ON THE GENERALIZED ROPER-SUFFRIDGE EXTENSION OPERATOR IN BANACH SPACES

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The generalized Roper-Suffridge extension operator in Banach spaces is introduced. We prove that this operator preserves the starlikeness on some domains in Banach spaces and does not preserve convexity in some cases. Furthermore, the growth theorem and covering theorem of the corresponding mappings are given. Some results of Roper and Suffridge and Graham et al. in $\mathbb{C}^n$ are extended to Banach spaces.

1. Introduction and preliminaries

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ and B_n denote the unit disc in $\mathbb{C}$ and the unit ball in $\mathbb{C}^n$, respectively. In [11], Roper and Suffridge introduced an extension operator, which is defined for normalized locally univalent function f on U by

$$\Phi_n(f)(z) = \left(f(z_1), \sqrt{f'(z_1)}z_0\right), \tag{1.1}$$

where $z_1 \in U$, $z_0 = (z_2,\dots,z_n) \in \mathbb{C}^{n-1}$ with $z = (z_1,z_0) \in B_n$, and we choose the branch of the square root such that $\sqrt{f'(0)} = 1$. This operator is known as the Roper-Suffridge extension operator. Roper and Suffridge [11] proved that if f is a normalized convex function on U, then $\Phi_n(f)$ is a normalized biholomorphic convex mapping on B_n. In [7], Graham and Kohr proved that (1) if f is a normalized starlike function on U, then $\Phi_n(f)$ is a normalized biholomorphic starlike mapping on B_n; (2) if f is a normalized Bloch function on U, then $\Phi_n(f)$ is a normalized Bloch mapping on B_n. Because Roper-Suffridge extension operator has these important properties, many authors are interested in this extension operator. They generalized this extension operator in $\mathbb{C}^n$ and discussed their properties (see [2, 3, 5, 6, 7, 8, 9, 10], etc.).

In [9], I. Graham, G. Kohr, and M. Kohr generalized the Roper-Suffridge extension operator to

$$\Phi_{n,\beta}(f)(z) = F_\beta(z) = \left(f(z_1), \left(f'(z_1)\right)^\beta z_0\right), \tag{1.2}$$

where $\beta \in [0,1/2]$, f, z_1, z_0, z are defined as above, and the branch of the power function such that $(f'(z))^\beta|_{z=0} = 1$ is chosen. In [7], Graham and Kohr posed the following open

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problem. Consider the Reinhardt domain

$$\Omega_{2,p}(f)(z) = \left\{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^p < 1\right\}, \tag{1.3}$$

where $p \geq 1$. Does the operator

$$\Phi_{2,1/p}(f)(z) = F_{1/p}(z) = \left(f(z_1), (f'(z_1))^{1/p} z_2\right) \tag{1.4}$$

extend convex functions on U to the convex mappings on $\Omega_{2,p}$?

In [2], Gong and Liu solved the above open problem of Graham and Kohr in the affirmative.

The purpose of the present paper is to extend the Roper-Suffridge extension operator from $\mathbb{C}^n$ to Banach spaces and discuss its properties. In particular, we will verify that the solution of the above open problem of Graham and Kohr still holds in Banach spaces (see Theorem 2.6).

Throughout this paper, let X be a complex Banach space with norm $\|\cdot\|$, let X^* be the dual space of X, and let $\Omega \subset X$ be a domain. Suppose that $f : \Omega \to X$ is a biholomorphic mapping and $0 \in f(\Omega)$. A biholomorphic mapping $f : \Omega \to X$ is said to be starlike, provided $f(\Omega)$ is starlike with respect to the origin. A holomorphic mapping $f : \Omega \to X$ is called normalized if $f(0) = 0$ and $Df(0) = I$, where I is the identity map on X and $Df(x)$ is the Fréchet derivative of f at $x \in \Omega$. The class of all normalized biholomorphic starlike mappings on Ω is denoted by $S^*(\Omega)$. Then $f \in S^*(\Omega)$ if and only if f is a normalized biholomorphic mapping on Ω and

$$\lambda f(x) \in f(\Omega) \tag{1.5}$$

for all $x \in \Omega$ and $0 \leq \lambda \leq 1$. A biholomorphic mapping $f : \Omega \to X$ is said to be convex, provided $f(\Omega)$ is a convex set. The class of all normalized biholomorphic convex mappings on Ω is denoted by $K(\Omega)$. Then $f \in K(\Omega)$ if and only if f is a normalized biholomorphic mapping on Ω and

$$(1-\lambda)f(x_1) + \lambda f(x_2) \in f(\Omega) \tag{1.6}$$

for all $x_1, x_2 \in \Omega$ and $0 \leq \lambda \leq 1$. In particular, let $S^*(U)$, $K(U)$, S be the class of all normalized starlike functions, convex functions, and univalent functions on U, respectively.

Suppose that n is a positive integer and $\dim X \geq n$. Let $x_1, x_2, \ldots, x_n$ be a linearly independent family in X with $\|x_j\| = 1$ $(j = 1, 2, \ldots, n)$. According to the Hahn-Banach theorem [12], there exist $x_j^* \in X^*$ such that $x_j^*(x_j) = 1$ and $x_j^*(x) = 0$ for all $x \in M_j$ $(j = 1, 2, \ldots, n)$, where $M_j = \overline{\operatorname{span}}\{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n\}$. Hence, we have $x_j^*(x_j) = 1$ and $x_j^*(x_i) = 0$ $(j \neq i)$.

Assume $n \geq 2$, $p_j \geq 1$ $(j = 2, 3, \ldots, n+1)$, and let

$$\Omega_n(p_2, \ldots, p_{n+1}) = \left\{x \in X : |x_1^*(x)|^2 + \sum_{j=2}^{n} |x_j^*(x)|^{p_j} + \left\|x - \sum_{j=1}^{n} x_j^*(x)x_j\right\|^{p_{n+1}} < 1\right\}. \tag{1.7}$$

In this paper, we consider the generalized Roper-Suffridge extension operator

$$\begin{aligned} F(x) &= \Phi_{\alpha_2,\beta_2,\dots,\alpha_{n+1},\beta_{n+1}}(f)(x) \\ &= f(x_1^*(x))x_1 + \sum_{j=2}^{n}\left(\frac{f(x_1^*(x))}{x_1^*(x)}\right)^{\alpha_j}(f'(x_1^*(x)))^{\beta_j}x_j^*(x)x_j \\ &\quad+\left(\frac{f(x_1^*(x))}{x_1^*(x)}\right)^{\alpha_{n+1}}(f'(x_1^*(x)))^{\beta_{n+1}}\left[x-\sum_{j=1}^{n}x_j^*(x)x_j\right], \end{aligned} \tag{1.8}$$

where $f \in S$ and $x \in \Omega_n(p_2,\dots,p_{n+1})$, $\alpha_j \in [0,1]$, $\beta_j \in [0,1/p_j]$, $\alpha_j+\beta_j \le 1$ $(j = 2,3,\dots, n+1)$. We may choose the branch of all power functions $(f(z)/z)^{\alpha}|_{z=0} = 1$ and $(f'(z))^{\beta}|_{z=0} = 1$ for $\alpha \ge 0, \beta \ge 0$. Below we apply this agreement.

