$\begin{aligned} &\limsup_{n\to\infty} \limsup_{m\to\infty} \int_0^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, T_\eta(H_m(u_n) - T_k(u)_\mu) \right\rangle dt \\ &\quad \geq \mu \int_Q (T_k(u) - T_k(u)_\mu) T_\eta(u - T_k(u)_\mu)\, dx\, dt = \epsilon(m,n) \geq 0. \end{aligned} \tag{2.26}$$

This implies that

$$\begin{aligned} &\int_{\{|H_m(u_n) - T_k(u)_\mu| < \eta\}} a(x,t,\nabla u_n)\nabla u_n h_m^2(u_n) - \nabla T_k(u)_\mu h_m(u_n)\, dx\, dt \\ &\quad + \int_Q |h(x,u_n)|^{n-1} h(x,u_n) |G(x,t,u_n,\nabla u_n)| h_m(u_n) T_\eta(H_m(u_n) - T_k(u)_\mu)\, dx\, dt \\ &\quad + \int_Q a(x,t,\nabla u_n)\nabla u_n h_m'(u_n) T_\eta(H_m(u_n) - T_k(u)_\mu)\, dx\, dt \\ &\quad \leq \int_Q f h_m(u_n) T_\eta(H_m(u_n) - T_k(u)_\mu)\, dx\, dt + \epsilon(m,n), \end{aligned} \tag{2.27}$$

which gives by using the fact that

$$\begin{aligned} &\int_Q |h(x,u_n)|^{n-1} h(x,u_n) |G(x,t,u_n,\nabla u_n)| h_m(u_n) T_\eta(H_m(u_n) - T_k(u)_\mu)\, dx\, dt \leq C\eta, \\ &\int_{\{|H_m(u_n) - T_k(u)_\mu| < \eta\}} a(x,t,\nabla u_n)\nabla u_n h_m^2(u_n) - \nabla T_k(u)_\mu h_m(u_n)\, dx\, dt \\ &\quad \leq C\eta + \epsilon(m,n) + \eta \int_{\{m \leq |u_n| \leq m+1\}} a(x,t,\nabla u_n)\nabla u_n\, dx\, dt, \end{aligned} \tag{2.28}$$

which gives as $m \to \infty$,

$$\int_{\{|u_n - T_k(u)_\mu| < \eta\}} a(x,t,\nabla u_n)\nabla u_n - \nabla T_k(u)_\mu\, dx\, dt \leq C\eta + \epsilon(n). \tag{2.29}$$

Finally from (2.22),

$$|I_1| \leq C\eta + \epsilon(n) - \int_{\{|T_k(u_n)-T_k(u)_\mu|<\eta\}} a(x,t,\nabla T_k(u)_\mu)(\nabla T_k(u_n) - \nabla T_k(u)). \tag{2.30}$$

Since $a(x,t,\nabla T_k(u)_\mu)\chi_{\{|T_k(u_n)-T_k(u)_\mu|<\eta\}} \to a(x,t,\nabla T_k(u)_\mu)\chi_{\{|T_k(u)-T_k(u)_\mu|<\eta\}}$ in $L^{p'}(Q)$ and $T_k(u_n) \to T_k(u)$ weakly in $L^p(0,T;W_0^{1,p}(\Omega))$, then

$$\begin{aligned} &-\int_{\{|T_k(u_n)-T_k(u)_\mu|<\eta\}} a(x,t,\nabla T_k(u)_\mu)(\nabla T_k(u_n) - \nabla T_k(u))\,dx\,dt \\ &\quad = -\int_{\{|T_k(u)-T_k(u)_\mu|<\eta\}} a(x,t,\nabla T_k(u)_\mu)(\nabla T_k(u) - \nabla T_k(u))\,dx\,dt + \epsilon(n). \end{aligned} \tag{2.31}$$

