TRANSFORMATION GROUPS ON REAL PLANE AND THEIR DIFFERENTIAL INVARIANTS

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Complete sets of bases of differential invariants, operators of invariant differentiation, and Lie determinants of continuous transformation groups acting on the real plane are constructed. As a necessary preliminary, realizations of finite-dimensional Lie algebras on the real plane are revisited.

1. Introduction

Differential invariants emerged as one of the most important tools in investigation of differential equations in the works of Lie. In [19, 23] he proved that any nonsingular invariant system of differential equations can be expressed in terms of differential invariants of the corresponding symmetry group. In the same paper he also applied differential invariants to integration of ODEs. If differential invariants of a Lie group are known, the differential equations admitting this group can be easily described and the special representation (so-called group foliation) of such differential equations can be constructed.

Differential invariants of all finite-dimensional local transformation groups on a space of two complex variables were described by Lie himself in [18]. A modern treatment of these results was adduced in [30]. Namely, functional bases of differential invariants, operators of invariant differentiation, and, Lie determinants were constructed for all inequivalent realizations of point and contact finite-dimensional transformation groups on the complex plane. The real finite-dimensional Lie algebras of contact vector fields and their differential invariants were completely classified in [9]. Differential invariants of a one-parameter group of local transformations in the case of arbitrary number of dependent and independent variables were studied in [37].

The subject of this paper is exhaustive description of differential invariants and Lie determinants of finite-dimensional Lie groups acting on the real plane. A necessary prerequisite to do it is classification of Lie algebra realizations in vector fields on the real plane up to local diffeomorphisms.

Realizations of Lie algebras by vector fields are widely applicable in the general theory of differential equations, integration of differential equations and their systems [29, 32], in group classification of ODEs and PDEs [2], in classification of gravity fields of a general form with respect to motion groups [36], in geometric control theory, and in the theory of systems with superposition principles [6, 40]. Such realizations are also applicable in the difference schemes for numerical solutions of differential equations [4]. Description of realizations is the first step for solving the Levine's problem [17] on the second-order time-independent Hamiltonian operators which lie in the universal enveloping algebra of a finite-dimensional Lie algebra of the first-order differential operators. The Levine's problem was posed in molecular dynamics. In such a way, realizations are relevant in the theory of quasi-exactly solvable problems of quantum mechanics through the so-called algebraic approach to scattering theory and molecular dynamics. The list of possible applications of realizations of Lie algebras is not exhausted by the above-mentioned subjects.

The plan of the paper is the following. In Section 2 we discuss and compare different classifications of realizations of finite-dimensional Lie algebras on the real and complex planes, which are available in literature. In particular, we thoroughly study the question of parametrization and equivalence in series of realizations. The realizations of finite-dimensional Lie algebras in vector fields on the real plane are arranged in the form of Table 1.1. The transformations that reduce real Lie algebras to complex ones are presented in Table 1.2. In Section 3 some definitions and results concerning differential invariants are collected and detailed example of calculation is adduced. Using the results of Table 1.1 and technique proposed in Section 2, we obtain complete sets of bases of differential invariants, operators of invariant differentiation and Lie determinants and collect them in Table 1.3. Short overview of the obtained results as well as their possible applications and development are presented in the conclusion.

2. Realizations of Lie algebras on real and complex planes

There are two important classification problems among a variety of others in the classical theory of Lie algebras.

The first one is classification of Lie algebra structures, that is, classification of possible commutation relations between basis elements. A list of isomorphism classes of the Lie algebras is in use of many authors for different purposes, for example, [1, 2, 5, 12, 33, 34, 36]. But the problem of unification and correction of the existing lists (see, e.g., [3, 8, 20–22, 24–28, 35, 43]) is a very laborious task, even in the case of low dimensions, because the number of entries in such lists rapidly increases with growing dimension and the problem of classification of Lie algebras includes a subproblem of reduction of pair of matrices to a canonical form [16]. Here we only remind that all possible complex Lie algebras of dimensions no greater than four were listed by Lie himself [20–22] and later the semisimple Lie algebras [15] and the Lie algebras of dimensions no greater than six [25–28, 43] over the complex and real fields were classified.

The other problem established by S. Lie is the problem of description of different Lie algebra representations and realizations, particularly, by vector fields up to local diffeomorphisms.

Table 1.1. Realizations of Lie algebras on the real plane.