It is obvious that if $x_1^*(x) = 0$ in (1.8), then $F(x) \equiv x$.

When $\alpha_j = 0$, $\beta_j = 1/2$, $p_j = 2$ $(j = 2,\dots,n)$, $X = \mathbb{C}^n$, $x^*(\cdot) = \langle \cdot, u_j\rangle \in X^*$ in (1.8), we obtain the Roper-Suffridge operator $\Phi_n(f)$, where u_j denotes the vector in $\mathbb{C}^n$ with 1 in the jth place and zeros elsewhere.

In order to derive our main results, we need the following lemmas.

Lemma 1.1 [1]. *Let $g : U \to U$ be an analytic function in U with $g(a) = b$, then*

$$|g'(a)| \le \frac{1-|b|^2}{1-|a|^2}. \tag{1.9}$$

Lemma 1.2. *Let $f \in K(U)$. Then for every $z_1, z_2 \in U$, $0 \le \lambda \le 1$,*

$$(1-\lambda)|f'(z_1)|\left(1-|z_1|^2\right)+\lambda|f'(z_2)|\left(1|z_2|^2\right) \le (1-|w|^2)|f'(w)|, \tag{1.10}$$

where $w = f^{-1}((1-\lambda)f(z_1)+\lambda f(z_2))$.

Proof. Let $\varphi_{z_j}(\zeta) = (\zeta+z_j)/(1+\overline{z_j}\zeta)$ $(j = 1,2)$, then the functions $\varphi_{z_j} : U \to U$ are analytic in U, and $\varphi_{z_j}(0) = z_j$, $\varphi'_{z_j}(0) = 1-|z_j|^2$ $(j = 1,2)$. Set

$$g(\zeta) = f^{-1}((1-\lambda)f(\varphi_{z_1}(\zeta e^{-i\theta_1}))+\lambda f(\varphi_{z_2}(\zeta e^{-i\theta_2}))), \tag{1.11}$$

where $\theta_j = \arg(f'(z_j)/f'(w))$ $(j = 1,2)$. Then g is analytic in U with

$$\begin{aligned} g(0) &= f^{-1}((1-\lambda)f(z_1)+\lambda f(z_2)) = w, \\ g'(0) &= (1-\lambda)\frac{|f'(z_1)|}{|f'(w)|}\left(1-|z_1|^2\right)+\lambda\frac{|f(z_2)|}{|f'(w)|}\left(1-|z_2|^2\right). \end{aligned} \tag{1.12}$$

Hence, (1.10) follows from Lemma 1.1, and the proof of Lemma 1.2 is complete. □

Lemma 1.3 [10]. *Suppose that φ, ψ are twice differentiable on $[0,1)$, and*

$$\begin{aligned} &\varphi(0) = \varphi'(0) - 1 = 0, \quad \varphi'(r) \ge 0, \quad \varphi''(r) \le 0 \quad \text{on } [0,1); \\ &\psi(0) = \psi'(0) - 1 = 0, \quad \psi'(r) \ge 0, \quad \psi''(r) \ge 0 \quad \text{on } [0,1). \end{aligned} \tag{1.13}$$

If $p \ge 1$, $\alpha \in [0,1]$, $\beta \in [0,1/p]$ with $\alpha + \beta \le 1$, then for fixed $r \in [0,1)$, the minimum of

$$(\varphi(t))^p + (r^p - t^p)\left(\frac{\varphi(t)}{t}\right)^{p\alpha}(\varphi'(t))^{p\beta} \tag{1.14}$$

for $t \in [0,r]$ occurs when $t = r$; the maximum of

$$(\psi(t))^p + (r^p - t^p)\left(\frac{\psi(t)}{t}\right)^{p\alpha}(\psi'(t))^{p\beta} \tag{1.15}$$

for $t \in [0,r]$ occurs when $t = r$.

Lemma 1.4 [4]. (1) *If $f(z) \in S^*(U)$, then for $|z| = r < 1$,*

$$\frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2}, \qquad \frac{1-r}{(1+r)^3} \le |f'(z)| \le \frac{1+r}{(1-r)^3}. \tag{1.16}$$

(2) *If $f(z) \in K(U)$, then for $|z| = r < 1$,*

$$\frac{r}{1+r} \le |f(z)| \le \frac{r}{1-r}, \qquad \frac{1}{(1+r)^2} \le |f'(z)| \le \frac{1}{(1-r)^2}. \tag{1.17}$$

2. Main results and their proofs

Theorem 2.1. *Suppose that $\alpha_j \in [0,1]$, $\beta_j \in [0,1/p_j]$ with $\alpha_j + \beta_j \le 1$ $(j = 2,3,\dots,n+1)$. If $f \in S$ and $F(x) = \Phi_{\alpha_2,\beta_2,\dots,\alpha_{n+1},\beta_{n+1}}(f)(x)$ is defined by (1.8), then $F \in S^*(\Omega)$ if and only if $f \in S^*(U)$, where $\Omega = \Omega_n(p_2,\dots,p_{n+1})$.*

Proof. First, we prove that $F : \Omega \to X$ is a normalized biholomorphic mapping on Ω. From (1.8), by a direct calculation and noting $f(0) = 0$, $f'(0) = 1$, we have $F(0) = 0$ and

$$DF(0) = x_1^*(\cdot)x_1 + \sum_{j=2}^{n} x_j^*(\cdot)x_j + \left(I - \sum_{j=1}^{n} x_j^*(\cdot)x_j\right) = I. \tag{2.1}$$

Hence, F is a normalized holomorphic mapping on Ω.

If $F(y_1) = F(y_2)$ for $y_1, y_2 \in \Omega$, then we have

$$f(x_1^*(y_1)) = x_1^*(F(y_1)) = x_1^*(F(y_2)) = f(x_1^*(y_2)). \tag{2.2}$$

Therefore, it follows from $f \in S$ that $x_1^*(y_1) = x_1^*(y_2)$. Since $F(y_1) = F(y_2), f \in S$, and

$$\begin{aligned} x_j^*(F(y_1)) &= \left(\frac{f(x_1^*(y_1))}{x_1^*(y_1)}\right)^{\alpha_j} (f'(x_1^*(y_1)))^{\beta_j} x_j^*(y_1), \\ x_j^*(F(y_2)) &= \left(\frac{f(x_1^*(y_2))}{x_1^*(y_2)}\right)^{\alpha_j} (f'(x_1^*(y_2)))^{\beta_j} x_j^*(y_2) \end{aligned} \tag{2.3}$$

for $j = 2,3,\dots,n$, then we obtain that $x_j^*(y_1) = x_j^*(y_2)$ $(j = 1,2,\dots,n)$. Hence, we have $y_1 = y_2$. It follows that F is a biholomorphic mapping on Ω.