So

$$|I_1| \leq C\eta + \epsilon(n). \tag{2.32}$$

For what concerns the term I_2, one has

$$I_2 = \epsilon(n,\mu), \tag{2.33}$$

since

$$\begin{aligned} &a(x,t,\nabla T_k(u_n))\chi_{\{|T_k(u_n)-T_k(u)_\mu|<\eta\}} \longrightarrow a(x,t,\nabla T_k(u))\chi_{\{|T_k(u)-T_k(u)_\mu|<\eta\}} \quad \text{in } (L^{p'}(Q))^N, \\ &(\nabla T_k(u)_\mu - \nabla T_k(u))\chi_{\{|T_k(u_n)-T_k(u)_\mu|<\eta\}} \longrightarrow (\nabla T_k(u)_\mu - \nabla T_k(u))\chi_{\{|T_k(u)-T_k(u)_\mu|<\eta\}}. \end{aligned} \tag{2.34}$$

In the same way, we show that

$$I_3 = \epsilon(n,\mu), \qquad I_4 = \epsilon(n,\mu), \qquad I_5 = \epsilon(n,\mu). \tag{2.35}$$

Combining the above estimates, we get

$$\lim_{n\to\infty} \int_Q \left([a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\right)^\theta dx\,dt = 0. \tag{2.36}$$

Then there exists a subsequence also denoted by (u_n) such that

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } Q. \tag{2.37}$$

Step 3. From (2.16), we deduce that

$$\int_Q |h(x,u_n)|^n |G(x,t,u_n,\nabla u_n)|\,dx\,dt \leq C, \tag{2.38}$$

which gives for every $\beta > 0$,

$$\int_{|h(x,T_\beta(u_n))|>k} |G(x,t,T_\beta(u_n),\nabla T_\beta(u_n))|\,dx\,dt \leq \frac{C}{k^n}, \tag{2.39}$$

where $k > 1$. Letting $n \to +\infty$ for k fixed, we deduce by using Fatou's lemma

$$\int_{|h(x,T_\beta(u))|>k} |G(x,t,T_\beta(u),\nabla T_\beta(u))|\,dx\,dt = 0, \tag{2.40}$$

and so, by (2.4)

$$|h(x,T_\beta(u))| \le 1 \quad \text{a.e. in } Q. \tag{2.41}$$

So

$$q_-(x) \le T_\beta(u(x)) \le q_+(x) \quad \text{a.e. in } Q. \tag{2.42}$$

Letting now $\beta \to +\infty$, we deduce also that

$$q_-(x) \le u(x) \le q_+(x) \quad \text{a.e. in } Q. \tag{2.43}$$

Step 4. Strong convergence of the truncations.

We will prove that

$$\lim_{n\to\infty}\int_Q [a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\,dx\,dt = 0. \tag{2.44}$$

Fix $k > 0$ and let $\varphi(s) = \exp(\delta s^2)$, $\delta > 0$. Let $l > k$ and define the function $R_l(s) = \int_0^s \rho_l(t)dt$. Let us consider $\omega_\mu^m = T_k(H_m(u)_\mu)$, where v_μ is the mollification with respect to time v. Letting $v_\mu^{m,n} = \rho_l(H_m(u_n))\varphi(T_k(H_m(u_n)) - \omega_\mu^m)$ as test function in the problem (R_n), we get

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, \rho_l(H_m(u_n))\varphi(T_k(H_m(u_n)) - \omega_\mu^m)\right\rangle dt \\
&\qquad + \int_Q a(x,t,\nabla u_n)\nabla u_n h^2(u_n)\rho_l'(H_m(u_n))\varphi(T_k(H_m(u_n)) - \omega_\mu^m)\,dx\,dt \\
&\qquad + \int_Q a(x,t,\nabla u_n)(\nabla T_k(H_m(u_n)) - \nabla \omega_\mu^m) \\
&\qquad \times h_m(u_n)\rho_l(H_m(u_n))\varphi'(T_k(H_m(u_n)) - \omega_\mu^m)\,dx\,dt \\
&\qquad + \int_Q a(x,t,\nabla u_n)\nabla u_n h_m'(u_n)\rho_l(H_m(u_n))\varphi(T_k(H_m(u_n)) - \omega_\mu^m)\,dx\,dt \\
&\qquad + \int_Q |h(x,u_n)|^{n-1}h(x,u_n)\,|G(x,t,u_n,\nabla u_n)| \\
&\qquad \times h_m(u_n)\rho_l(H_m(u_n))\varphi(T_k(H_m(u_n)) - \omega_\mu^m)\,dx\,dt \\
&\quad = \int_Q f v_\mu^{m,n} h_m(u_n)\,dx\,dt.
\end{aligned} \tag{2.45}$$

We deal now with the estimate of each term of the last equalities.