N	Realizations	N_1	N_0	N_3
1	∂_x	9	57, (1)	$R(A_1,1)$
2	∂_x,∂_y	22	57, (2)	$R(2A_1,1)$
3	$\partial_x,y\partial_x$	20	57, (4)	$R(2A_1,2)$
4	$\partial_x,x\partial_x+y\partial_y$	—	57, (3)	$R(A_{2.1},1)$
5	$\partial_x,x\partial_x$	10	57, (5)	$R(A_{2.1},2)$
6	$\partial_y,x\partial_y,\xi_1(x)\partial_y$	20	57, (14)	$R(3A_1,5)$
7	$\partial_y,y\partial_y,\partial_x$	23	73, (10)	$R(A_{2.1}\oplus A_1,3)$
8	$e^{-x}\partial_y,\partial_x,\partial_y$	22	57, (8)	$R(A_{2.1}\oplus A_1,4)$
9	$\partial_y,\partial_x,x\partial_y$	22	57, (9)	$R(A_{3.1},3)$
10	$\partial_y,\partial_x,x\partial_x+(x+y)\partial_y$	25	57, (11)	$R(A_{3.2},2)$
11	$e^{-x}\partial_y,-xe^{-x}\partial_y,\partial_x$	22	57, (7)	$R(A_{3.2},3)$
12	$\partial_x,\partial_y,x\partial_x+y\partial_y$	12	57, (10)	$R(A_{3.3},2)$
13	$\partial_y,x\partial_y,y\partial_y$	21	57, (15)	$R(A_{3.3},4)$
14	$\partial_x,\partial_y,x\partial_x+ay\partial_y,0<|a|\le 1,a\ne 1$	12	57, (10)	$R(A_{3.4}^a,2)$
15	$e^{-x}\partial_y,e^{-ax}\partial_y,\partial_x,0<|a|\le 1,a\ne 1$	22	57, (6)	$R(A_{3.4}^a,3)$
16	$\partial_x,\partial_y,(bx+y)\partial_x+(by-x)\partial_y,b\ge 0$	1	$\overset{\mathbb{C}}{\sim}$ 57, (10)	$R(A_{3.5}^b,2)$
17	$e^{-bx}\sin x\partial_y,\ e^{-bx}\cos x\partial_y,\partial_x,\ b\ge 0$	22	$\overset{\mathbb{C}}{\sim}$ 57, (6)	$R(A_{3.5}^b,3)$
18	$\partial_x,x\partial_x+y\partial_y,(x^2-y^2)\partial_x+2xy\partial_y$	2	$\overset{\mathbb{C}}{\sim}$ 57, (13); 73, (4)	$R(sl(2,\mathbb{R}),2)$
19	$\partial_x+\partial_y,x\partial_x+y\partial_y,x^2\partial_x+y^2\partial_y$	17	57, (13); 73, (4)	$R(sl(2,\mathbb{R}),3)$
20	$\partial_x,x\partial_x+\frac{1}{2}y\partial_y,x^2\partial_x+xy\partial_y$	18	57, (16); 72, (10)	$R(sl(2,\mathbb{R}),4)$
21	$\partial_x,x\partial_x,x^2\partial_x$	11	$\overset{\mathbb{C}}{\sim}$ 57, (16); 72,(10)	$R(sl(2,\mathbb{R}),5)$
22	$y\partial_x-x\partial_y,(1+x^2-y^2)\partial_x+2xy\partial_y,$ $2xy\partial_x+(1+y^2-x^2)\partial_y$	3	$\overset{\mathbb{C}}{\sim}$ 57, (13); 73, (4)	$R(so(3),1)$
23	$\partial_y,x\partial_y,\xi_1(x)\partial_y,\xi_2(x)\partial_y$	20	58, (8)	$R(4A_1,11)$
24	$\partial_x,x\partial_x,\partial_y,y\partial_y$	13	58, (6)	$R(2A_{2.1},5)$
25	$e^{-x}\partial_y,\partial_x,\partial_y,y\partial_y$	23	58, (1)	$R(2A_{2.1},7)$
26	$e^{-x}\partial_y,-xe^{-x}\partial_y,\partial_x,\partial_y$	22	57, (21)	$R(A_{3.2}\oplus A_1,9)$
27	$e^{-x}\partial_y,e^{-ax}\partial_y,\partial_x,\partial_y,0<|a|\le 1,\ a\ne 1$	22	57, (20)	$R(A_{3.4}^a\oplus A_1,9)$
28	$e^{-bx}\sin x\partial_y,\ e^{-bx}\cos x\partial_y,\partial_x,\partial_y,\ b\ge 0$	22	$\overset{\mathbb{C}}{\sim}$ 57, (20)	$R(A_{3.5}^b\oplus A_1,8)$
29	$\partial_x,x\partial_x,y\partial_y,x^2\partial_x+xy\partial_y$	19	58, (7)	$R(sl(2,\mathbb{R})\oplus A_1,8)$
30	$\partial_x,\partial_y,x\partial_x,x^2\partial_x$	14	58, (3)	$R(sl(2,\mathbb{R})\oplus A_1,9)$
31	$\partial_y,-x\partial_y,\frac{1}{2}x^2\partial_y,\partial_x$	22	57, (23)	$R(A_{4.1},8)$
32	$e^{-bx}\partial_y,e^{-x}\partial_y,-xe^{-x}\partial_y,\partial_x$	22	57, (18)	$R(A_{4.2}^{b\ne 1},8)$
33	$e^{-x}\partial_y,-x\partial_y,\partial_y,\partial_x$	22	57, (22)	$R(A_{4.3},8)$
34	$e^{-x}\partial_y,-xe^{-x}\partial_y,\frac{1}{2}x^2e^{-x}\partial_y,\partial_x$	22	57, (19)	$R(A_{4.4},7)$
35	$\partial_y,x\partial_y,\xi_1(x)\partial_y,y\partial_y$	21	58, (9)	$R(A_{4.5}^{1,1,1},10)$
36	$e^{-ax}\partial_y,e^{-bx}\partial_y,e^{-x}\partial_y,\partial_x,-1\le a<b<1,ab\ne 0$	22	57, (17)	$R(A_{4.5}^{a,b,1},7)$
37	$e^{-ax}\partial_y,\ e^{-bx}\sin x\partial_y,e^{-bx}\cos x\partial_y,\partial_x,a>0$	22	$\overset{\mathbb{C}}{\sim}$ 57, (17)	$R(A_{4.6}^{a,b},6)$
38	$\partial_x,\partial_y,x\partial_y,x\partial_x+(2y+x^2)\partial_y$	25	58, (5)	$R(A_{4.7},5)$

Table 1.1. Continued.

N	Realizations	N_1	N_0	N_3
39	$\partial_y,\ \partial_x,\ x\partial_y,\ (1+b)x\partial_x+y\partial_y,\ \lvert b\rvert\le 1$	24	58, (4)	$R(A_{4.8}^b,5)$
40	$\partial_y,\ -x\partial_y,\ \partial_x,\ y\partial_y$	23	58, (2); 72, (7)	$R(A_{4.8}^0,7)$
41	$\partial_x,\ \partial_y,\ x\partial_x+y\partial_y,\ y\partial_x-x\partial_y$	4	$\overset{\mathbb{C}}{\sim}$ 58, (6)	$R(A_{4.10},6)$
42	$\sin x\partial_y,\ \cos x\partial_y,\ y\partial_y,\ \partial_x$	23	$\overset{\mathbb{C}}{\sim}$ 58, (1)	$R(A_{4.10},7)$
43	$\partial_x,\ \partial_y,\ x\partial_x-y\partial_y,\ y\partial_x,\ x\partial_y$	5	71, (3)	$\dim A = 5$
44	$\partial_x,\ \partial_y,\ x\partial_x,\ y\partial_y,\ y\partial_x,\ x\partial_y$	6	71, (2)	$\dim A = 6$
45	$\partial_x,\ \partial_y,\ x\partial_x+y\partial_y,\ y\partial_x-x\partial_y,$ $(x^2-y^2)\partial_x-2xy\partial_y,\ 2xy\partial_x-(y^2-x^2)\partial_y$	7	$\overset{\mathbb{C}}{\sim}$ 73, (3)	$\dim A = 6$
46	$\partial_x,\ \partial_y,\ x\partial_x,\ y\partial_y,\ x^2\partial_x,\ y^2\partial_y$	16	73, (3)	$\dim A = 6$
47	$\partial_x,\ \partial_y,\ x\partial_x,\ y\partial_y,\ y\partial_x,\ x\partial_y,$ $x^2\partial_x+xy\partial_y,\ xy\partial_x+y^2\partial_y$	8	71, (1)	$\dim A = 8$
48	$\partial_y,\ x\partial_y,\ \xi_1(x)\partial_y,\ldots,\xi_r(x)\partial_y,\ r\ge 3$	20	73, (2)	$\dim A \ge 5$
49	$y\partial_y,\ \partial_y,\ x\partial_y,\ \xi_1(x)\partial_y,\ldots,\xi_r(x)\partial_y,\ r\ge 2$	21	72, (8)	$\dim A \ge 5$
50	$\partial_x,\ \eta_1(x)\partial_y,\ldots,\eta_r(x)\partial_y,\ r\ge 4$	22	73, (1)	$\dim A \ge 5$
51	$\partial_x,\ y\partial_y,\ \eta_1(x)\partial_y,\ldots,\eta_r(x)\partial_y,\ r\ge 3$	23	72, (7)	$\dim A \ge 5$
52	$\partial_x,\ \partial_y,\ x\partial_x+cy\partial_y,\ x\partial_y,\ldots,x^r\partial_y,\ r\ge 2$	24	72, (5)	$\dim A \ge 5$
53	$\partial_x,\ \partial_y,\ x\partial_y,\ldots,x^{r-1}\partial_y,x\partial_x+(ry+x^r)\partial_y,\ r\ge 3$	25	72, (6)	$\dim A \ge 5$
54	$\partial_x,\ x\partial_x,\ y\partial_y,\ \partial_y,\ x\partial_y,\ldots,x^r\partial_y,\ r\ge 1$	26	72, (4)	$\dim A \ge 5$
55	$\partial_x,\ \partial_y,\ 2x\partial_x+ry\partial_y,\ x^2\partial_x+rxy\partial_y,$ $x\partial_y,\ x^2\partial_y,\ldots,\ x^r\partial_y,\ r\ge 1$	27	71, (4); 72, (1)	$\dim A \ge 5$
56	$\partial_x,\ x\partial_x,\ y\partial_y,\ x^2\partial_x+rxy\partial_y,$ $\partial_y,\ x\partial_y,x^2\partial_y,\ldots,x^r\partial_y,r\ge 0$	15; 28	73, (5); 72, (2)	$\dim A \ge 5$

