

# ON THE 2-ORTHOGONAL POLYNOMIALS AND THE GENERALIZED BIRTH AND DEATH PROCESSES

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We discuss the connections between the 2-orthogonal polynomials and the generalized birth and death processes. Afterwards, we find the sufficient conditions to give an integral representation of the transition probabilities from these processes.

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## 1. Introduction

The birth and death processes are closely related to the orthogonal polynomials. The latter allows determining the stochastic matrix associated with these processes.

Let us also note that these processes are stationary Markov processes whose state space is the nonnegative integers.

Many authors treated the question of the existing relationship between the birth and death processes and the orthogonal polynomials, in particular, in the works of Karlin and McGregor [6] and Ismail et al. [5].

The properties of these processes and of the orthogonal polynomials were the subject of other works, we can quote by the way of example Ismail et al. [3, 4], Maki [8], and Letessier and Valent [7].

In this paper, we will consider not only the orthogonal polynomials, but the 2-orthogonal polynomials and we will try to establish a bond between the latter and certain birth and death processes that will be called “generalized.” These processes will have like transition probabilities

$$P_{ij}(t) = \Pr \{X(t+s) = j/X(s) = i\} = \Pr \{X(t) = j, X(0) = i\} \quad (1.1)$$

which satisfy, when  $h \rightarrow 0$ ,

$$P_{ij}(t) = \begin{cases} \lambda_i h + o(h), & j = i + 1, \\ 1 - (\lambda_i + \mu_i + \bar{\mu}_i)h + o(h), & j = i, \\ \mu_i h + o(h), & j = i - 1, \\ \bar{\mu}_i h + o(h), & j = i - 2, \\ o(h), & j \neq i - 2, i - 1, i, i + 1, \end{cases} \quad (1.2)$$

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$$P_{ij}(0) = \delta_{ij}, \quad \forall i, j \in \mathbb{N}, \quad (1.3)$$

where  $\lambda_i$  are the birth rates and  $\mu_i$  and  $\bar{\mu}_i$  are the death rates. It is also assumed that

$$\lambda_i > 0, \quad \mu_i \geq 0, \quad \bar{\mu}_{i+1} > 0; \quad i \geq 0, \quad \bar{\mu}_0 \geq 0. \quad (1.4)$$

These processes being stationary,  $P_{ij}(t)$  does not depend on the way taken by the system to reach the state  $j$ , but depends only on the states  $i$  and  $j$  and of the laps of time  $t$  taken while going from state  $i$  towards state  $j$ . This is equivalent to

$$\mathcal{P}(s+t) = \mathcal{P}(s)\mathcal{P}(t). \quad (1.5)$$

In this work, we start initially by giving some properties of the “generalized” birth and death processes by determining the sequence of 2-orthogonal polynomials associated with this type of processes. Then, the sufficient conditions are given, which allow giving an integral representation of the transition probabilities from these processes.

We can quote as an example that this type of processes can be the modelling of problems met while studying the kinetics of enzymes, in particular, those which catalyze reactions to a substrate in the presence of noncompetitive inhibitors.

We will treat this type of model in future, when we study the generalized linear processes.

### 2. The generalized Chapman-Kolmogorov equations

**PROPOSITION 2.1.** *Let be a “generalized” birth and death process, where the transition probabilities are given by (1.2), then these probabilities satisfy two systems of differential recurrence relations called “Chapman-Kolmogorov (or C-K) equations.” The forward C-K equations are*

$$\begin{aligned} \frac{d}{dt}P_{ij}(t) &= \bar{\mu}_{j+2}P_{i,j+2}(t) + \mu_{j+1}P_{i,j+1}(t) + \lambda_{j-1}P_{i,j-1}(t), \\ &- (\bar{\mu}_j + \mu_j + \lambda_j)P_{ij}(t). \end{aligned} \quad (2.1)$$

The backward C-K equations are

$$\begin{aligned} \frac{d}{dt}P_{ij}(t) &= \lambda_i P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) + \bar{\mu}_i P_{i-2,j}(t), \\ &- (\bar{\mu}_i + \mu_i + \lambda_i)P_{ij}(t). \end{aligned} \quad (2.2)$$