Since $H_m(u_n) \in L^p(0,T;W_0^{1,p}(\Omega))$ and $\partial H_m(u_n)/\partial t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$, there exists a smooth function $H_m(u_n)_\sigma$ such that as $\sigma \to 0$,

$$\begin{gathered} H_m(u_n)_\sigma \longrightarrow H_m(u_n) \quad \text{strongly in } L^p(0,T;W_0^{1,p}(\Omega)), \\ \frac{\partial H_m(u_n)_\sigma}{\partial t} \longrightarrow \frac{\partial H_m(u_n)}{\partial t} \quad \text{strongly in } L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q). \end{gathered} \tag{2.46}$$

This implies that

$$\begin{aligned} I &= \int_0^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, \rho_l(H_m(u_n))\varphi(T_k(H_m(u_n)) - \omega_\mu^m) \right\rangle dt \\ &= \lim_{\sigma\to 0^+} \int_Q (H_m(u_n)_\sigma)' \rho_l(H_m(u_n)_\sigma)\varphi(T_k(H_m(u_n)_\sigma) - \omega_\mu^m)\,dx\,dt \\ &= \lim_{\sigma\to 0^+} \int_Q [R_l(H_m(u_n)_\sigma)]' \varphi(T_k(H_m(u_n)_\sigma) - \omega_\mu^m)\,dx\,dt \\ &= \lim_{\sigma\to 0^+} \int_Q [R_l(H_m(u_n)_\sigma) - T_k(H_m(u_n)_\sigma)]' \varphi(T_k(H_m(u_n)_\sigma) - \omega_\mu^m)\,dx\,dt \\ &\quad + \int_Q [T_k(H_m(u_n)_\sigma)]' \varphi(T_k(H_m(u_n)_\sigma) - \omega_\mu^m)\,dx\,dt \\ &= \lim_{\sigma\to 0^+} \bigg\{ \int_\Omega [R_l(H_m(u_n)_\sigma) - T_k(H_m(u_n)_\sigma)\varphi(T_k(H_m(u_n)_\sigma) - \omega_\mu^m)]_0^T\,dx\,dt \\ &\qquad - \int_Q [R_l(H_m(u_n)_\sigma) - T_k(H_m(u_n)_\sigma)]' \varphi'(T_k(H_m(u_n)_\sigma) - \omega_\mu^m)(T_k(H_m(u_n)_\sigma) \\ &\qquad - \omega_\mu^m)'\,dx\,dt + \int_Q [T_k(H_m(u_n)_\sigma)]' \varphi(T_k(H_m(u_n)_\sigma) - \omega_\mu^m)\,dx\,dt \bigg\} \\ &= \lim_{\sigma\to 0^+} \{I_1(\sigma) + I_2(\sigma) + I_3(\sigma)\}. \end{aligned} \tag{2.47}$$

Observe that for $|s| \le k$ we have $R_l(s) = T_k(s) = s$ and for $|s| > k$ we have $|R_l(s)| \ge |T_k(s)|$ and, since both $R_l(s)$ and $T_k(s)$ have the same sign of s, we deduce that $\operatorname{sign}(s)(R_l(s) - T_k(s)) \ge 0$. Consequently,

$$I_1(\sigma) = \int_{\{|H_m(u_n)_\sigma| > k\}} [R_l(H_m(u_n)_\sigma) - T_k(H_m(u_n)_\sigma)\varphi(T_k(H_m(u_n)_\sigma) - \omega_\mu^m)]_0^T\,dx\,dt \ge 0. \tag{2.48}$$