Next, we prove that $F = \Phi_{\alpha_2,\beta_2,\dots,\alpha_{n+1},\beta_{n+1}}(f)$ is a starlike mapping on Ω when $f \in S^*(U)$. Let $y \in \Omega$ and $0 \le \lambda \le 1$. Since $f \in S^*(U)$, there exists a point $z \in U$ such that

$$f(z) = \lambda f(x_1^*(y)). \tag{2.4}$$

Set

$$\begin{aligned} v = {} & \lambda \sum_{j=2}^{n} \frac{(f(x_1^*(y))/x_1^*(y))^{\alpha_j}(f'(x_1^*(y)))^{\beta_j}}{(f(z)/z)^{\alpha_j}(f'(z))^{\beta_j}} x_j^*(y)x_j \\ & + \lambda \frac{(f(x_1^*(y))/x_1^*(y))^{\alpha_{n+1}}(f'(x_1^*(y)))^{\beta_{n+1}}}{(f(z)/z)^{\alpha_{n+1}}(f'(z))^{\beta_{n+1}}} \left[y - \sum_{j=1}^{n} x_j^*(y)x_j\right], \end{aligned} \tag{2.5}$$

where $(f(x_1^*(y))/x_1^*(y))^{\alpha_j} = (f'(x_1^*(y)))^{\beta_j} = 1$ if $x_1^*(y) = 0$ and $(f(z)/z)^{\alpha_j} = (f'(z))^{\beta_j} = 1$ if $z = 0$ for $j = 1,\dots,n+1$.

Note that $x_j^*(x_j) = 1$ and $x_j^*(x_i) = 0$ $(j \neq i)$, we have $x_1^*(v) = 0$ and

$$x_j^*(v) = \lambda \frac{(f(x_1^*(y))/x_1^*(y))^{\alpha_j}(f'(x_1^*(y)))^{\beta_j}}{(f(z)/z)^{\alpha_j}(f'(z))^{\beta_j}} x_j^*(y), \tag{2.6}$$

for $2 \le j \le n$. Hence,

$$v - \sum_{j=1}^{n} x_j^*(v)x_j = \lambda \frac{(f(x_1^*(y))/x_1^*(y))^{\alpha_{n+1}}(f'(x_1^*(y)))^{\beta_{n+1}}}{(f(z)/z)^{\alpha_{n+1}}(f'(z))^{\beta_{n+1}}} \left[y - \sum_{j=1}^{n} x_j^*(y)x_j\right]. \tag{2.7}$$

From (2.4), we have

$$z = f^{-1}(\lambda f(u)), \tag{2.8}$$

where $u = x_1^*(y)$. Let $g(\xi) = f^{-1}(\lambda f(\xi))$ for $\xi \in U$. Then $g : U \to U$ is analytic in U with $g(0) = 0$ and $z = g(u)$. By Schwarz's lemma and Lemma 1.1, we obtain that $|z| = |g(u)| \le |u| = |x_1^*(y)|$ and

$$\left|g'(u)\right| \le \frac{1-|z|^2}{1-|u|^2} = \frac{1-|z|^2}{1-\left|x_1^*(y)\right|^2}. \tag{2.9}$$

On the other hand,

$$g'(u) = \lambda \frac{f'(x_1^*(y))}{f'(z)}. \tag{2.10}$$

According to (2.6), (2.7), (2.8), (2.9), and (2.10) and $\alpha_j + \beta_j \le 1$, $p_j\beta_j \le 1$ $(j = 2,3,\ldots,n+1)$, $|z| \le |x_1^*(y)|$, we have

$$\begin{aligned}
&\sum_{j=2}^{n}\left|x_j^*(v)\right|^{p_j} + \left\|v - \sum_{j=1}^{n} x_j^*(v)x_j\right\|^{p_{n+1}} \\
&\quad \le \sum_{j=2}^{n}\left|g'(u)\right|^{\beta_j p_j}\left|x_j^*(y)\right|^{p_j} + \left|g'(u)\right|^{\beta_{n+1}p_{n+1}}\left\|y - \sum_{j=1}^{n} x_j^*(y)x_j\right\|^{p_{n+1}} \\
&\quad \le \sum_{j=2}^{n}\left(\frac{1-|z|^2}{1-\left|x_1^*(y)\right|^2}\right)^{\beta_j p_j}\left|x_j^*(y)\right|^{p_j} + \left(\frac{1-|z|^2}{1-\left|x_1^*(y)\right|^2}\right)^{\beta_{n+1}p_{n+1}}\left\|y - \sum_{j=1}^{n} x_j^*(y)x_j\right\|^{p_{n+1}} \\
&\quad \le \sum_{j=2}^{n}\frac{1-|z|^2}{1-\left|x_1^*(y)\right|^2}\left|x_j^*(y)\right|^{p_j} + \frac{1-|z|^2}{1-\left|x_1^*(y)\right|^2}\cdot\left\|y - \sum_{j=1}^{n} x_j^*(y)x_j\right\|^{p_{n+1}} \\
&\quad < 1-|z|^2.
\end{aligned} \tag{2.11}$$

Let $z_0 = v + zx_1$. Then we have $x_1^*(z_0) = z$, $x_j^*(z_0) = x_j^*(v)$ for $2 \le j \le n$, and

$$z_0 - \sum_{j=1}^{n} x_j^*(z_0)x_j = v - \sum_{j=1}^{n} x_j^*(v)x_j, \tag{2.12}$$

where $x_1^*(v) = 0$. Hence, we obtain

$$\sum_{j=2}^{n}\left|x_j^*(z_0)\right|^{p_j} + \left\|z_0 - \sum_{j=1}^{n} x_j^*(z_0)x_j\right\|^{p_{n+1}} < 1-|z|^2 = 1-\left|x_1^*(z_0)\right|^2. \tag{2.13}$$

This implies $z_0 \in \Omega$. From (2.6), (2.7), and (2.12), direct computation yields

$$\lambda F(y) = F(z_0). \tag{2.14}$$

Hence, $F \in S^*(\Omega)$.

Conversely, if $F = \Phi_{\alpha_2,\beta_2,\dots,\alpha_{n+1},\beta_{n+1}}(f) \in S^*(\Omega)$, we prove that $f \in S^*(U)$.

In fact, for every $z_1 \in U$ and $t \in [0,1]$, if we let $x = z_1 x_1$, using the fact that $x_1^*(x) = z_1$ and $x_j^*(x) = 0$ $(j = 2,3,\dots,n)$, then we have $x \in \Omega$ and $x_1^*(F(x)) = f(z_1)$. Since $F \in S^*(\Omega)$, then $tF(x) \in F(\Omega)$. It follows that there exists $x_0 \in \Omega$ such that $tF(x) = F(x_0)$. This implies $|x_1^*(x_0)| < 1$ and

$$tf(z_1) = x_1^*(tF(x)) = x_1^*(F(x_0)) = f(x_1^*(x_0)) \in f(U). \tag{2.15}$$

Hence, $f \in S^*(U)$ and the proof of Theorem 2.1 is complete. □

Example 2.2. Let $p_{n+1} = p \geq 1$, $p_j \geq 1$ $(j = 2,3,\dots,n)$, let e_j denote the vector in l_p with 1 in the jth place and zeros elsewhere, $x_j^*(\cdot) = \langle \cdot, e_j\rangle$, and let

$$\Omega_p^l = \left\{ x = (x_1,x_2,\dots,x_n,\dots) \in l_p : |x_1|^2 + \sum_{j=2}^{n} |x_j|^{p_j} + \sum_{j=n+1}^{+\infty} |x_j|^{p} < 1 \right\}. \tag{2.16}$$

Since $f_1(\zeta) = \zeta/(1-\zeta)^2 \in S^*(U)$, then we have

$$F(x) = \frac{x_1 e_1}{(1-x_1)^2} + \sum_{j=2}^{n} \frac{x_j e_j (1+x_1)^{\beta_j}}{(1-x_1)^{2\alpha_j + 3\beta_j}} + \frac{\left(x - \sum_{j=1}^{n} x_j e_j\right)(1+x_1)^{\beta_{n+1}}}{(1-x_1)^{2\alpha_{n+1}+3\beta_{n+1}}} \in S^*\left(\Omega_p^l\right), \tag{2.17}$$

where $x = (x_1,x_2,\dots,x_n,\dots) \in \Omega_p^l$, $\alpha_j \in [0,1]$, $\beta_j \in [0,1/p_j]$, and $\alpha_j + \beta_j \leq 1$ $(j = 2,3,\dots, n+1)$.