*Proof.* Since the process is stationary, on one hand we can write

$$P_{ij}(t+s) = \sum_{k \geq 0} P_{ik}(t)P_{kj}(s), \quad s \rightarrow 0, \quad (2.3)$$

it follows from (1.2) that

$$\begin{aligned} \frac{d}{dt}P_{ij}(t) &= \lim_{s \rightarrow 0} \frac{P_{ij}(t+s) - P_{ij}(t)}{s} \\ &= \lim_{s \rightarrow 0} \frac{\sum_{k \geq 0} (P_{ik}(t)P_{kj}(s) - P_{ij}(t))}{s} \\ &= \lim_{s \rightarrow 0} \left[ \bar{\mu}_{j+2}P_{ij+2}(t) + \mu_{j+1}P_{ij+1}(t) + \lambda_{j-1}P_{ij-1}(t) - (\lambda_j + \mu_j + \bar{\mu}_j)P_{ij}(t) + \frac{o(s)}{s} \right], \end{aligned} \tag{2.4}$$

then we have (2.1).

On the other hand, by applying the semigroup property, we have also

$$P_{ij}(s+t) = \sum_{k \geq 0} P_{ik}(s)P_{kj}(t), \quad s \rightarrow 0, \tag{2.5}$$

then from (1.2), we have

$$\begin{aligned} \frac{d}{dt}P_{ij}(t) &= \lim_{s \rightarrow 0} \frac{P_{ij}(t+s) - P_{ij}(t)}{s} \\ &= \lim_{s \rightarrow 0} \left[ \lambda_i P_{i+1j}(t) + \mu_i P_{i-1j}(t) + \bar{\mu}_i P_{i-2j}(t) - (\lambda_i + \mu_i + \bar{\mu}_i) P_{ij}(t) + \frac{o(s)}{s} \right]. \end{aligned} \tag{2.6}$$

So, we get (2.2). □

*Definition 2.2.* The matrix  $\mathcal{A} = (a_{ij})_{i,j \in \mathbb{N}}$  defined by

$$a_{ij} = \begin{cases} \lambda_i & \text{if } j = i + 1, \\ \mu_i & \text{if } j = i - 1, \\ \bar{\mu}_i & \text{if } j = i - 2, \\ -(\lambda_i + \mu_i + \bar{\mu}_i) & \text{if } j = i, \\ 0 & \text{otherwise,} \end{cases} \tag{2.7}$$

is called the infinitesimal generator of the process.

**LEMMA 2.3.** *A generalized birth and death process has the following properties:*

- (i)  $\mathcal{P}'(t) = \mathcal{P}(t)\mathcal{A}$ ,
- (ii)  $\mathcal{P}'(t) = \mathcal{A}\mathcal{P}(t)$ ,
- (iii)  $\mathcal{P}(0) = I = (\delta_{ij})_{i,j \in \mathbb{N}}$ ,
- (iv)  $P_{ij}(t) \geq 0$ ,
- (v)  $\sum_{j \geq 0} P_{ij}(t) \leq 1; i \geq 0, t \geq 0$ ,
- (vi)  $\mathcal{P}(t+s) = \mathcal{P}(t)\mathcal{P}(s)$ ,

where  $\mathcal{P}(t) = (P_{ij}(t))_{i,j \in \mathbb{N}}$  is the transition matrix.

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Now, we will seek a solution of the Chapman-Kolmogorov equations, by using the method of separation of variables. So if we put

$$P_{ij}(t) = f(t)F_iQ_j, \quad (2.8)$$

we will get the following lemma.

LEMMA 2.4. *If the transition probabilities satisfy (1.2), then*

$$f(t) = \exp(-xt), \quad (2.9)$$

$$\begin{aligned} F_0(x) &= 1; & F_{-2}(x) &\equiv F_{-1}(x) \equiv 0, \\ -xF_i(x) &= \lambda_i F_{i+1}(x) + \mu_i F_{i-1}(x) + \bar{\mu}_i F_{i-2}(x) \\ &- (\bar{\mu}_i + \mu_i + \lambda_i) F_i(x), \quad i \geq 0, \end{aligned} \quad (2.10)$$

$$\begin{aligned} Q_{-1}(x) &\equiv 0; & Q_0(x) &= 1; & Q_1(x) &= -\frac{\mu_0}{\bar{\mu}_1}; \\ -xQ_j(x) &= \bar{\mu}_{j+2} Q_{j+2}(x) + \mu_{j+1} Q_{j+1}(x) + \lambda_{j-1} Q_{j-1}(x) \\ &- (\bar{\mu}_j + \mu_j + \lambda_j) Q_j(x), \quad j \geq 0. \end{aligned} \quad (2.11)$$