We have, since $(R_l(s) - T_k(s))(T_k(s))' = 0$, for all s,

$$I_2(\sigma) = \int_{\{|H_m(u_n)_\sigma|>k\}} \big[R_l(H_m(u_n)_\sigma) - T_k(H_m(u_n)_\sigma)\varphi'(T_k(H_m(u_n)_\sigma) - \omega_\mu^m)\big](\omega_\mu^m)'\,dx\,dt$$
$$= \mu \int_{\{|H_m(u_n)_\sigma|>k\}} \big[R_l(H_m(u_n)_\sigma) - T_k(H_m(u_n)_\sigma)\varphi'(T_k(H_m(u_n)_\sigma) - \omega_\mu^m)\big](T_k(H_m(u_n)_\sigma) - \omega_\mu^m)\,dx\,dt, \tag{2.49}$$

by using the fact that $\varphi' \geq 0$ and that

$$(R_l(H_m(u_n)_\sigma) - T_k(H_m(u_n)_\sigma))(T_k(H_m(u_n)_\sigma) - \omega_\mu^m)\chi_{\{|H_m(u_n)_\sigma|>k\}} \geq 0, \tag{2.50}$$

the last integral is of the form $\epsilon(m,n)$. We deduce that

$$\limsup_{\sigma\to 0^+} I_2(\sigma) \geq \epsilon(m,n). \tag{2.51}$$

For $I_3(\sigma)$, one has

$$I_3(\sigma) = \int_Q \big[T_k(H_m(u_n)_\sigma)\big]'\varphi(T_k(H_m(u_n)_\sigma) - \omega_\mu^m)\,dx\,dt$$
$$= \int_Q \big[T_k(H_m(u_n)_\sigma) - \omega_\mu^m\big]'\varphi(T_k(H_m(u_n)_\sigma) - \omega_\mu^m)\,dx\,dt$$
$$+ \int_Q (\omega_\mu^m)'\varphi(T_k(H_m(u_n)_\sigma) - \omega_\mu^m)\,dx\,dt. \tag{2.52}$$

Let $\Phi(s) = \int_0^s \varphi(t)dt$. Remark that $(T_k(H_m(u_n)_\sigma) - \omega_\mu^m)\varphi(T_k(H_m(u_n)_\sigma) - \omega_\mu^m) \geq 0$.

Integrating by parts, using the fact that $\Phi \geq 0$, and following the same way as above, we have

$$\limsup_{\sigma\to 0^+} I_3(\sigma) \geq \epsilon(m,n). \tag{2.53}$$

Combining these estimates, we conclude that

$$\int_0^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, \rho_l(H_m(u_n))\varphi(T_k(H_m(u_n)) - \omega_\mu^m) \right\rangle dt \geq \epsilon(m,n). \tag{2.54}$$

We set

$$I_4(m) = \int_Q |h(x,u_n)|^{n-1} h(x,u_n)\rho_l(H_m(u_n))\varphi(T_k(H_m(u_n)_\sigma) - \omega_\mu^m)|G(x,t,u_n,\nabla u_n)|, \tag{2.55}$$

so we have

$$\limsup_{m\to\infty} I_4(m) \geq I_4^1 + I_4^2, \tag{2.56}$$

where

$$\begin{aligned} I_4^1 &= \int_{\{|u_n|<k, 0\leq u_n \leq T_k(u)_\mu\}} \\ &\quad \times |h(x,u_n)|^{n-1} h(x,u_n)\varphi(T_k(u_n) - T_k(u)_\mu)\rho_l(u_n)|G(x,t,u_n,\nabla u_n)|\,dx\,dt, \\ I_4^2 &= \int_{\{|u_n|<k, T_k(u)_\mu \leq u_n \leq 0\}} \\ &\quad \times |h(x,u_n)|^{n-1} h(x,u_n)\varphi(T_k(u_n) - T_k(u)_\mu)\rho_l(u_n)|G(x,t,u_n,\nabla u_n)|\,dx\,dt. \end{aligned} \tag{2.57}$$

Since $q_- \leq T_k(u)_\mu \leq q_+$ (recall that $q_- \leq T_k(u) \leq q_+$) and $0 \leq \rho_l(u_n) \leq 1$, one easily has

$$\begin{aligned} |I_4^1| &\leq \int_{\{|u_n|<k\}} c(x,t)|\varphi(T_k(u_n) - T_k(u)_\mu)| \\ &\quad + \frac{b(k)}{\alpha}\int_{\{|u_n|<k\}} |\nabla u_n|^p |\varphi(T_k(u_n) - T_k(u)_\mu)| \\ &\leq \frac{b(k)}{\alpha}\int_Q [a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u)_\mu)][\nabla T_k(u_n) - \nabla T_k(u)_\mu] \\ &\quad \times |\varphi(T_k(u_n) - T_k(u)_\mu)|\,dx\,dt + \epsilon(n,\mu), \end{aligned} \tag{2.58}$$

and also we have the same estimation of I_4^2.