Remark 2.3. Let x_j $(j = 1,2,\dots,n+1)$ be the vector in $\mathbb{C}^n$ with 1 in the jth place and zeros elsewhere. Setting $\alpha_j = 0, \beta_j = 1/2, X = \mathbb{C}^n, x_j^*(\cdot) = \langle \cdot, x_j\rangle \in X^*, p_j = 2$ $(j = 2,3,\dots,n)$ in Theorem 2.1, we obtain [7, Theorem 2.2] from the sufficient condition of Theorem 2.1; [8, Corollary 3.3], [9, Corollary 2.2], [6, Corollary 2.2], and [10, Theorem 3.1] are all the special cases of the sufficient condition of Theorem 2.1.

THEOREM 2.4. *Suppose that $\alpha_j \geq 0$, $\beta_j \geq 0$, and $p_j \geq 1$ $(j = 2,3,\dots,n+1)$. If* $\dim X \geq n+1$ *and $\Phi_{\alpha_2,\beta_2,\dots,\alpha_{n+1},\beta_{n+1}}(K(U)) \subset K(\Omega_n(p_2,\dots,p_{n+1}))$, where $\Phi_{\alpha_2,\beta_2,\dots,\alpha_{n+1},\beta_{n+1}}(f)$ is defined by (1.8), then $\beta_j \leq 1/p_j$ for $j = 2,3,\dots,n+1$. Furthermore, if $\beta_{j_0} = 1/p_{j_0}$ for some $j_0 \in \{2,3,\dots,n+1\}$, then $\alpha_{j_0} = 0$.*

Proof. Let $M = \{x \in X : x_j^*(x) = 0,\ j = 1,2,\dots,n\}$. For every $x \in X$, setting $x'_{n+1} = x - \sum_{j=1}^{n} x_j^*(x) x_j$, we have

$$x_j^*(x'_{n+1}) = x_j^*(x) - x_j^*(x) x_j^*(x_j) = 0, \quad j = 1,2,\dots,n. \tag{2.18}$$

This implies $x'_{n+1} \in M$. Since $x_j^*(x_i) = 0$ $(j \neq i)$ and $x_j^*(x_j) = 1$ $(i,j = 1,2,\dots,n)$, we obtain $X = M \oplus \{\lambda x_1 : \lambda \in \mathbb{C}\} \oplus \cdots \oplus \{\lambda x_n : \lambda \in \mathbb{C}\}$. Because $\dim X \geq n+1$, then there exists $x_{n+1} \in M$ with $\|x_{n+1}\| = 1$.

Suppose that there exists $\beta_k > 1/p_k$ $(2 \le k \le n+1)$. For $0 < \varepsilon < 1$, we let $r = \sqrt{1-\varepsilon^{p_k}}$, $x = rx_1 + (\varepsilon/2)x_k$, $w = -rx_1 + (\varepsilon/2)x_k$, then we have $x_1^*(x) = r$, $x_1^*(w) = -r$.

Case 1. When $k = n+1$, we have $x_j^*(x) = x_j^*(w) = 0$ $(j = 2,\dots,n)$, and

$$x - \sum_{j=1}^{n} x_j^*(x)x_j = \frac{\varepsilon}{2}x_{n+1}, \qquad w - \sum_{j=1}^{n} x_j^*(w)x_j = \frac{\varepsilon}{2}x_{n+1}. \tag{2.19}$$

Hence, $x, w \in \Omega_n(p_2,\dots,p_{n+1})$.

Taking $f(z_1) = (1/2)\log((1+z_1)/(1-z_1))$ with $\log 1 = 0$, we have $f \in K(U)$. Setting $F(x) = \Phi_{\alpha_2,\beta_2,\dots,\alpha_{n+1},\beta_{n+1}}(f)(x)$, since $F \in \Phi_{\alpha_2,\beta_2,\dots,\alpha_{n+1},\beta_{n+1}}(K(U)) \subset K(\Omega_n(p_2,\dots,p_{n+1}))$, we have

$$\frac{1}{2}[F(x)+F(w)] \in F(\Omega_n(p_2,\dots,p_{n+1})). \tag{2.20}$$

Hence, there exists $x_0 \in \Omega_n(p_2,\dots,p_{n+1})$ such that $F(x_0) = (1/2)[F(x)+F(w)]$. Using the fact that $f(-r) = -f(r)$ for $0 < r < 1$, we obtain

$$\begin{aligned} f(x_1^*(x_0)) = x_1^*(F(x_0)) &= \frac{1}{2}[x_1^*(F(x)) + x_1^*(F(w))] \\ &= \frac{1}{2}[f(x_1^*(x)) + f(x_1^*(w))] \\ &= \frac{1}{2}[f(r)+f(-r)] \\ &= 0. \end{aligned} \tag{2.21}$$

Since f is univalent on U, then we obtain $x_1^*(x_0) = 0$ and $F(x_0) = x_0$. Hence,

$$x_j^*(x_0) = x_j^*(F(x_0)) = \frac{1}{2}\Big[x_j^*(F(x)) + x_j^*(F(w))\Big] = 0, \quad j = 2,3,\dots,n. \tag{2.22}$$

On the other hand, from (1.8), we have

$$\begin{aligned} F(x_0) &= \frac{1}{2}[F(x)+F(w)] \\ &= \frac{1}{2}\left(\frac{1}{2r}\log\frac{1+r}{1-r}\right)^{\alpha_{n+1}}\left(\frac{1}{1-r^2}\right)^{\beta_{n+1}}\frac{\varepsilon}{2}x_{n+1} \\ &\quad + \frac{1}{2}\left(-\frac{1}{2r}\log\frac{1-r}{1+r}\right)^{\alpha_{n+1}}\left(\frac{1}{1-r^2}\right)^{\beta_{n+1}}\frac{\varepsilon}{2}x_{n+1} \\ &= \frac{1}{2}\left(\frac{2\log(1+\sqrt{1-\varepsilon^{p_{n+1}}}) - p_{n+1}\log\varepsilon}{2\sqrt{1-\varepsilon^{p_{n+1}}}}\right)^{\alpha_{n+1}} \varepsilon^{1-p_{n+1}\beta_{n+1}}x_{n+1}. \end{aligned} \tag{2.23}$$

If $\beta_{n+1} > 1/p_{n+1}$, letting $\varepsilon \to 0^+$, we obtain

$$\|F(x_0)\| = \frac{1}{2}\left(\frac{2\log\left(1+\sqrt{1-\varepsilon^{p_{n+1}}}\right) - p_{n+1}\log\varepsilon}{2\sqrt{1-\varepsilon^{p_{n+1}}}}\right)^{\alpha_{n+1}} \varepsilon^{1-p_{n+1}\beta_{n+1}} \longrightarrow +\infty, \tag{2.24}$$

which contradicts $\|F(x_0)\| = \|x_0\| = \|x_0 - \sum_{j=1}^{n} x_j^*(x_0)x_j\| < 1$. Hence, $\beta_{n+1} \le 1/p_{n+1}$.