*Proof.* By differentiating with respect to  $t$  and taking into account that  $F_i$  and  $Q_j$  cannot vanish identically, we get

$$\begin{aligned} \frac{f'(t)}{f(t)} &= \frac{1}{Q_j} [\bar{\mu}_{j+2} Q_{j+2} + \mu_{j+1} Q_{j+1} + \lambda_{j-1} Q_{j-1} - (\lambda_j + \mu_j + \bar{\mu}_j) Q_j] = -x, \\ \frac{f'(t)}{f(t)} &= \frac{1}{F_i} [\lambda_i F_{i+1} + \mu_i F_{i-1} + \bar{\mu}_i F_{i-2} - (\lambda_i + \mu_i + \bar{\mu}_i) F_i] = -x, \end{aligned} \quad (2.12)$$

where  $x$  is a separation constant. As  $P_{00}(0) = 1$ , we get (2.9), (2.10), and (2.11).  $\square$

*Remark 2.5.* (a) We have written  $F_i(x)$  and  $Q_j(x)$  to exhibit the dependence of  $F_i$  and  $Q_j$  on the constant  $x$ .

(b) It is easy to see that the functions  $F_i(x)$  and  $Q_j(x)$  defined, respectively, by the recurrence formulas (2.10) and (2.11) are polynomials. Moreover, let us note that  $\deg F_k = k$  for all  $k \in \mathbb{N}$ .

To characterize the sequence of polynomials  $\{F_i(x)\}_{i \in \mathbb{N}}$ , we will introduce  $d$ -orthogonality notion.

### 3. $d$ -orthogonality

*Definition 3.1.* Let  $\{B_n(x)\}_{n \geq 0}$  be a monic sequence of polynomials and  $\{\mathcal{L}_n\}_{n \geq 0}$  a sequence of linear forms.  $\{\mathcal{L}_n\}_{n \geq 0}$  is called the dual sequence of  $\{B_n(x)\}_{n \geq 0}$  if and only if

$$\mathcal{L}_n(B_m) = \langle \mathcal{L}_n, B_m \rangle = \delta_{mn}; \quad n, m \geq 0. \quad (3.1)$$

*Definition 3.2* [2, 9]. Let  $\Gamma = (\Gamma^1, \Gamma^2, \dots, \Gamma^d)^T$  be the  $d$  linear forms ( $d \geq 1$ ). A sequence of polynomials  $\{B_n(x)\}_{n \geq 0}$  is said to be  $d$ -orthogonal with respect to  $\Gamma$  if it fulfills

$$\begin{aligned} \Gamma^\sigma (x^m B_n) &= 0; \quad n \geq md + \sigma, \quad m \geq 0, \\ \Gamma^\sigma (x^m B_{dm+\sigma-1}) &\neq 0; \quad m \geq 0, \quad 1 \leq \sigma \leq d. \end{aligned} \quad (3.2)$$

**PROPOSITION 3.3** [2, 9]. *Let  $\{B_n(x)\}_{n \geq 0}$  be a monic sequence of polynomials, then the following statements are equivalent.*

(a) *The sequence  $\{B_n(x)\}_{n \geq 0}$  satisfies a recurrence relation with four terms:*

$$\begin{aligned} B_0(x) &= 1; \quad B_1(x) = x - \beta_0; \quad B_2(x) = (x - \beta_1)B_1(x) - \gamma_1; \\ B_{n+3}(x) &= (x - \beta_{n+2})B_{n+2}(x) - \gamma_{n+2}B_{n+1}(x) - \delta_{n+1}B_n(x), \quad n \geq 0, \end{aligned} \quad (3.3)$$

where  $\delta_{n+1} \neq 0$  for all  $n \in \mathbb{N}$ .