Realizations of Lie algebras by vector fields in one real, one and two complex variables were classified by Lie [20–22]. Gonzalez-Lopez et al. ordered the Lie's classification of realizations of complex Lie algebras [13] and extended it to the real case [14]. A complete set of inequivalent realizations of real Lie algebras of dimension no greater than four in vector fields on a space of an arbitrary (finite) number of variables was constructed in [38]. The mentioned works do not exhaust all papers devoted to realizations of Lie algebras, but only them will be used in the present paper.

An extended overview on both these subjects is contained in the preprint math-ph/ 0301029v7.

Starting from the above results, we detailed and amended the classification of realizations of finite-dimensional Lie algebras on the real plane. The obtained classification is

Table 1.2. Transformations of real realizations to complex ones.

N_1	Transformation of space variables	Transformation of basis elements	N_2
1	$\widetilde{x} = x - iy,\ \widetilde{y} = x + iy$	$\widetilde{e}_1 = \dfrac{1+i}{2}(e_1 + e_2),\ \widetilde{e}_2 = \dfrac{1}{c+i}e_3,$ $\widetilde{e}_3 = \dfrac{1-i}{2}(e_1 - e_2)$	2.7, $k = 1$
2	$\widetilde{x} = x - iy,\ \widetilde{y} = \dfrac{1}{2iy}$	$\widetilde{e}_1 = e_1,\ \widetilde{e}_2 = e_2,\ \widetilde{e}_3 = e_3$	2.2
3	$\widetilde{x} = -\dfrac{1}{ix+y},\ \widetilde{y} = \dfrac{ix+y}{1+x^2+y^2}$	$\widetilde{e}_1 = \dfrac{1}{2}(ie_2 + e_3),\ \widetilde{e}_2 = ie_1,\ \widetilde{e}_3 = \dfrac{1}{2}(e_3 - ie_2)$	2.2
4	$\widetilde{x} = \dfrac{y - ix}{2},\ \widetilde{y} = -\dfrac{y + ix}{2}$	$\widetilde{e}_1 = ie_1 - e_2,\ \widetilde{e}_2 = ie_1 + e_2,$ $\widetilde{e}_3 = \dfrac{e_3 + ie_4}{2},\ \widetilde{e}_4 = \dfrac{e_3 - ie_4}{2}$	2.9, $k = 1$
7	$\widetilde{x} = y + ix,\ \widetilde{y} = y - ix$	$\widetilde{e}_1 = \dfrac{e_1 + ie_2}{2i},\ \widetilde{e}_2 = \dfrac{e_3 - ie_4}{2},\ \widetilde{e}_3 = \dfrac{e_6 + ie_5}{2},$ $\widetilde{e}_4 = \dfrac{ie_2 - e_1}{2i},\ \widetilde{e}_5 = \dfrac{e_3 + ie_4}{2},\ \widetilde{e}_6 = \dfrac{e_6 - ie_5}{2}$	2.4
17	$\widetilde{x} = y,\ \widetilde{y} = \dfrac{1}{x - y}$	$\widetilde{e}_1 = e_1,\ \widetilde{e}_2 = e_2,\ \widetilde{e}_3 = e_3$	2.2
18	$\widetilde{x} = x,\ \widetilde{y} = \dfrac{1}{y^2}$	$\widetilde{e}_1 = e_1,\ \widetilde{e}_2 = \dfrac{1}{2}e_2,\ \widetilde{e}_3 = e_3$	2.1
19	$\widetilde{x} = x,\ \widetilde{y} = \dfrac{1}{y}$	$\widetilde{e}_1 = e_1,\ \widetilde{e}_2 = e_2,\ \widetilde{e}_3 = -e_3,\ \widetilde{e}_4 = e_4$	2.3

compared with existing classifications on the real [14] and complex [20–22] planes and arranged in Table 1.1.

The nontrivial transformations over the complex field that reduce realizations from [14] to realizations from [30] are adduced in Table 1.2.

Notation. We denote ∂/∂_x, ∂/∂_y,... as ∂_x, ∂_y,.... The indices i and j run from 1 to r, where variation range for r is to be determined additionally in each case. The label N_0 consists of two parts which denote the page (from 57 to 73) and realization numbers in [22] correspondingly. The labels N_1 and N_2 coincide with the numerations of real and complex realizations in [14, 30]. N_3 corresponds to the numeration of realizations introduced in [38], namely, $R(A, n)$ denotes the nth realization of the Lie algebra A from [38], or, if the dimension of the algebra is larger than four, the corresponding dimension is indicated in the column entitled N_3. The symbol N without subscripts corresponds to the numeration used in the present paper.