Furthermore, if $\beta_{n+1} = 1/p_{n+1}$, from (2.24), we have $\|F(x_0)\| \to +\infty(\varepsilon \to 0^+)$ when $\alpha_{n+1} > 0$. This is impossible. Hence, we have $\alpha_{n+1} = 0$, and the proof of Case 1 is complete.

Case 2. When $2 \le k \le n$, we have $x_k^*(x) = \varepsilon/2$, $x_k^*(w) = \varepsilon/2$, $x_j^*(x) = x_j^*(w) = 0$ ($j = 2,\dots,k-1,k+1,\dots,n$), and

$$x - \sum_{j=1}^{n} x_j^*(x)x_j = 0, \qquad w - \sum_{j=1}^{n} x_j^*(w)x_j = 0. \tag{2.25}$$

Hence, $x, w \in \Omega_n(p_2,\dots,p_{n+1})$.

Similarly, it can be shown that there exists $x_0 \in \Omega_n(p_2,\dots,p_{n+1})$ such that $F(x_0) = (1/2)[F(x)+F(w)]$, $x_1^*(x_0) = 0$, and $F(x_0) = x_0$.

On the other hand, by (1.8), we have

$$\begin{aligned} F(x_0) &= \frac{1}{2}[F(x)+F(w)] \\ &= \frac{1}{2}\left(\frac{1}{2r}\log\frac{1+r}{1-r}\right)^{\alpha_k}\left(\frac{1}{1-r^2}\right)^{\beta_k}\frac{\varepsilon}{2}x_k \\ &\quad + \frac{1}{2}\left(-\frac{1}{2r}\log\frac{1-r}{1+r}\right)^{\alpha_k}\left(\frac{1}{1-r^2}\right)^{\beta_k}\frac{\varepsilon}{2}x_k \\ &= \frac{1}{2}\left(\frac{2\log\left(1+\sqrt{1-\varepsilon^{p_k}}\right) - p_k\log\varepsilon}{2\sqrt{1-\varepsilon^{p_k}}}\right)^{\alpha_k}\varepsilon^{1-p_k\beta_k}x_k. \end{aligned} \tag{2.26}$$

Since $\beta_k > 1/p_k$, letting $\varepsilon \to 0^+$, we obtain

$$|x_k^*(x_0)| = |x_k^*(F(x_0))| = \frac{1}{2}\left(\frac{2\log\left(1+\sqrt{1-\varepsilon^{p_k}}\right) - p_k\log\varepsilon}{2\sqrt{1-\varepsilon^{p_k}}}\right)^{\alpha_k}\varepsilon^{1-p_k\beta_k} \longrightarrow +\infty, \tag{2.27}$$

which contradicts $|x_k^*(x_0)| < 1$. Hence, $\beta_k \le 1/p_k$.

Furthermore, if $\beta_k = 1/p_k$ ($2 \le k \le n$), from (2.27), we have $x_k^*(x_0) \to +\infty(\varepsilon \to 0^+)$ when $\alpha_k > 0$. This is impossible. Hence, we have $\alpha_k = 0$, this completes the proof of Case 2. □

Remark 2.5. Let x_j $(j = 1,2,\dots,n+1)$ be the vector in $\mathbb{C}^n$ with 1 in the jth place and zeros elsewhere. Setting $p_j = 2$, $\alpha_j = \alpha$, $\beta_j = \beta$, $X = \mathbb{C}^n$, $x_j^*(\cdot) = \langle \cdot, x_j \rangle \in X^*$, $j = 2,3,\dots,n+1$, in Theorem 2.4, we obtain the partial result of [6]. Furthermore, when $\alpha_j = 0$, $j = 2,3,\dots,n+1$, Theorem 2.4 provides the necessary condition of preserving convexity, and the following result provides the sufficient condition of preserving convexity.

Theorem 2.6. *If $f \in K(U)$, and*

$$\begin{aligned} G(x) &= \Psi_{p_2,\dots,p_{n+1}}(f)(x) \\ &= f\left(x_1^*(x)\right)x_1 + \sum_{j=2}^{n}\left(f'\left(x_1^*(x)\right)\right)^{1/p_j} x_j^*(x)x_j + \left(f'\left(x_1^*(x)\right)\right)^{1/p_{n+1}}\left[x - \sum_{j=1}^{n} x_j^*(x)x_j\right], \end{aligned} \tag{2.28}$$

then $G \in K(\Omega)$, where $\Omega = \Omega_n(p_2,\dots,p_{n+1})$.

Proof. First, since $f \in S$, according to the proof of Theorem 2.1, by straightforward calculation from (2.28), we obtain that G is a normalized biholomorphic mapping on Ω.

Next, we prove that $G = \Psi_{p_2,\dots,p_{n+1}}(f) \in K(\Omega)$ for $f \in K(U)$.

For every $y_1, y_2 \in \Omega$ and $0 < \lambda < 1$, there exists a point $w \in U$ such that

$$f(w) = (1-\lambda)f\left(x_1^*(y_1)\right) + \lambda f\left(x_1^*(y_2)\right). \tag{2.29}$$

Let

$$\begin{aligned} v &= (1-\lambda)\sum_{j=2}^{n}\left(\frac{f'\left(x_1^*(y_1)\right)}{f'(w)}\right)^{1/p_j} x_j^*(y_1)x_j \\ &\quad + (1-\lambda)\left(\frac{f'\left(x_1^*(y_1)\right)}{f'(w)}\right)^{1/p_{n+1}}\left[y_1 - \sum_{j=1}^{n} x_j^*(y_1)x_j\right] \\ &\quad + \lambda\sum_{j=2}^{n}\left(\frac{f'\left(x_1^*(y_2)\right)}{f'(w)}\right)^{1/p_j} x_j^*(y_2)x_j \\ &\quad + \lambda\left(\frac{f'\left(x_1^*(y_2)\right)}{f'(w)}\right)^{1/p_{n+1}}\left[y_2 - \sum_{j=1}^{n} x_j^*(y_2)x_j\right]. \end{aligned} \tag{2.30}$$

Then for $1 \le j \le n$, note that $x_j^*(x_j) = 1$ and $x_j^*(x_i) = 0$ $(j \ne i)$, we have $x_1^*(v) = 0$ and

$$x_j^*(v) = (1-\lambda)\left(\frac{f'\left(x_1^*(y_1)\right)}{f'(w)}\right)^{1/p_j} x_j^*(y_1) + \lambda\left(\frac{f'\left(x_1^*(y_2)\right)}{f'(w)}\right)^{1/p_j} x_j^*(y_2) \tag{2.31}$$

for $2 \le j \le n$. Hence, we obtain

$$\begin{aligned} v-\sum_{j=1}^{n} x_j^*(v)x_j &= (1-\lambda)\left(\frac{f'(x_1^*(y_1))}{f'(w)}\right)^{1/p_{n+1}}\left(y_1-\sum_{j=1}^{n} x_j^*(y_1)x_j\right)\\ &\quad +\lambda\left(\frac{f'(x_1^*(y_2))}{f'(w)}\right)^{1/p_{n+1}}\left(y_2-\sum_{j=1}^{n} x_j^*(y_2)x_j\right). \end{aligned} \tag{2.32}$$

In the following, we prove that

$$|x_j^*(v)|^{p_j} \le (1-\lambda)\left|\frac{f'(x_1^*(y_1))}{f'(w)}\right| |x_j^*(y_1)|^{p_j} + \lambda\left|\frac{f'(x_1^*(y_2))}{f'(w)}\right| |x_j^*(y_2)|^{p_j} \tag{2.33}$$

for $p_j \ge 1$ $(j = 2,3,\dots,n)$.