(b) *The sequence  $\{B_n(x)\}_{n \geq 0}$  is 2-orthogonal with respect to  $\Gamma = (\Gamma^1, \Gamma^2)^T$  or with respect to  $\mathcal{L} = (\mathcal{L}_0, \mathcal{L}_1)^T$ , because*

$$\Gamma^\sigma = \sum_{k \geq 0}^{\alpha-1} \alpha_k^\sigma \mathcal{L}_k, \quad (\alpha_{\sigma-1}^\sigma \neq 0) \text{ equivalently } \mathcal{L}_k = \sum_{\sigma=1}^k \xi_\sigma^k \Gamma^\sigma, \quad (\xi_k^k \neq 0). \quad (3.4)$$

(c) *For any couple  $(n, \sigma)$ ,  $\sigma = 0, 1$ , and  $n \geq 0$ , there exist two polynomials  $\Lambda^\mu(n, \sigma)$ ,  $\mu = 0, 1$ , such that*

$$\mathcal{L}_{2n+\sigma} = \Lambda^0(n, \sigma) \mathcal{L}_0 + \Lambda^1(n, \sigma) \mathcal{L}_1 \quad \text{for } \sigma = 0, 1, \quad n \in \mathbb{N}, \quad (3.5)$$

and satisfying  $\deg \Lambda^\mu(n, \mu) = n$  for  $\mu = 0, 1$ ,  $\deg \Lambda^0(n, 1) \leq n$  for  $n \geq 0$ , and  $\deg \Lambda^1(n, 0) \leq n - 1$  for  $n \geq 1$ .

*Remark 3.4.* From this proposition, we deduce that the sequence of polynomials  $\{F_n\}_{n \in \mathbb{N}}$  is 2-orthogonal with respect to  $\mathcal{L} = (\mathcal{L}_0, \mathcal{L}_1)^T$ .

#### 4. Integral representation

In this section, we try to give an integral representation of the  $P_{ij}(t)$ , which are the solutions of the Chapman-Kolmogov equations. First, we give the following lemma.

**LEMMA 4.1.** *Let there be a generalized birth and death process, where the transition probabilities are given by (1.2), then the sequence  $\{F_n\}_{n \geq 0}$  given by (2.10) satisfies the following properties.*

(A) *Putting  $F_n(x) = \sum_{k \geq 0} a_{n,k} x^k$  for all  $n \geq 0$ , then*

$$a_{0,0} = 1, \quad a_{n,n} = (-1)^n \prod_{i=0}^{n-1} \lambda_i; \quad n \geq 1. \quad (4.1)$$

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The moments of the two orthogonality forms  $c_n^\sigma = \mathcal{L}_\sigma(x^n)$ ,  $n \geq 0$ , and  $\sigma = 0, 1$  are given by

$$\begin{aligned} c_0^0 &= 1, & c_0^1 &= \lambda_0 + \mu_0 + \bar{\mu}_0, & c_{n+2}^0 &= \frac{1}{a_{n+2, n+2}} \sum_{i=0}^{n+1} a_{n+2, i} c_i^0, & n &\geq 0. \\ c_0^1 &= 0, & c_1^1 &= -\lambda_0, & c_{n+2}^1 &= \frac{1}{a_{n+2, n+2}} \sum_{i=0}^{n+1} a_{n+2, i} c_i^1, & n &\geq 0. \end{aligned} \quad (4.2)$$

(B) The following equation holds:

$$e^{-xt} F_n(x) = \sum_{j \geq 0} P_{nj}(t) F_j(x), \quad x \in \mathbb{R}, n \in \mathbb{N}. \quad (4.3)$$

(C) The sequence  $\{F_n(0)\}_{n \geq 0}$  is a nondecreasing sequence and  $F_n(0) \geq 1$  ( $n \geq 0$ ).

*Proof.* (A) If we put  $F_n(x) = \sum_{k=0}^n a_{n,k} x^k$ ,  $n \geq 0$ , then from (2.10) we will deduce that

$$a_{n+1, n+1} = -\frac{1}{\lambda_n} a_{n, n}, \quad (4.4)$$

where

$$a_{0,0} = 1. \quad (4.5)$$

So, we get (4.1). From the definition of the dual sequence we find (4.2).

(B) Let  $\{f_n\}_{n \in \mathbb{N}}$  be a sequence of functions defined by

$$f_n(x, t) = \sum_{j \geq 0} P_{nj}(t) F_j(x), \quad n \in \mathbb{N}, \quad (4.6)$$

or  $f(x, t) = \mathcal{P}(t) \mathcal{F}(x)$ , where  $f(x, t) = (f_n(x, t))_{n \in \mathbb{N}}$  and  $\mathcal{F}(x) = (F_n(x))_{n \in \mathbb{N}}$ , then

$$\frac{\partial}{\partial t} f(x, t) = \mathcal{P}'(t) \mathcal{F}(x) = \mathcal{P}(t) \mathcal{A} \mathcal{F}(x). \quad (4.7)$$

From (2.10), we have  $-x \mathcal{F}(x) = \mathcal{A} \mathcal{F}(x)$ . So,

$$\begin{aligned} \frac{\partial}{\partial t} f_n(x, t) &= -x f_n(x, t), \\ f_n(x, 0) &= F_n(x), \quad n \in \mathbb{N}. \end{aligned} \quad (4.8)$$

Then, we get (4.3).