Remark 2.1. The realization of rank two of the non-Abelian two-dimensional real Lie algebra $\langle \partial_x, x\partial_x + y\partial_y \rangle$ (case $N = 4$) is missed in [14] from the formal point of view. But it can be joined to the realization series $\langle \partial_x, \partial_y, x\partial_x + cy\partial_y, x\partial_y, \ldots, x^r\partial_y \rangle$, $r \geq 1$ (case $N_1 = 24$), written in the form $\langle \partial_x, x\partial_x + cy\partial_y, x^k\partial_y, k = -1, 0, \ldots, r \rangle$ under the supposition for $k = -1$, $x^{-1} = 0$ and $c = 1$.

Table 1.3. Differential invariants, operators of invariant differentiation, and Lie determinants of real-izations of Lie algebras on the real plane.

N	Basis of differential invariants	Operator	Lie determinant
1	y	D_x	const
1*	x, y'	D_x	const
2	y', y''	D_x	const
3	$y, \dfrac{y''}{y'^3}$	$\dfrac{1}{y'}D_x$	$-(y')^2$
3*	x, y''	D_x	const
4	y'	yD_x	y
5	$y, \dfrac{y''}{(y')^2}$	$\dfrac{1}{y'}D_x$	y'
5*	$x, \dfrac{y''}{y'}$	D_x	y'
6	$x, y''\xi_1'''(x) - y'''\xi_1''(x)$	D_x	$\xi_1''(x)$
7	$\dfrac{y''}{y'}$	D_x	y'
8	$y'' + y'$	D_x	$-e^{-x}$
9	y''	D_x	const
10	$y''e^{y'}$	$e^{y'}D_x$	const
11	$y'' + 2y' + y$	D_x	$-e^{-2x}$
12	$\dfrac{y'''}{(y'')^2}$	$\dfrac{1}{y''}D_x$	$-y''$
13	$x, \dfrac{y'''}{y''}$	D_x	$-y''$
14	$y''y'^{(2-a)/(a-1)}$	$(y')^{(1)/(a-1)}D_x$	$(a-1)y'$
15	$y'' + (a+1)y' + ay$	D_x	$(1-a)e^{-(1+a)x}$
16	$y''e^{-c\arctan y'}B_1^{-3/2}$	$e^{-c\arctan y'}B_1^{-1/2}D_x$	B_1
17	$y'' + 2by' + (b^2+1)y$	D_x	$-e^{-2bx}$
18	$(y''y + (y')^2 + 1)B_1^{-3/2}$	$2yB_1^{-1/2}D_x$	$2y^2B_1$
19	$(y''(x-y) + 2y'(1+y'))(y')^{-3/2}$	$(x-y)(y')^{-1/2}D_x$	$2y'(x-y)^2$
20	y^3y''	y^2D_x	y^2
21	$x, (y')^{-2}Q_3$	D_x	$y(y-x)y'$
21*	$y, (3y''^2 - 2y'y''')(y')^{-4}$	$\dfrac{1}{y'}D_x$	y'
22	$y''B_0B_1^{-3/2} + 2(y-xy')B_1^{-1/2}$	$B_0B_1^{-1/2}D_x$	$B_0^2B_1$
23	$x, y'''P_{2,4}(\xi_1,\xi_2) + y''P_{4,3}(\xi_1,\xi_2) + y^{(4)}$	D_x	$P_{2,3}(\xi_1,\xi_2)$
24	$\dfrac{y'y'''}{(y'')^2}$	$\dfrac{y'}{y''}D_x$	$y'y''$
25	$\dfrac{y''' + y''}{y'' + y'}$	D_x	$-e^{-x}(y'' + y')$
26	$y''' + 2y'' + y'$	D_x	$-e^{-2x}$
27	$y''' + (1+a)y'' + ay'$	D_x	$a(a-1)e^{-(1+a)x}$

Table 1.3. Continued.

N	Basis of differential invariants	Operator	Lie determinant
28	$y''' + 2by'' + (1+b^2)y'$	D_x	$-(1+b^2)e^{-2bx}$
29	$S_3 Q_2^{-3/2}$	$\sqrt{\dfrac{y}{y''}}D_x$	$-2y^2y''$
30	$Q_3(y')^{-4}$	$\dfrac{1}{y'}D_x$	$2y'^2$
31	y'''	D_x	const
32	$y''' + (b+2)y'' + (2b+1)y' + by$	D_x	$(b-1)^2 e^{-(b+2)x}$
33	$y''' + y''$	D_x	$-e^{-x}$
34	$y''' + 3y'' + 3y' + y$	D_x	$-e^{-3x}$
35	$x, \dfrac{P_{2,4}(\xi_1, y)}{P_{2,3}(\xi_1, y)}$	D_x	$P_{2,3}(\xi_1, y)$
36	$y''' + (a+b+1)y'' + (ab+a+b)y' + aby$	D_x	$\dfrac{(b-a)(1-a)(1-b)}{e^{(a+b+1)x}}$
37	$y''' + (2b+a)y'' + (b^2+2ab+1)y' + a(b^2+1)y$	D_x	$((b-a)^2+1)e^{-(2b+a)x}$
38	$y''' e^{y''/2}$	$e^{y''/2}D_x$	const
39	$b=1: y'', \; y^{\mathrm{iv}}(y''')^{-2}$	$\dfrac{1}{y'''}D_x$	y'''
	$b \neq 1: (y'')^{(2-b)/(b-1)}$	$y''^{(1)/(b-1)}D_x$	$(1-b)y''$
40	$\dfrac{y'''}{y''}$	D_x	$-y''$
41	$(y'')^{-2}B_1 y''' - 3y'$	$\dfrac{B_1}{y''}D_x$	$3y''B_1$
42	$\dfrac{y'''+y'}{y''+y}$	D_x	$y''+y$
43	$(3y''y^{\mathrm{iv}} - 5(y''')^2)(y'')^{-8/3}$	$(y'')^{-1/3}D_x$	y''
44	$S_5 R_4^{-3/2}$	$y'' R_4^{-1/2}D_x$	$(y'')^2 R_4$
45	$\tilde{U}_5 \tilde{Q}_3^{-3}$	$B_1 \tilde{Q}_3^{-1/2}D_x$	$-16B_1 \tilde{Q}_3^2$
46	$U_5 Q_3^{-3}$	$y' Q_3^{-1/2}D_x$	$-4y' Q_3^{-2}$
47	$V_7 S_5^{-8/3}$	$y'' S_5^{-1/3}D_x$	$-2y'' S_5^2$
48	$x, W(y'', \xi_1'', \xi_2'', \ldots, \xi_r'')$	D_x	$W(\xi_1'', \xi_2'', \ldots, \xi_r'')$
49	$x, D_x \ln |W(y'', \xi_1'', \xi_2'', \ldots, \xi_r'')|$	D_x	$W(y'', \xi_1'', \ldots, \xi_r'')$
50	$K_r(\eta_1, \ldots, \eta_r)$	D_x	$W(\eta_1, \eta_2, \ldots, \eta_r)$
51	$D_x \ln |K_r(\eta_1, \ldots, \eta_r)|$	D_x	$W(y, \eta_1, \ldots, \eta_r)$
52	$c \neq r+1: (y^{(r+1)})^{(2-c+r)/(c-r-1)} y^{(r+2)}$	$(y^{(r+1)})^{1/(c-r-1)}D_x$	$y^{(r+1)}$
	$c = r+1: y^{(r+1)}, \; \dfrac{y^{(r+3)}}{(y^{(r+2)})^2}$	$\dfrac{1}{y^{(r+2)}}D_x$	$y^{(r+2)}$
53	$y^{(r+1)} e^{y^{(r)}/r!}$	$e^{y^{(r)}/r!}D_x$	const
54	$\dfrac{y^{(r+1)} y^{(r+3)}}{(y^{(r+2)})^2}$	$\dfrac{y^{(r+1)}}{y^{(r+2)}}D_x$	$y^{(r+1)} y^{(r+2)}$
55	$Q_{r+3}(y^{(r+1)})^{-(2r+8)/(r+2)}$	$(y^{(r+1)})^{-2/r+2}D_x$	$y^{(r+1)}$
56	$S_{r+4} Q_{r+3}^{-3/2}$	$y^{(r+1)} Q_{r+3}^{-1/2}D_x$	$y^{(r+1)} Q_{r+3}$