Case 1. Suppose $p_j > 1$. Taking $q_j > 1$ such that $1/p_j + 1/q_j = 1$, by Hölder's inequality, we have

$$\begin{aligned} &|x_j^*(v)|^{p_j}\\ &\quad\le\left[(1-\lambda)^{1/q_j}\left((1-\lambda)\left|\frac{f'(x_1^*(y_1))}{f'(w)}\right| |x_j^*(y_1)|^{p_j}\right)^{1/p_j}\right.\\ &\qquad\left.+\lambda^{1/q_j}\left(\lambda\left|\frac{f'(x_1^*(y_2))}{f'(w)}\right| |x_j^*(y_2)|^{p_j}\right)^{1/p_j}\right]^{p_j}\\ &\quad\le\left[(1-\lambda)\left|\frac{f'(x_1^*(y_1))}{f'(w)}\right| |x_j^*(y_1)|^{p_j}+\lambda\left|\frac{f'(x_1^*(y_2))}{f'(w)}\right| |x_j^*(y_2)|^{p_j}\right](1-\lambda+\lambda)^{p_j/q_j}\\ &\quad\le(1-\lambda)\left|\frac{f'(x_1^*(y_1))}{f'(w)}\right| |x_j^*(y_1)|^{p_j}+\lambda\left|\frac{f'(x_1^*(y_2))}{f'(w)}\right| |x_j^*(y_2)|^{p_j}. \end{aligned} \tag{2.34}$$

The proof of Case 1 is complete.

Case 2. Suppose $p_j = 1$. By the triangle inequality, we have

$$\begin{aligned} |x_j^*(v)|^{p_j} = |x_j^*(v)| &\le (1-\lambda)\left|\frac{f'(x_1^*(y_1))}{f'(w)}\right| |x_j^*(y_1)| + \lambda\left|\frac{f'(x_1^*(y_2))}{f'(w)}\right| |x_j^*(y_2)|\\ &= (1-\lambda)\left|\frac{f'(x_1^*(y_1))}{f'(w)}\right| |x_j^*(y_1)|^{p_j} + \lambda\left|\frac{f'(x_1^*(y_2))}{f'(w)}\right| |x_j^*(y_2)|^{p_j}. \end{aligned} \tag{2.35}$$

The proof of Case 2 is complete.

Hence, the inequality (2.33) holds for $p_j \ge 1$, $j = 2,3,\dots,n$.

Similarly, we may obtain

$$\begin{aligned}\left\|v-\sum_{j=1}^{n}x_j^*(v)x_j\right\|^{p_{n+1}} &\le (1-\lambda)\left|\frac{f'(x_1^*(y_1))}{f'(w)}\right|\left\|y_1-\sum_{j=1}^{n}x_j^*(y_1)x_j\right\|^{p_{n+1}}\\ &\quad+\lambda\left|\frac{f'(x_1^*(y_2))}{f'(w)}\right|\left\|y_2-\sum_{j=1}^{n}x_j^*(y_2)x_j\right\|^{p_{n+1}}\\ &< (1-\lambda)\left|\frac{f'(x_1^*(y_1))}{f'(w)}\right|\left(1-\sum_{j=1}^{n}|x_j^*(y_1)|^{p_j}\right)\\ &\quad+\lambda\left|\frac{f'(x_1^*(y_2))}{f'(w)}\right|\left(1-\sum_{j=1}^{n}|x_j^*(y_2)|^{p_j}\right).\end{aligned} \tag{2.36}$$

According to (2.33), (2.36), and Lemma 1.2, we have

$$\begin{aligned}&\sum_{j=2}^{n}|x_j^*(v)|^{p_j}+\left\|v-\sum_{j=1}^{n}x_j^*(v)x_j\right\|^{p_{n+1}}\\ &\quad< (1-\lambda)\left|\frac{f'(x_1^*(y_1))}{f'(w)}\right|\left(1-|x_1^*(y_1)|^2\right)+\lambda\left|\frac{f'(x_1^*(y_2))}{f'(w)}\right|\left(1-|x_1^*(y_2)|^2\right)\\ &\quad\le 1-|w|^2.\end{aligned} \tag{2.37}$$

Let $z_0 = v + wx_1$. Then we have $x_1^*(z_0) = w$, $x_j^*(z_0) = x_j^*(v)$ for $2 \le j \le n$, and

$$z_0-\sum_{j=1}^{n}x_j^*(z_0)x_j = v-\sum_{j=1}^{n}x_j^*(v)x_j, \tag{2.38}$$

where $x_1^*(v) = 0$. Hence, we obtain

$$\sum_{j=2}^{n}|x_j^*(z_0)|^{p_j}+\left\|z_0-\sum_{j=1}^{n}x_j^*(z_0)x_j\right\|^{p_{n+1}} < 1-|w|^2 = 1-|x_1^*(z_0)|^2. \tag{2.39}$$

This implies $z_0 \in \Omega$. From (2.31), (2.32), and (2.38), straightforward calculation yields

$$(1-\lambda)G(y_1)+\lambda G(y_2) = G(z_0). \tag{2.40}$$

Hence, $F \in K(\Omega)$ and the proof is complete. □

Remark 2.7. Theorem 2.6 tell us that the solution of the open problem of Graham and Kohr [7] mentioned in Section 1 still holds in Banach Spaces; [11, Theorem 1], [7, Theorem 2.1] and [3, Theorem 2] are all the special cases of Theorem 2.6.