(C) We show by induction that  $\{F_n(0)\}_{n \geq 0}$  is a nondecreasing sequence.

Indeed, we have  $F_1(0) = 1 + (\mu_0 + \bar{\mu}_0)/\lambda_0 \geq F_0(0)$ .

Assume that  $F_{n+1}(0) \geq F_n(0) \geq 1$  for all  $0 \leq n \leq k-1$ , then from (2.10) we get

$$\lambda_k (F_{k+1}(0) - F_k(0)) = \mu_k (F_k(0) - F_{k-1}(0)) + \bar{\mu}_k (F_k(0) - F_{k-2}(0)) \geq 0. \quad (4.9)$$

So,

$$F_{n+1}(0) \geq F_n(0) \geq 1; \quad n \geq 0. \quad (4.10)$$

□

**THEOREM 4.2** [1, Boas's theorem]. *Let  $\{\alpha_n\}_{n \geq 0}$  be a real sequence, then there exists a bounded variation function  $\alpha(x)$  defined on  $[0, +\infty[$  such that*

$$\int_0^{+\infty} x^n d\alpha(x) = \alpha_n, \quad n \in \mathbb{N}. \quad (4.11)$$

*Remark 4.3.* For any element  $\mathcal{L}_m$  of the dual sequence, there exists a bounded variation function  $\Psi_m$ , such that

$$\mathcal{L}_m(x^n) = \int_0^{+\infty} x^n d\Psi_m; \quad n, m \in \mathbb{N}. \quad (4.12)$$

**THEOREM 4.4.** *Let  $\mathcal{P}(t) = (P_{ij}(t))_{i,j \in \mathbb{N}}$  be a solution of (1.3), (2.1), and (2.2), then if*

$$\sum_{j=0}^m |F_j(x)| \leq g_\sigma(x) \quad \forall x \in \mathbb{R}_+, \quad \forall m \in \mathbb{N}, \quad \sigma = 0, 1, \quad (4.13)$$

where  $g_\sigma$  ( $\sigma = 0, 1$ ) is an integrable function with respect to  $\Psi_\sigma$ , the following integral representation holds:

$$P_{mn}(t) = \int_0^{+\infty} e^{-xt} F_m(x) d\Psi_n \quad n, m \in \mathbb{N}, \quad (4.14)$$

where  $\Psi_n$ ,  $n \geq 0$  is the bounded variation function given in Remark 4.3.

*Proof.* From (4.3), we deduce that  $\{P_{ni}(t)F_i(x)\}_{i \geq 0}$  is a bounded sequence for all  $n \in \mathbb{N}$  and  $x, t \geq 0$ . In particular, there exists a sequence of functions  $M_n(t)$  such that

$$|P_{ni}(t)| \leq \frac{M_n(t)}{F_n(0)} \leq M_n(t) \quad \forall i \in \mathbb{N}, \quad (4.15)$$

then we have

$$\left| \sum_{i=0}^m P_{ni}(t)F_i(x) \right| \leq \sum_{i=0}^m |P_{ni}(t)F_i(x)| \leq M_n(t) \sum_{i=0}^m |F_i(x)|. \quad (4.16)$$

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Hence  $\{\sum_{i=0}^m P_{ni}(t)F_i(x)\}_{m \geq 0}$  is dominated by an integrable function with respect to  $\Psi_\sigma$  ( $\sigma = 0, 1$ ). Then, we can write

$$\begin{aligned} \int_0^{+\infty} e^{-xt} F_m d\Psi_n(x) &= \sum_{j \geq 0} P_{mj}(t) \int_0^{+\infty} F_j d\Psi_n(x) \\ &= P_{mn}(t) \int_0^{+\infty} F_n d\Psi_n(x) = P_{mn}(t). \end{aligned} \quad (4.17)$$