Remark 2.2. There are two different approaches to classification of Lie algebra realizations by vector fields. According to the first approach, one should start from classification of Lie algebras and then look for basis vector fields that satisfy the given commutation relations. The second approach consists in direct construction of finite-dimensional spaces of vector fields, which are closed with respect to the standard Lie bracket. If a complete list of realizations of a fixed dimension is constructed, then the problem of separation of the realizations for a given Lie algebra from others arises and becomes nontrivial in the case of parameterized series of realizations.

Example 2.3. Consider the series $\{A_{4.8}^b\}$ [27] of real four-dimensional Lie algebras parameterized with the parameter $|b| \leq 1$. For a fixed value of b, the basis elements of $A_{4.8}^b$ satisfy the canonical commutation relations

$$[e_2, e_3] = e_1, \qquad [e_1, e_4] = (1+b)e_1, \qquad [e_2, e_4] = e_2, \qquad [e_3, e_4] = be_3. \qquad (2.1)$$

In the framework of the first approach we obtain two inequivalent realizations in vector fields on a space of two variables [38]

$$\langle \partial_x, \partial_y, y\partial_x, (1+b)x\partial_x + y\partial_y \rangle, \qquad \langle \partial_x, y\partial_x, -\partial_y, (1+b)x\partial_x + by\partial_y \rangle \qquad (2.2)$$

of the algebra $A_{4.8}^b$ if $|b| < 1$. There is a unique inequivalent realization in the case $b = \pm 1$ since under this condition realizations (2.2) are equivalent and we have to choose only one of them.

Lie [20–22] used the second approach to construct all possible realizations of finite-dimensional Lie algebras on the plane. The algebras from the series $\{A_{4.8}^b\}$ are represented in the obtained list by the following realizations:

$$\langle \partial_y, \partial_x, x\partial_y, x\partial_x + \tilde{b}y\partial_y \rangle, \quad \tilde{b} \in \mathbb{R}, \qquad \langle \partial_y, -x\partial_y, \partial_x, y\partial_y \rangle. \qquad (2.3)$$

In fact, the sets of realizations (2.2) and (2.3) coincide. To show this, we redenote the variables x and y in (2.3) (namely, $x \leftrightarrow y$) at first and shift the parameter $\tilde{b}$: $\tilde{b} = 1 + b'$. After reordering the basis in the first realization from (2.3) in the case $|b'| \leq 1$, we obtain the first realization from (2.2), where $b = b'$. If $|b'| > 1$, the first realization from (2.3) is reduced to the second realization from (2.2) with $b = 1/b'$ by the additional simultaneous transformations of the basis and realization variables: $\tilde{e}_1 = b'e_1$, $\tilde{e}_2 = e_3$, $\tilde{e}_3 = -b'e_2$, $\tilde{e}_4 = be_4$; $\tilde{x} = bx$, $\tilde{y} = by$. The second realization from (2.3) coincides with the second one from (2.2), where $b = 0$.

The above consideration explains in some way why the parameter values $b = \pm 1$ are singular for the Lie algebra series $\{A_{4.8}^b\}$ from the viewpoint of number of realizations.

Remark 2.4. It is clear that the realizations from different series adduced in Table 1.1 are inequivalent each to other, but there can exist equivalent realizations belonging to the same series.

Example 2.5. Consider the series of realizations

$$N = 6, 23, 48 : \langle \xi_1(x)\partial_y, \xi_2(x)\partial_y, \ldots, \xi_{r+2}(x)\partial_y \rangle, \quad r \geq 1, \tag{2.4}$$

$$N = 35, 49 : \langle y\partial_y, \xi_1(x)\partial_y, \xi_2(x)\partial_y, \ldots, \xi_{r+2}(x)\partial_y \rangle, \quad r \geq 1 \tag{2.5}$$

parameterized with arbitrary linearly independent real-valued functions ξ_i.

Any realization from series (2.4) or (2.5) pass into realizations from the same series under the basis transformations with nonsingular constant matrices (c_{ij}) and the nonsingular variable transformations $\tilde{x} = \varphi(x)$, $\tilde{y} = \psi(x)y$. By means of these equivalence transformations the parameter functions ξ_i change in the following way $\tilde{\xi}_i(\tilde{x}) = c_{ij}\psi(x)\xi_j(x)|_{\tilde{x}=\varphi(x)}$. Consequently, without loss of generality we can put $\tilde{\xi}_{r+1} = 1$ and $\tilde{\xi}_{r+2} = \tilde{x}$. Hence, the series of realizations (2.4) and (2.5) takes the form adduced in Table 1.1, namely,

$$\langle \partial_{\tilde{y}}, \tilde{x}\partial_{\tilde{y}}, \tilde{\xi}_1(\tilde{x})\partial_{\tilde{y}}, \ldots, \tilde{\xi}_r(\tilde{x})\partial_{\tilde{y}} \rangle, \qquad \langle \tilde{y}\partial_{\tilde{y}}, \partial_{\tilde{y}}, \tilde{x}\partial_{\tilde{y}}, \tilde{\xi}_1(\tilde{x})\partial_{\tilde{y}}, \ldots, \tilde{\xi}_r(\tilde{x})\partial_{\tilde{y}} \rangle. \tag{2.6}$$

Accurately speaking, the series with normalized forms (2.6) also contain equivalent realizations, and the corresponding equivalence transformations are restrictions of the aforesaid ones.