Theorem 2.8. *Suppose that $f \in S$ satisfies*

$$\varphi(r) \leq |f(z_1)| \leq \psi(r), \quad |z_1| = r, \tag{2.41}$$
$$\varphi'(r) \leq |f'(z_1)| \leq \psi'(r), \quad |z_1| = r, \tag{2.42}$$

where φ, ψ are twice differentiable on $[0,1)$, and

$$\begin{aligned} &\varphi(0) = \varphi'(0) - 1 = 0, \quad \varphi'(r) \geq 0, \quad \varphi''(r) \leq 0 \quad \text{on } [0,1); \\ &\psi(0) = \psi'(0) - 1 = 0, \quad \psi'(r) \geq 0, \quad \psi''(r) \geq 0 \quad \text{on } [0,1). \end{aligned} \tag{2.43}$$

Let $\alpha_j \geq 0$, $\beta_j \geq 0$, $q_j \geq 1$ $(j = 1,2,\dots,n+1)$, $\alpha \in [0,1]$, $\beta \in [0,1/q_1]$, where $\alpha = \max_{j=2,3,\dots,n+1}\{\alpha_j\}$, $\beta = \max_{j=2,3,\dots,n+1}\{\beta_j\}$, and $F(x) = \Phi_{\alpha_2,\beta_2,\dots,\alpha_{n+1},\beta_{n+1}}(f)(x)$ is defined by (1.8), and

$$\Omega' = \Omega_{q_1,q_2,\dots,q_n} = \{x \in X : \rho(x) < 1\}, \tag{2.44}$$

where $\rho(x) = (\sum_{j=1}^n |x_j^*(x)|^{q_j} + \|x - \sum_{j=1}^n x_j^*(x)x_j\|^{q_{n+1}})^{1/q_1}$. If $q_1 = \max_{j=1,2,\dots,n+1}\{q_j\}$ and $\alpha+\beta \leq 1$, then F is a normalized biholomorphic mapping on Ω', and

$$\varphi(r) \leq \rho(F(x)) \leq \psi(r), \tag{2.45}$$

for $\rho(x) = r < 1$.

Furthermore, if for some f the lower (resp., upper) estimate (2.41) is sharp at $z_1 \in U$, then the lower (resp., upper) estimate (2.45) is sharp for $\Phi_{\alpha_2,\beta_2,\dots,\alpha_{n+1},\beta_{n+1}}(f)(x)$ at $x = z_1x_1$.

Proof. Since $f \in S$, according to the proof of Theorem 2.1, by straightforward calculation from (1.8), we obtain that F is a normalized biholomorphic mapping on Ω'.

Now we prove that the inequalities (2.45) hold for $\rho(x) = r < 1$.

Let $t = |x_1^*(x)|$, using the fact that $0 \leq \varphi(t)/t \leq 1$, $0 \leq \varphi'(t) \leq 1$ for $t \in (0,1)$ and

$$\begin{gathered} x_1^*(F(x)) = f(x_1^*(x)), \qquad x_j^*(F(x)) = \left(\frac{f(x_1^*(x))}{x_1^*(x)}\right)^{\alpha_j} (f'(x_1^*(x)))^{\beta_j} x_j^*(x) \quad (j=2,3,\dots,n), \\ F(x) - \sum_{j=1}^n x_j^*(F(x))x_j = \left(\frac{f(x_1^*(x))}{x_1^*(x)}\right)^{\alpha_{n+1}} (f'(x_1^*(x)))^{\beta_{n+1}} \left[x - \sum_{j=1}^n x_j^*(x)x_j\right], \\ q_1 = \max_{j=1,2,\dots,n+1}\{q_j\}, \quad \alpha = \max_{j=2,3,\dots,n+1}\{\alpha_j\}, \quad \beta = \max_{j=2,3,\dots,n+1}\{\beta_j\}, \end{gathered} \tag{2.46}$$

we obtain

$$
\begin{aligned}
\rho(F(x))^{q_1} &= \sum_{j=1}^{n} \left| x_j^*(F(x)) \right|^{q_j} + \left\| F(x) - \sum_{j=1}^{n} x_j^*(F(x))x_j \right\|^{q_{n+1}} \\
&= \left| f(x_1^*(x)) \right|^{q_1} + \sum_{j=2}^{n} \left| \frac{f(x_1^*(x))}{x_1^*(x)} \right|^{\alpha_j q_j} \left| f'(x_1^*(x)) \right|^{\beta_j q_j} \left| x_j^*(x) \right|^{q_j} \\
&\quad + \left| \frac{f(x_1^*(x))}{x_1^*(x)} \right|^{\alpha_{n+1} q_{n+1}} \left| f'(x_1^*(x)) \right|^{\beta_{n+1} q_{n+1}} \left\| x - \sum_{j=1}^{n} x_j^*(x)x_j \right\|^{q_{n+1}} \\
&\geq \varphi(t)^{q_1} + \sum_{j=2}^{n} \left(\frac{\varphi(t)}{t} \right)^{\alpha_j q_j} (\varphi'(t))^{\beta_j q_j} \left| x_j^*(x) \right|^{q_j} \\
&\quad + \left(\frac{\varphi(t)}{t} \right)^{\alpha_{n+1} q_{n+1}} (\varphi'(t))^{\beta_{n+1} q_{n+1}} \left\| x - \sum_{j=1}^{n} x_j^*(x)x_j \right\|^{q_{n+1}} \\
&\geq \varphi(t)^{q_1} + \left(\frac{\varphi(t)}{t} \right)^{\alpha q_1} (\varphi'(t))^{\beta q_1} (r^{q_1} - t^{q_1}).
\end{aligned}
\tag{2.47}
$$

By Lemma 1.3, we have

$$\rho(F(x))^{q_1} \geq \varphi(r)^{q_1}, \tag{2.48}$$

hence $\varphi(r) \leq \rho(F(x))$.

Similarly, we may prove that $\rho(F(x)) \leq \psi(r)$. This completes the proof. □

Setting $q_1 = q_2 = \cdots = q_{n+1} = p$ in Theorem 2.8, we obtain the following corollary.

Corollary 2.9. *Suppose that f, φ, ψ satisfy the hypothesis of Theorem 2.8. Let $p \geq 1$, $\alpha_j \geq 0$, $\beta_j \geq 0$ $(j = 2,3,\ldots,n+1)$, $\alpha \in [0,1]$, $\beta \in [0,1/p]$, where $\alpha = \max_{j=2,3,\ldots,n+1}\{\alpha_j\}$, $\beta = \max_{j=2,3,\ldots,n+1}\{\beta_j\}$. Let $F(x) = \Phi_{\alpha_2,\beta_2,\ldots,\alpha_{n+1},\beta_{n+1}}(f)(x)$ be defined by (1.8), and*

$$\Omega_p = \{x \in X : \|x\|_p < 1\}, \tag{2.49}$$

where $\|x\|_p = (\sum_{j=1}^{n} |x_j^*(x)|^p + \|x - \sum_{j=1}^{n} x_j^*(x)x_j\|^p)^{1/p}$. If $\alpha + \beta \leq 1$, then F is a normalized biholomorphic mapping on Ω_p, and

$$\varphi(r) \leq \|F(x)\|_p \leq \psi(r), \tag{2.50}$$

for $\|x\|_p = r < 1$.

Furthermore, if for some f the lower (resp., upper) estimate (2.41) is sharp at $z_1 \in U$, then the lower (resp., upper) estimate (2.50) is sharp for $\Phi_{\alpha_2,\beta_2,\ldots,\alpha_{n+1},\beta_{n+1}}(f)(x)$ at $x = z_1 x_1$.

Remark 2.10. Setting $p = 2$, $\alpha_j = \alpha$, $\beta_j = \beta$ $(j = 2,3,\ldots,n)$, $X = \mathbf{C}^n$, $x_1 = (1,0,\ldots,0)$, $x_2 = (0,1,0,\ldots,0),\ldots$, $x_n = (0,\ldots,0,1)$, $x_j^*(\cdot) = \langle \cdot, x_j \rangle \in X^*$ $(j = 1,2,\ldots,n)$ in Corollary 2.9, we obtain [6, Theorem 3.1].