□

**COROLLARY 4.5.** *Let  $\mathcal{P}(t) = (P_{ij}(t))_{i,j \in \mathbb{N}}$  be a solution of (1.3), (2.1), and (2.2). Then the  $P_{ij}(t)$  given by relation (4.14) can be written in the following form:*

$$\begin{aligned} P_{n\sigma}(t) &= \int_0^{+\infty} e^{-xt} F_n(x) d\Psi_\sigma(x); \quad n \geq 0, \sigma = 0, 1, \\ P_{n,2k+\sigma}(t) &= \frac{1}{\prod_{j=1}^k \bar{\mu}_{2j+\sigma}} \left[ \int_0^{+\infty} (-x)^k e^{-xt} F_n(x) d\Psi_\sigma(x) \right. \\ &\quad \left. - \sum_{j=0}^{2k+\sigma-1} P_{nj}(t) \int_0^{+\infty} (-x)^k F_j(x) d\Psi_\sigma(x) \right]; \quad k \geq 0, n \geq 0, \sigma = 0, 1. \end{aligned} \quad (4.18)$$

*Proof.* We have for  $t \geq 0$  and  $\sigma = 0, 1$

$$\begin{aligned} \int_0^{+\infty} (-x)^m e^{-xt} F_n(x) d\Psi_\sigma(x) &= \sum_{j \geq 0} P_{n,2j}(t) \int_0^{+\infty} (-x)^m P_{nj}(t) F_j(x) d\Psi_\sigma(x) \\ &\quad + \sum_{j \geq 0} P_{n,2j+1}(t) \int_0^{+\infty} (-x)^m P_{nj}(t) F_j(x) d\Psi_\sigma(x), \end{aligned} \quad (4.19)$$

taking into account that

$$\begin{aligned} \mathcal{L}_0((-x)^m F_{2j+1}(x)) &= 0, \quad j \geq m, \\ \mathcal{L}_0((-x)^m F_{2j}(x)) &= 0, \quad j \geq m+1, \\ \mathcal{L}_1((-x)^m F_{2j+1}(x)) &= 0, \quad j \geq m+1, \\ \mathcal{L}_1((-x)^m F_{2j}(x)) &= 0, \quad j \geq m+1. \end{aligned} \quad (4.20)$$

It follows that

$$\begin{aligned} \sum_{j=0}^m P_{n,2j}(t) \mathcal{L}_0((-x)^m F_{2j}(x)) + \sum_{j=0}^{m-1} P_{n,2j+1}(t) \mathcal{L}_0((-x)^m F_{2j+1}(x)) &= \mathcal{L}_0((-x)^m e^{-xt} F_n), \\ \sum_{j=0}^m P_{n,2j}(t) \mathcal{L}_1((-x)^m F_{2j}(x)) + \sum_{j=0}^m P_{n,2j+1}(t) \mathcal{L}_1((-x)^m F_{2j+1}(x)) &= \mathcal{L}_1((-x)^m e^{-xt} F_n), \end{aligned} \quad (4.21)$$



for all  $n$  and  $m \in \mathbb{N}$ , which is a system with lower triangular matrix of the form  $\mathcal{B}_m \mathcal{P}_n(t) = Q_{nm}$ , where  $\mathcal{B}_m = (b_{i,j})_{i,j=0}^{2m+1}$  and  $Q_{nm} = (q_{n,i})_{i=0}^{2m+1}$  such that

$$\begin{aligned} b_{\sigma,\sigma} &= \mathcal{L}_\sigma(F_\sigma) = 1, & b_{10} &= b_{01} = 0, & \sigma &= 0, 1, \\ b_{2i+\sigma,j} &= \mathcal{L}_\sigma((-x)^i F_j), & i &= \overline{0,m}; j = \overline{0,2i+\sigma} & \sigma &= 0, 1, \\ q_{n,2i+\sigma} &= q_{n2i+\sigma}(t) = \mathcal{L}_\sigma((-x)^i e^{-xt} F_n) & i &= \overline{0,m} & \sigma &= 0, 1, \end{aligned} \tag{4.22}$$

where the determinant of  $\mathcal{B}_m$  is

$$\Delta_m = \prod_{j=1}^m \mathcal{L}_0((-x)^j F_{2j}) \mathcal{L}_1((-x)^j F_{2j+1}) > 0, \tag{4.23}$$

because

$$\mathcal{L}_\sigma((-x)^j F_{2j}) = \prod_{i=1}^j \overline{\mu}_{2i+\sigma} > 0, \quad j \geq 1 \quad \sigma = 0, 1. \tag{4.24}$$

So, this system admits a unique solution given by (4.18) for all  $n \geq 0$  and  $\sigma = 0, 1$ .  $\square$

Now, we give the sufficient conditions so that the  $P_{ij}(t)$  given by the integral representation (4.14) is indeed a transition probability.