Example 2.6. Another example is given by two series of realizations

$$N = 50 : \langle \partial_x, \eta_1(x)\partial_y, \ldots, \eta_r(x)\partial_y \rangle, \quad r \geq 4,$$
$$N = 51 : \langle \partial_x, y\partial_y, \eta_1(x)\partial_y, \ldots, \eta_r(x)\partial_y \rangle, \quad r \geq 3, \tag{2.7}$$

parameterized with real functions η_i which form a fundamental system of solutions for an r-order ordinary differential equation with constant coefficients

$$\eta^{(r)}(x) + c_1\eta^{(r-1)}(x) + \cdots + c_r\eta(x) = 0. \tag{2.8}$$

The transformations that reduce any realization from the series $N = 50$ and $N = 51$ to a realization from the same series are generated by the changes of basis with nonsingular constant matrices (c_{ij}) and the variable transformations $\tilde{x} = a_1x + a_0$, $\tilde{y} = by + f(x)$, where $f(x) = b_0\eta_0(x) + b_1\eta_1(x) + \cdots + b_r\eta_r(x)$, $a_1, a_0, b, b_0, \ldots, b_r \in \mathbb{R}$, $a_1b \neq 0$. The function $\eta_0(x)$ is a solution of the ODE $\eta_0^{(r)}(x) + c_1\eta_0^{(r-1)}(x) + \cdots + c_r\eta_0(x) = 1$ and in the case $N = 51$ additionally $b_0 = 0$. These equivalence transformations act on the functions η_i as follows: $\tilde{\eta}_i(a_1x + a_0) = c_{ij}\eta_j(x)$.

3. Differential invariants

Foundations of the theory of differential invariants were established in classical works of Tresse [41, 42], Lie, and Cartan, and are developed in our days [10, 11, 30–32]. We shortly formulate necessary definitions and statements following [31, 32] in general outlines.

Consider a local r-parametric transformation group G acting on $M \subset X \times Y = \mathbb{R} \times \mathbb{R}$ and denote a prolonged transformation group acting on the subset of jet space $M^{(n)} = M \times \mathbb{R}^n$ as $\mathrm{pr}^{(n)} G$. Let $\mathfrak{g}$ be the r-dimensional Lie algebra with basis of infinitesimal operators $\{e_i = \xi_i(x, y)\partial_x + \eta_i(x, y)\partial_y\}$ which corresponds to G. Then the prolonged algebra $\mathrm{pr}^{(n)} \mathfrak{g}$ is generated by the prolonged first-order differential operators [31, 32]:

$$e_i^{(n)} = \xi_i(x, y)\partial_x + \eta_i(x, y)\partial_y + \sum_{k=1}^{n} \eta_i^k(x, y_{(k)})\partial_{y^{(k)}}. \tag{3.1}$$

Hereafter $n, k \in \mathbb{N}$, $i = 1, \ldots, r$, the symbol $y_{(k)}$ denotes the tuple $(y, y', \ldots, y^{(k)})$ of the dependent variable y and its derivatives with respect to x of order no greater than k.

Definition 3.1. A smooth function $I = I(x, y_{(n)}) : M^{(n)} \to \mathbb{R}$ is called a *differential invariant* of order n of the group G if I is an invariant of the prolonged group $\mathrm{pr}^{(n)} G$, namely,

$$I(\mathrm{pr}^{(n)} g \cdot (x, y_{(n)})) = I(x, y_{(n)}), \quad (x, y_{(n)}) \in M^{(n)}, \tag{3.2}$$

for all $g \in G$ for which $\mathrm{pr}^{(n)} g \cdot (x, y_{(n)})$ is defined.

In infinitesimal terms, $I(x, y_{(n)})$ is an nth-order differential invariant of the group G if $e_i^{(n)} I(x, y_{(n)}) = 0$ for any prolonged basis infinitesimal generators $e_i^{(n)}$ of $\mathrm{pr}^{(n)} \mathfrak{g}$.

Consider the series of the ranks $r_k = \mathrm{rank}\{(\xi_i, \eta_i, \eta_i^1, \ldots, \eta_i^k), \ i = 1, \ldots, r\}$. For further statements we introduce the number $\nu = \min\{k \in \mathbb{Z} \,|\, r_k = r\}$. Since the sequence $\{r_k\}$ is nondecreasing, bounded by r, and reaches the value r, the number ν exists and the relation $r_\nu = r_{\nu+1} = \cdots = r$ holds true.

Definition 3.2. Let $\mathrm{pr}^{(\nu)} \mathfrak{g}$ be generated by the set of the prolonged infinitesimal operators $\{e_i^{(\nu)}\}$, and let L be the matrix formed by their coefficients:

$$L = \begin{pmatrix} \xi_1 & \eta_1 & \eta_1^1 & \cdots & \eta_1^{(\nu)} \\ \xi_2 & \eta_2 & \eta_2^1 & \cdots & \eta_2^{(\nu)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_r & \eta_r & \eta_r^1 & \cdots & \eta_r^{(\nu)} \end{pmatrix}. \tag{3.3}$$

A maximal minor of L, which does not vanish identically, is called a *Lie determinant*.

Importance of the adduced notions is explained by the following fact. If a system of ordinary differential equations is invariant under action of the prolonged group $\mathrm{pr}^{(n)} G$ then it can be locally presented as a union of conditions of vanishing Lie determinants and equations written in terms of differential invariants of G.

A natural question is whether it is possible to choose a minimal set of differential invariants that allows us to obtain all differential invariants of the given order by a finite number of certain operations. The answer to this question is affirmative.

Below we will briefly state several results concerning differential invariants of Lie groups acting on the plane. The presented statements are well known and adduced only on purpose to complete the picture of differential invariants on the plane. Detailed definitions and statements on the theory of differential invariants in general cases, review

of main results, and approaches to differential invariants that are different from the presented one (such as differential one-forms and moving coframes) and possibly more convenient for other applications could be found, for example, in [10, 11, 31, 32].