According to Corollary 2.9 and Lemma 1.4, we have the following corollary.

Corollary 2.11. *Suppose that X is a Banach space, $p \geq 1$, $\alpha_j \geq 0$, $\beta_j \geq 0$ $(j = 2,3,\dots,n+1)$, $\alpha \in [0,1]$, $\beta \in [0,1/p]$ with $\alpha+\beta \leq 1$, where $\alpha = \max_{j=2,3,\dots,n+1}\{\alpha_j\}$, $\beta = \max_{j=2,3,\dots,n+1}\{\beta_j\}$. Let $F(x) = \Phi_{\alpha_2,\beta_2,\dots,\alpha_{n+1},\beta_{n+1}}(f)(x)$ be defined by (1.8).*

(1) *If $f \in S^*(U)$, then for $\|z\|_p = r < 1$,*

$$\frac{r}{(1+r)^2} \leq \left\|F(z)\right\|_p \leq \frac{r}{(1-r)^2}. \tag{2.51}$$

(2) *If $f \in K(U)$, then for $\|z\|_p = r < 1$,*

$$\frac{r}{1+r} \leq \left\|F(z)\right\|_p \leq \frac{r}{1-r}. \tag{2.52}$$

These estimates are sharp.

From Corollary 2.11, we have the following corollary.

Corollary 2.12 (covering theorem). *Suppose that X is a Banach space, $p \geq 1$, $\alpha_j \geq 0$, $\beta_j \geq 0$ $(j = 2,3,\dots,n+1)$, $\alpha \in [0,1]$, $\beta \in [0,1/p]$ with $\alpha+\beta \leq 1$, where $\alpha = \max_{j=2,3,\dots,n+1}\{\alpha_j\}$, $\beta = \max_{j=2,3,\dots,n+1}\{\beta_j\}$. Let $F(x) = \Phi_{\alpha_2,\beta_2,\dots,\alpha_{n+1},\beta_{n+1}}(f)(x)$ be defined by (1.8) and Ω_p defined by (2.49).*

(1) *If $f \in S^*(U)$, then $F(\Omega_p) \supset (1/4)\Omega_p$.*

(2) *If $f \in K(U)$, then $F(\Omega_p) \supset (1/2)\Omega_p$.*

Remark 2.13. Setting $X = \mathbb{C}^n$, $x_1 = (1,0,\dots,0)$, $x_2 = (0,1,0,\dots,0),\dots,$ $x_n = (0,\dots,0,1)$, $x_j^*(\cdot) = \langle \cdot, x_j\rangle \in X^*$ $(j = 1,2,\dots,n)$ in Corollary 2.11, we obtain [10, Theorem 3.3]. Setting $X = \mathbb{C}^n$, $x_j^*(\cdot) = \langle \cdot, x_j\rangle \in X^*(j = 1,2,\dots,n)$ in Corollary 2.12, we obtain [10, Corollary 3.4].

Suppose \$ is a nonempty subclass of normalized biholomorphic mappings on $\Omega = \Omega_n(p_2,\dots,p_{n+1})$. Let $n \geq 2$, $p_j \geq 1$ $(j = 2,3,\dots,n+1)$, $r > 0$, and let

$$\Omega_n^r(p_2,\dots,p_{n+1}) = \left\{x \in X : \left|x_1^*\left(\frac{x}{r}\right)\right|^2 + \sum_{j=2}^{n}\left|x_j^*\left(\frac{x}{r}\right)\right|^{p_j} + \left\|\frac{x}{r} - \sum_{j=1}^{n} x_j^*\left(\frac{x}{r}\right)x_j\right\|^{p_{n+1}} < 1\right\}, \tag{2.53}$$

we define

$$r^*(\$) = \sup\left\{r : F \text{ is a starlike mapping on } \Omega_n^r(p_2,\dots,p_{n+1}),\ F \in \$\right\}. \tag{2.54}$$

For every $f \in S$, according to the proof of Theorem 2.1, we obtain that $F(x) = \Phi_{\alpha_2,\beta_2,\dots,\alpha_{n+1},\beta_{n+1}}(f)(x)$ is a biholomorphic mapping on Ω. Hence, the mapping family $\$_1 = \{\Phi_{\alpha_2,\beta_2,\dots,\alpha_{n+1},\beta_{n+1}}(f)(x) : f \in S\}$ is given. Consequently, we derive the following result from Theorem 2.1.

Theorem 2.14. *Let $n \geq 2$, $p_j \geq 1$ $(j = 2,3,\dots,n+1)$, $\alpha_j \in [0,1]$, $\beta_j \in [0,1/p_j]$, and $\alpha_j + \beta_j \leq 1$ $(j = 2,3,\dots,n+1)$, then $r^*(\$_1) = \tanh(\pi/4)$.*

Proof. Since the radius of starlikeness for the set S is $r = \tanh(\pi/4)$ (see [4]), for any $f \in S$, $g(z) = (1/r)f(rz)$ is a normalized biholomorphic starlike mapping. According to Theorem 2.1, we obtain that

$$\begin{aligned}\Phi_{\alpha_2,\beta_2,\ldots,\alpha_{n+1},\beta_{n+1}}(g)(x)\\ = g(x_1^*(x))x_1 + \sum_{j=2}^{n}\left(\frac{g(x_1^*(x))}{x_1^*(x)}\right)^{\alpha_j}(g'(x_1^*(x)))^{\beta_j}x_j^*(x)x_j\\ + \left(\frac{g(x_1^*(x))}{x_1^*(x)}\right)^{\alpha_{n+1}}(g'(x_1^*(x)))^{\beta_{n+1}}\left[x - \sum_{j=1}^{n}x_j^*(x)x_j\right]\end{aligned} \tag{2.55}$$

is a starlike mapping on Ω, thus

$$\begin{aligned}f(x_1^*(rx))x_1 + \sum_{j=2}^{n}\left(\frac{f(x_1^*(rx))}{x_1^*(rx)}\right)^{\alpha_j}(f'(x_1^*(rx)))^{\beta_j}x_j^*(rx)x_j\\ + \left(\frac{f(x_1^*(rx))}{x_1^*(rx)}\right)^{\alpha_{n+1}}(f'(x_1^*(rx)))^{\beta_{n+1}}\left[rx - \sum_{j=1}^{n}x_j^*(rx)x_j\right]\end{aligned} \tag{2.56}$$

is a starlike mapping on Ω, too. Set $y = rx$, then

$$\begin{aligned}f(x_1^*(y))x_1 + \sum_{j=2}^{n}\left(\frac{f(x_1^*(y))}{x_1^*(y)}\right)^{\alpha_j}(f'(x_1^*(y)))^{\beta_j}x_j^*(y)x_j\\ + \left(\frac{f(x_1^*(y))}{x_1^*(y)}\right)^{\alpha_{n+1}}(f'(x_1^*(y)))^{\beta_{n+1}}\left[y - \sum_{j=1}^{n}x_j^*(y)x_j\right]\end{aligned} \tag{2.57}$$

is a starlike mapping on $\Omega_n^r(p_2,\ldots,p_{n+1})$. From Theorem 2.1, we have $r^*(\$_1) = \tanh(\pi/4) = (e^{\pi/2}-1)/(e^{\pi/2}+1) \approx 0.65579$, and the proof is complete. $\square$

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