First, we give the following lemma.

LEMMA 4.6. *Let there be a system with a lower triangular matrix of the form  $M_n X_n = C_n$ , where*

$$\begin{aligned} M_n &= \begin{pmatrix} m_{00} & 0 & 0 & \cdots & 0 \\ m_{10} & m_{11} & 0 & \cdots & 0 \\ m_{20} & m_{21} & m_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{n0} & m_{n1} & m_{n2} & \cdots & m_{nn} \end{pmatrix}, \\ X_n &= (x_i)_{i=0}^n, & C_n &= (c_i)_{i=0}^n. \end{aligned} \tag{4.25}$$

Then the following properties are equivalent.

- (A)  $\sum_{i=0}^n x_i \leq 1$ .
- (B)  $\det D_n$  and  $\det M_n$  have the same sign, where  $D_n$  is the matrix defined by

$$D_n = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ c_0 & m_{00} & 0 & 0 & \cdots & 0 \\ c_1 & m_{10} & m_{11} & 0 & \cdots & 0 \\ c_2 & m_{20} & m_{21} & m_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & m_{n0} & m_{n1} & m_{n2} & \cdots & m_{nn} \end{pmatrix}. \tag{4.26}$$

*Proof.* Using Cramer's rule, on one hand we can get

$$\begin{aligned}
 x_i &= \frac{1}{\det M_n} \begin{vmatrix} m_{00} & 0 & \cdots & 0 & c_0 & \cdots & 0 & 0 \\ m_{10} & m_{11} & \cdots & 0 & c_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{i-10} & m_{i-11} & \cdots & m_{i-1i-1} & c_{i-1} & \cdots & 0 & 0 \\ m_{i0} & m_{i1} & \cdots & m_{ii-1} & c_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{n-10} & m_{n-11} & \cdots & m_{n-1i-1} & c_{n-1} & \cdots & m_{n-1n-1} & 0 \\ m_{n0} & m_{n1} & \cdots & m_{ni-1} & c_n & \cdots & \cdots m_{nn-1} & m_{nn} \end{vmatrix} \\
 &= \frac{(-1)^i}{\det M_n} \begin{vmatrix} c_0 & m_{00} & 0 & \cdots & 0 & \cdots & 0 & 0 \\ c_1 & m_{10} & m_{11} & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{i-1} & m_{i-10} & m_{i-11} & \ddots & m_{i-1i-1} & \cdots & 0 & 0 \\ c_i & m_{i0} & m_{i1} & \cdots & m_{ii-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1} & m_{n-10} & m_{n-11} & \cdots & m_{n-1i-1} & \cdots & m_{n-1n-1} & 0 \\ c_n & m_{n0} & m_{n1} & \cdots & m_{ni-1} & \cdots & m_{nn-1} & m_{nn} \end{vmatrix}, \tag{4.27}
 \end{aligned}$$

and on the other hand, we have

$$\begin{aligned}
 \det D_n &= \det M_n \\
 &- \sum_{i=0}^n (-1)^i \begin{vmatrix} c_0 & m_{00} & 0 & \cdots & 0 & \cdots & 0 & 0 \\ c_1 & m_{10} & m_{11} & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{i-1} & m_{i-10} & m_{i-11} & \ddots & m_{i-1i-1} & \cdots & 0 & 0 \\ c_i & m_{i0} & m_{i1} & \cdots & m_{ii-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1} & m_{n-10} & m_{n-11} & \cdots & m_{n-1i-1} & \cdots & m_{n-1n-1} & 0 \\ c_n & m_{n0} & m_{n1} & \cdots & m_{ni-1} & \cdots & m_{nn-1} & m_{nn} \end{vmatrix}. \tag{4.28}
 \end{aligned}$$

So,  $\det D_n = \det M_n (1 - \sum_{i=0}^n x_i)$  with  $\det M_n \neq 0$ , consequently we deduce that (A) and (B) are equivalent.  $\square$

From Corollary 4.5 and Lemma 4.1, we have the following result.