Then

$$\begin{aligned} |I_4^1| + |I_4^2| &\leq 2\frac{b(k)}{\alpha}\int_Q [a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u)_\mu)][\nabla T_k(u_n) - \nabla T_k(u)_\mu] \\ &\quad \times |\varphi(T_k(u_n) - T_k(u)_\mu)|\,dx\,dt + \epsilon(n,\mu). \end{aligned} \tag{2.59}$$

By denoting by J_1 the third term of (2.45), one can write

$$\begin{aligned}
J_1 = &\int_Q a(x,t,\nabla u_n)(\nabla T_k(H_m(u_n)) - \nabla T_k(H_m(u))_\mu) \\
&\quad \times h_m(u_n)\rho_l(H_m(u_n))\varphi'(T_k(H_m(u_n)) - T_k(H_m(u))_\mu)\,dx\,dt \\
= &\int_Q a(x,t,\nabla T_k(u_n))(\nabla T_k(H_m(u_n)) - \nabla T_k(H_m(u))_\mu)h_m(u_n) \\
&\quad \times \rho_l(H_m(u_n))\varphi'(T_k(H_m(u_n)) - T_k(H_m(u))_\mu)\,dx\,dt \\
&+ \int_{\{|u_n|>k\}} a(x,t,\nabla u_n)(\nabla T_k(H_m(u_n)) - \nabla T_k(H_m(u))_\mu)h_m(u_n) \\
&\qquad \times \rho_l(H_m(u_n))\varphi'(T_k(H_m(u_n)) - T_k(H_m(u))_\mu)\,dx\,dt, \\
J_1 = &\int_Q a(x,t,\nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(u)_\mu)\rho_l(u_n)\varphi'(T_k(u_n) - T_k(u)_\mu)\,dx\,dt \\
&- \int_{\{|u_n|>k\}} \rho_l(u_n)a(x,t,\nabla u_n)(\nabla T_k(u)_\mu)\varphi'(T_k(u_n) - T_k(u)_\mu)\,dx\,dt + \epsilon(m).
\end{aligned} \tag{2.60}$$

Since $a(x,t,\nabla u_n)\rho_l(u_n)$ is bounded in $L^{p'}(Q)$, we deduce that

$$a(x,t,\nabla u_n)\rho_l(u_n) \longrightarrow a(x,t,\nabla u)\rho_l(u) \quad \text{weakly in } L^{p'}(Q), \tag{2.61}$$

and so

$$\begin{aligned}
J_1 = &\int_Q (a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u)_\mu))(\nabla T_k(u_n) - \nabla T_k(u)_\mu) \\
&\quad \times \varphi'(T_k(u_n) - T_k(u)_\mu)\,dx\,dt + \epsilon(m,n,\mu).
\end{aligned} \tag{2.62}$$

Concerning the second term of (2.45), one easily has

$$\begin{aligned}
&\int_Q a(x,t,\nabla u_n)\nabla u_n h_m^2(u_n)\rho_l'(H_m(u_n))\varphi(T_k(H_m(u_n)) - \omega_\mu^m)\,dx\,dt \\
&\quad \le \varphi(2k)\int_{\{l\le|u_n|<l+1\}} a(x,t,\nabla u_n)\nabla u_n\,dx\,dt,
\end{aligned} \tag{2.63}$$

and since

$$\int_{\{l\le|u_n|<l+1\}} a(x,t,\nabla u_n)\nabla u_n\,dx\,dt \le \int_{|u_n|>l} |f|\,dx\,dt, \tag{2.64}$$

we deduce that

$$\left| \int_Q a(x,t,\nabla u_n)\nabla u_n h_m^2(u_n)\rho_l'(H_m(u_n))\varphi(T_k(H_m(u_n)) - \omega_\mu^m)\,dx\,dt \right| \le \varphi(2k)\int_{|u_n|>l} |f|\,dx\,dt = \epsilon(n,l). \tag{2.65}$$

The same result can be obtained for the fourth term of (2.45).