Definition 3.3. A maximal set I_n of functionally independent differential invariants of order no greater than n (i.e., invariants of the prolonged group $\mathrm{pr}^{(n)}G$) is called a *universal differential invariant* of order n of the group G.

Note that the dimension of the jet space $M^{(n)}$ is $\dim M^{(n)} = n + 2$ and the number of functionally independent differential invariants of order n is $d_n = n + 2 - r_n$. Any nth-order differential invariant I of G is necessarily an $(n + l)$th-order differential invariant of G, $l \geq 0$. Therefore for any $n, l \geq 0$, a universal differential invariant I_{n+l} can be obtained by extension of a universal differential invariant I_n.

Definition 3.4. A vector field (or a differential operator) δ on the infinitely prolonged jet space $M^{(\infty)}$ is called an *operator of invariant differentiation* of the group G if for any differential invariant I of G the expression δI is also a differential invariant of G.

Any operator δ commuting with all formally infinitely prolonged basis infinitesimal generators e_i^∞ of the corresponding Lie algebra is an operator of invariant differentiation of G. For any Lie group acting on the real or complex planes, there exists exactly one independent (over the field of invariants of this group) operator of invariant differentiation.

For any Lie group G there exists a finite *basis of differential invariants*, that is, a finite set of functionally independent differential invariants such that any differential invariant of G can be obtained from it via a finite number of functional operations and operations of invariant differentiation. A basis of differential invariants of the group G is always contained in a universal differential invariant $I_{\nu+1}$.

To describe completely differential invariants of all transformation groups acting on the real plane, we obtain a functional basis of differential invariants and operators of invariant differentiation for each algebra from the known list of inequivalent Lie algebras of vector fields on the plane.

Bases of differential invariants are constructed as a part of the $(\nu + 1)$th-order universal differential invariants using the infinitesimal approach. The constructive procedure for finding invariant differentiation operators is directly derived from the condition of their commutation with formally infinitely prolonged elements of the algebra. Namely, we look for an operator of invariant differentiation as the operator of total differentiation D_x with a multiplier λ depending on x and $y_{(\nu)}$: $X = \lambda(x, y_{(\nu)})D_x$, where $\lambda : M^{(\nu)} \to \mathbb{R}$. The function λ is implicitly determined by the equation $\varphi(x, y_{(\nu)}, \lambda) = 0$, where φ satisfies the condition

$$\overline{\zeta_i^\gamma}\varphi = 0, \quad \overline{\zeta_i^\gamma} = \xi_i \partial_x + \eta_i \partial_y + \eta_i^1 \partial_{y'} + \cdots + \eta_i^\gamma \partial_{y^{(\gamma)}} + (\lambda D_x)\xi_i \partial_\lambda. \tag{3.4}$$

In other words, $\varphi(x, y_{(\nu)}, \lambda)$ should be an invariant of the flows generated by vector fields $\overline{\zeta_i^\gamma}$. Let us note that $\mathrm{rank}\{(\overline{\zeta_i^\gamma}), \ i = 1, \ldots, r\} = r$. A universal invariant $\overline{I}$ of $\overline{\zeta_i^\gamma}$ can be presented as $\overline{I} = (I_\nu, \hat{I})$, where $\hat{I} : M^{(\nu)} \times \mathbb{R} \to \mathbb{R}$, $\partial\hat{I}/\partial\lambda \neq 0$. So, the unknown function λ can be found from the condition $\hat{I}(x, y_{(\nu)}, \lambda) = C$ for a fixed constant C.

All the obtained results obtained are presented in Table 1.3 and may be used for group classification of ODEs of any finite order. In a similar way one can describe the differential

invariants of the transformations groups acting in the spaces of more than two variables and having a low number of parameters by means of using the classification of realizations of real low-dimensional Lie algebras [38] and then applying them to investigation of systems of ODEs or PDEs.

Example 3.5. Let us illustrate the above statements by considering the Lie algebra $A_{3.5}^b$, $b \geq 0$, generated by the basis elements (case $N = 17$ in Tables 1.1 and 1.3)

$$e_1 = \partial_y, \qquad e_2 = x\partial_y, \qquad e_3 = -(1+x^2)\partial_x + (b-x)y\partial_y. \qquad (3.5)$$

The second prolongations of these operators are

$$e_1^{(2)} = \partial_y,$$

$$e_2^{(2)} = x\partial_y + \partial_{y'}, \qquad (3.6)$$

$$e_3^{(2)} = -(1+x^2)\partial_x + (b-x)y\partial_y - (y-(b+x)y')\partial_{y'} + (b+3x)y''\partial_{y''}.$$

Inasmuch as dimension of the algebra is $r = 3$ and the rank of the first prolongation is $r_1 = 3$, the ranks of all other prolongations also equal to 3: $r_1 = r_2 = \cdots = r = 3$. In this case $\nu = 1$. Hence, the basis of differential invariants is contained in the universal differential invariant $I_{\nu+1} = I_2$. The Lie determinant $\mathcal{L}$ is calculated as the determinant of the matrix formed by the coefficients of $e_i^{(2)}$, $i = 1,2,3$:

$$\mathcal{L} = \det \begin{pmatrix} 0 & 1 & 0 \\ 0 & x & 1 \\ -(1+x^2) & (b-x)y & -(y-(b+x)y') \end{pmatrix} = -(1+x^2). \qquad (3.7)$$

It produces no invariant differential equations.

Let us look for a basis of differential invariants. There are no differential invariants of orders 0 and 1 (because of $d_0 = 0 + 2 - 2 = 0$ and $d_1 = 1 + 2 - r_1 = 0$). The universal differential invariant I_2, as well as the basis of differential invariants, is formed by a single (because of $d_2 = 2 + 2 - r_2 = 1$) function $I = I(x,y,y',y'')$. It is defined by the conditions $e_i^{(2)}I = 0$, $i = 1,2,3$, which are equivalent to the overdetermined system of first-order linear PDEs

$$\frac{\partial I}{\partial y} = 0,$$

$$x\frac{\partial I}{\partial y} + \frac{\partial I}{\partial y'} = 0, \qquad (3.8)$$

$$-(1+x^2)\frac{\partial I}{\partial x} + (b-x)y\frac{\partial I}{\partial y} - (y-(b+x)y')\frac{\partial I}{\partial y'} + (b+3x)y''\frac{\partial I}{\partial y''} = 0.$$

It is follows from the first two equations that $I = I(x,y'')$. The basis of differential invariants of the considered realization is obtained as the set of functionally independent

integrals for the reduced characteristic system of the third equation:

$$I_2 = \left\{ y''(1+x^2)^{3/2} e^{b\arctan x} \right\}. \tag{3.9}$$

The last task which should be solved in order to describe all differential invariants of this realization is construction of an operator of invariant differentiation.