**THEOREM 4.7.** *Let  $\mathcal{P}(t) = (P_{ij}(t))_{i,j \in \mathbb{N}}$  be a solution of (1.3), (2.1), and (2.2). If the following conditions are satisfied:*

- (i)  $\int_0^{+\infty} e^{-xt} F_n(x) d\Psi_\sigma \geq 0$ , for all  $n \in \mathbb{N}$  and  $\sigma = 0, 1$ ,

(ii)  $\det \mathcal{D}_{nm} \geq 0$  for all  $n, m \in \mathbb{N}$ , where  $\mathcal{D}_{nm}$  is the following matrix:

$$D_{nm} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ q_{n0} & b_{00} & 0 & 0 & \cdots & 0 \\ q_{n1} & b_{10} & b_{11} & 0 & \cdots & 0 \\ q_{n2} & b_{20} & b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{n2m+1} & b_{2m+10} & b_{2m+11} & b_{2m+12} & \cdots & b_{2m+12m+1} \end{pmatrix}, \quad (4.29)$$

where  $q_{nj}$  and  $b_{ij}$  are given by (4.22),

(iii)

$$\int_0^{+\infty} (-x)^k F_j(x) d\Psi_\sigma \leq \int_0^{+\infty} (-x)^k e^{-xt} F_n(x) d\Psi_\sigma; \quad j = \overline{0, 2k + \sigma - 1} \quad (4.30)$$

for all  $n \in \mathbb{N}$ ,  $k \in \mathbb{N}^*$  and  $t > 0$ , then

$$P_{nm}(t) \geq 0; \quad n, m \geq 0, \quad \sum_{i=0}^m P_{ni}(t) \leq 1 \quad \forall n, m \in \mathbb{N}, t \geq 0. \quad (4.31)$$

*Proof.* If condition (ii) is satisfied, we deduce from Lemma 4.1 that

$$\sum_{i=0}^m P_{ni}(t) \leq 1 \quad \forall n, m \in \mathbb{N}, t \geq 0. \quad (4.32)$$

From (4.14), we can get  $P_{nm}(0) = \delta_{nm}$ .

As ( $t > 0$ )

$$P_{n,2k+\sigma}(t) = \frac{1}{\prod_{j=1}^k \bar{\mu}_{2j+\sigma}} \left[ \int_0^{+\infty} (-x)^k e^{-xt} F_n(x) d\Psi_\sigma - \sum_{j=0}^{2k+\sigma-1} P_{nj}(t) \int_0^{+\infty} (-x)^k F_j(x) d\Psi_\sigma \right], \quad k \geq 1, \quad (4.33)$$

then  $P_{n,2k+\sigma}(t) \geq 0$ , that is,

$$\int_0^{+\infty} (-x)^k e^{-xt} F_n(x) d\Psi_\sigma \geq \sum_{j=0}^{2k+\sigma-1} P_{nj}(t) \int_0^{+\infty} (-x)^k F_j(x) d\Psi_\sigma \quad \forall k \geq 1. \quad (4.34)$$

We will show by induction on  $m$  that  $P_{nm}(t) \geq 0$  for all  $n \geq 0$ . From (i) we have  $P_{n,0}(t) \geq 0$  and  $P_{n1}(t) \geq 0$ .

Assume that

$$P_{nm}(t) \geq 0; \quad 0 \leq m \leq j. \quad (4.35)$$

For any  $j$ , there exists  $k$  such that  $j = 2k + \sigma - 1$ , it follows that

$$\begin{aligned} \sum_{m=0}^{2k+\sigma-1} P_{nm}(t) \int_0^{+\infty} (-x)^k F_m(x) d\Psi_\sigma &\leq \max_{0 \leq m \leq 2k+\sigma-1} \left\{ \int_0^{+\infty} (-x)^k F_m(x) d\Psi_\sigma \right\} \sum_{m=0}^{2k+\sigma-1} P_{nm}(t) \\ &\leq \max_{0 \leq m \leq 2k+\sigma-1} \left\{ \int_0^{+\infty} (-x)^k F_m(x) d\Psi_\sigma \right\}. \end{aligned} \quad (4.36)$$

Since (iii) is satisfied, we obtain  $P_{n,j+1}(t) \geq 0$ .

Hence, we get  $P_{nm}(t) \geq 0$  for all  $n$  and  $m \geq 0$ .  $\square$

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