Combining (2.45)–(2.65), using the fact that $\phi' - 2(b(k)/\alpha)|\phi| \ge 1/2$ for $\delta \ge (b(k)/\alpha)^2$, we deduce that

$$\lim_{n\to\infty}\int_Q [a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u)_\mu)][\nabla T_k(u_n) - \nabla T_k(u)_\mu]\,dx\,dt = 0. \tag{2.66}$$

On the other hand, we have

$$\begin{aligned}
&\int_Q [a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\,dx\,dt \\
&\qquad - \int_Q [a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u)_\mu)][\nabla T_k(u_n) - \nabla T_k(u)_\mu]\,dx\,dt \\
&\quad = \int_Q a(x,t,\nabla T_k(u_n))(\nabla T_k(u)_\mu - \nabla T_k(u))\,dx\,dt \\
&\qquad - \int_Q a(x,t,\nabla T_k(u))(\nabla T_k(u_n) - \nabla T_k(u))\,dx\,dt \\
&\qquad + \int_Q a(x,t,\nabla T_k(u)_\mu)(\nabla T_k(u_n) - \nabla T_k(u)_\mu)\,dx\,dt = \epsilon(n,\mu).
\end{aligned} \tag{2.67}$$

Consequently by [1, Lemma 5], we obtain

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{strongly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ for every } k>0. \tag{2.68}$$

Step 5 (passage to the limit). Letting $v \in D \cap \mathcal{H} \cap L^\infty(Q)$, and using $T_k(H_m(u_n) - \theta v)$ as test function in the problem (R_n), we obtain

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, T_k(H_m(u_n) - \theta v)\right\rangle dt + \int_Q a(x,t,\nabla u_n)\nabla T_k(H_m(u_n) - \theta v)h_m(u_n)\,dx\,dt \\
&\qquad + \int_Q a(x,t,\nabla u_n)\nabla u_n T_k(H_m(u_n) - \theta v)h_m'(u_n)\,dx\,dt \\
&\qquad + \int_Q |h(x,u_n)|^{n-1}h(x,u_n)\,|G(x,t,u_n,\nabla u_n)|\,h_m(u_n)T_k(H_m(u_n) - \theta v)\,dx\,dt \\
&\quad \le \int_Q f\,T_k(H_m(u_n) - \theta v)h_m(u_n)\,dx\,dt.
\end{aligned} \tag{2.69}$$

We have

$$\begin{aligned}\int_Q |h(x,u_n)|^{n-1}h(x,u_n)|G(x,t,u_n,\nabla u_n)|h_m(u_n)T_k(H_m(u_n)-\theta v)\,dx\,dt \\ \geq \int_{\{0\leq H_m(u_n)\leq \theta v\}} |h(x,u_n)|^{n-1} \\ \times h(x,u_n)|G(x,t,u_n,\nabla u_n)|h_m(u_n)T_k(H_m(u_n)-\theta v)\,dx\,dt \\ + \int_{\{\theta v\leq H_m(u_n)\leq 0\}} |h(x,u_n)|^{n-1} \\ \times h(x,u_n)|G(x,t,u_n,\nabla u_n)|h_m(u_n)T_k(H_m(u_n)-\theta v)\,dx\,dt.\end{aligned} \tag{2.70}$$

Now we deal with the estimation of the last two terms in the right-hand side of the last inequality which we denote, respectively, by $J_1'(m,n)$ and $J_2'(m,n)$. Let us define

$$\delta_1(x,t) = \sup_{0\leq s\leq \theta v} h(x,s), \tag{2.71}$$

then we get $0 \leq \delta_1(x,t) < 1$ a.e. in Q.