We look for the operator of invariant differentiation in the form $\lambda(x, y, y')D_x$. Here D_x is the operator of total differentiation. The function λ is implicitly given by the equation $\varphi(x, y, y', \lambda) = 0$, where φ is a nonconstant solution of the overdetermined system of first-order linear PDEs

$$\frac{\partial \varphi}{\partial y} = 0, \qquad x\frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial y'} = 0,$$

$$(1+x^2)\frac{\partial \varphi}{\partial x} - (b-x)y\frac{\partial \varphi}{\partial y} + (y - (b+x)y')\frac{\partial \varphi}{\partial y'} + 2x\lambda\frac{\partial \varphi}{\partial \lambda} = 0. \tag{3.10}$$

The first two equations result in the condition $\varphi = \varphi(x, \lambda)$. Then the latter implies $\varphi = \varphi(\omega)$, where $\omega = \lambda(1+x^2)^{-1}$, that is, the function λ can be found from the equation $\lambda(1+x^2)^{-1} = 1$. The corresponding operator of invariant differentiation is $(1+x^2)D_x$.

Remark 3.6. The form of differential invariants essentially depends on explicit form of realizations. Therefore, to construct an optimal set of invariants, one should choose an optimal form of realizations. For example, for the realization $\langle e^{-bx}\sin x\partial_y, e^{-bx}\cos x\partial_y, \partial_x \rangle$, where $b \geq 0$ ($N = 17$), a fundamental differential invariant, an operator of invariant differentiation and a Lie determinant have the following form:

$$I_2 = y'' + 2by' + (b^2+1)y, \qquad X = D_x, \qquad L = -e^{-2bx}. \tag{3.11}$$

For the equivalent form $\langle \partial_y, x\partial_y, -(1+x^2)\partial_x + (b-x)y\partial_y \rangle$ which is considered in Example 3.5, the corresponding invariant objects are more complicated:

$$I_2 = y''(1+x^2)^{3/2}e^{b\arctan x}, \qquad X = (1+x^2)D_x, \qquad L = -(1+x^2). \tag{3.12}$$

Remark 3.7. The cases marked with "$*$" in Table 1.3 differ from the cases with the same numbers by changes of dependent and independent variables. They are adduced simultaneously because different forms may be convenient for different applications.

4. Concluding remarks

In this paper we provide a complete description of differential invariants and Lie determinants of finite-dimensional Lie groups acting on the real plane. Obtained results are presented in Table 1.3. As preliminaries of the above problem, known results on classification of realizations of real Lie algebras in vector fields in two variables were reviewed and amended (Table 1.1). Namely, the low-dimensional Lie algebras were extracted from the general cases and presented in explicit form, which made this classification more convenient for applications. Additionally, the problem of equivalence of realizations belonging to the same series of Lie algebra was discussed.

Results of the paper could be applied to group classification of ODEs of any finite order over the real field. In the future we plan to review and to generalize results of group classification of the third- and fourth-order ODEs that were obtained in [7, 39]. In a similar way one can describe the differential invariants of the transformations groups acting in the spaces of more than two variables and having a low number of parameters by means of using the classification of realizations of real low-dimensional Lie algebras and then applying them to investigation of systems of ODEs or PDEs.

In Table 1.1 the functions $1, x, \xi_1, \ldots, \xi_r$ are linearly independent. The functions $\eta_1, \ldots, \eta_r$ form a fundamental system of solutions for an r-order ordinary differential equation with constant coefficients $\eta^{(r)}(x) + c_1 \eta^{(r-1)}(x) + \cdots + c_r \eta(x) = 0$.

In Table 1.3 we use the following notations:

$$S_{k+3} = (k+1)^2 (y^{(k)})^2 y^{(k+3)} - 3(k+1)(k+3) y^{(k)} y^{(k+1)} y^{(k+2)} + 2(k+2)(k+3)(y^{(k+1)})^3,$$

$$Q_{k+2} = (k+1) y^{(k)} y^{(k+2)} - (k+2)(y^{(k+1)})^2, \qquad \widetilde{Q_3} = y''' B_1 - 3y'(y'')^2,$$

$$B_0 = 1 + x^2 + y^2, \qquad B_1 = 1 + (y')^2,$$

$$P_{i,j}(\varphi, \psi) = \varphi^{(i)} \psi^{(j)} - \varphi^{(j)} \psi^{(i)}, \qquad R_4 = 3y'' y^{\mathrm{iv}} - 5(y''')^2,$$

$$\widetilde{U_5} = 4y^{\mathrm{v}} B_1^3 Q + 10 y^{\mathrm{iv}} y'' B_1^3 (4y''' y' + 3(y'')^2) - 5(y^{\mathrm{iv}})^2 B_1^4 + 40(y''')^2 (y'')^2 ((y')^2 - 2) B_1^2$$

$$- 40(y''')^3 y' B_1^3 - 180 y''' y' (y'')^4 ((y')^2 - 1) B_1^2 - (y'')^6 (45(6(y')^2 + 1) - 135(y')^4),$$

$$U_5 = (y')^2 \left(Q_3 D_x^2 Q_3 - \frac{5}{4} (D_x Q_3)^2 \right) + y' y'' Q_3 D_x Q_3 - (2y' y''' - (y'')^2) Q_3^2,$$

$$V_7 = (y'')^2 \left(S_5 D_x^2 S_5 - \frac{7}{6} (D_x S_5)^2 \right) + y'' y''' S_5 D_x S_5 - \frac{1}{2} (9y'' y^{\mathrm{iv}} - 7(y''')^2) S_5^2,$$

$$W(f_1, f_2, \ldots, f_r) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_r(x) \\ f_1'(x) & f_2'(x) & \cdots & f_r'(x) \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(r-1)}(x) & f_2^{(r-1)}(x) & \cdots & f_r^{(r-1)}(x) \end{vmatrix},$$

$$K_r(\eta_1, \eta_2, \ldots, \eta_r) = y^{(r)} + c_1 y^{(r-1)} + c_1 y^{(r-1)} + \cdots + c_r y, \tag{4.1}$$

where $c_1, \ldots, c_r$ are the constant coefficients of the rth-order ODE,

$$\eta^{(r)}(x) + c_1 \eta^{(r-1)}(x) + \cdots + c_r \eta(x) = 0, \tag{4.2}$$

which is satisfied by the functions $\eta_1(x), \ldots, \eta_r(x)$.

Acknowledgments

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