We have

$$\begin{aligned}\limsup_{m\to\infty} |J_1'(m,n)| &\leq k\int_{\{0\leq u_n\leq \theta v\}} (\delta(x,t))^n(c(x,t)+|\nabla u_n|^p) \\ &\leq \int_{\{|u_n|\leq \|v\|_\infty\}} (\delta(x,t))^n(c(x,t)+|\nabla u_n|^p),\end{aligned} \tag{2.72}$$

and by using the strong convergence of $T_{\|v\|_\infty}(u_n)$ in $L^p(0,T;W_0^{1,p}(\Omega))$, we deduce that

$$\limsup_{n\to\infty}\limsup_{m\to\infty} |J_1'(m,n)| = 0, \tag{2.73}$$

with the same technique (taking $\delta_2(x,t) = \inf_{\theta v\leq s\leq 0} h(x,s)$), we can see that

$$\limsup_{m\to\infty} |J_2'(n,m)| \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty. \tag{2.74}$$

On the other hand,

$$\begin{aligned}\int_Q a(x,t,\nabla u_n)\nabla T_k(H_m(u_n)-\theta v)h_m(u_n)\,dx\,dt \\ = \int_Q a(x,t,\nabla u_n)\nabla(H_m(u_n)-\theta v)\chi_{\{|H_m(u_n)-\theta v|\leq k\}}h_m(u_n)\,dx\,dt \\ = \int_Q (a(x,t,\nabla u_n)-a(x,t,\theta\nabla v))\nabla(H_m(u_n)-\theta v)\chi_{\{|H_m(u_n)-\theta v|\leq k\}}h_m(u_n)\,dx\,dt \\ + \int_Q a(x,t,\theta\nabla v)\nabla(H_m(u_n)-\theta v)\chi_{\{|H_m(u_n)-\theta v|\leq k\}}h_m(u_n)\,dx\,dt.\end{aligned} \tag{2.75}$$

Since $a(x,t,\theta v)$ belongs to $(L^{p'}(Q))^N$, using Fatou's lemma in the first term of the last side gives

$$\liminf_{n,m\to+\infty}\int_0^T \langle Au_n, T_k(H_m(u_n)-\theta v)\rangle dt \geq \int_0^T \langle Au, T_k(u)-\theta v\rangle dt. \tag{2.76}$$

Go back to (2.69) and pass to the limit as $m,n\to\infty$ to obtain

$$\int_0^T \left\langle \theta\frac{\partial v}{\partial t}, T_k(u-\theta\nabla v)\right\rangle dt + \int_Q a(x,t,\nabla u)\nabla T_k(u-\theta v)\,dx\,dt \leq \int_Q f\,T_k(u-\theta v)\,dx\,dt. \tag{2.77}$$

Letting now θ tend to 1, we get

$$\int_0^T \left\langle \frac{\partial v}{\partial t}, T_k(u-\theta v)\right\rangle dt + \int_Q a(x,t,\nabla u)\nabla T_k(u-v)\,dx\,dt \leq \int_Q f\,T_k(u-v)\,dx\,dt, \tag{2.78}$$

which completes the proof. □

Remark 2.4. The same technique allows us to prove an existence result for solutions of the following parabolic inequalities:

$$\begin{gathered} q_-(x) \leq u(x,t) \leq q_+(x) \quad \text{a.e. in } Q, \\ T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega)), \end{gathered}$$

$$\begin{aligned} &\int_0^T \left\langle \frac{\partial v}{\partial t}, T_k(u-v)\right\rangle dt + \int_Q a(x,t,\nabla u)\nabla T_k(u-v)\,dx\,dt + \int_Q H(x,t,u,\nabla u)T_k(u-v)\,dx\,dt \\ &\qquad \leq \int_Q f\,T_k(u-v)\,dx\,dt, \quad \forall v\in \mathcal{K}\cap D\cap L^\infty(Q), \end{aligned} \tag{2.79}$$

where H is a given Carathéodory function satisfying, for all $(s,\zeta)\in\mathbb{R}\times\mathbb{R}^N$ and a.e. $(x,t)\in Q$, the following conditions:

$$\begin{gathered} |H(x,t,s,\zeta)| \leq \lambda(|s|)(\delta(x,t)+|\zeta|^p), \\ H(x,t,s,\zeta)s \geq 0, \end{gathered} \tag{2.80}$$

with $\lambda:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous increasing function and $\delta(x,t)$ is a given positive function in $L^1(Q)$.

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