

# ASYMPTOTIC ANALYSIS OF SINGULAR SOLUTIONS OF THE SCALAR AND MEAN CURVATURE EQUATIONS

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We show that positive solutions of a semilinear elliptic problem in the Sobolev critical exponent with Neumann conditions, related to conformal deformation of metrics in  $\mathbb{R}_+^n$ , are asymptotically symmetric in a neighborhood of the origin. As a consequence, we prove for a related problem of conformal deformation of metrics in  $\mathbb{R}_+^n$  that if a solution satisfies a Kazdan-Warner-type identity, then the conformal metric can be realized as a smooth metric on  $S_+^n$ .

## 1. Introduction

In recent years there has been a huge interest in studying properties of the positive singular solutions  $u$  of the scalar curvature equation

$$\begin{aligned} \Delta u + k(x)u^{(n+2)/(n-2)} &= 0 \quad \text{in } B_1 \setminus \{0\}, \\ u(x) &> 0, \quad u \in C^2(B_1 \setminus \{0\}), \end{aligned} \tag{1.1}$$

where  $k(x)$  is a smooth positive function and  $B_1$  is the unit ball of  $\mathbb{R}^n$ , with  $n \geq 3$ .

We observe that the interest for studying singular solutions of (1.1) comes from the study of asymptotic behavior of positive solutions of the following problem in  $\mathbb{R}^n$ ,  $n \geq 3$ :

$$\Delta v + \tilde{k}(x)v^{(n+2)/(n-2)} = 0 \quad \text{in } \mathbb{R}^n. \tag{1.2}$$

Using the Kelvin transformation, the study of the asymptotic behavior of solutions  $v$  of problem (1.2) at infinity is reduced to the study of the asymptotic behavior at zero of solutions of problem (1.1).

For  $k(x) \equiv 1$ , Caffarelli et al. in [2] proved that if zero is a nonremovable singularity of a solution  $u$  of (1.1), then  $u$  is asymptotically symmetric with respect to the origin. Furthermore, they showed the existence of a radial solution  $u_0$  of the problem

$$\begin{aligned} \Delta u_0 + u_0^{(n+2)/(n-2)}(x) &= 0, \quad u_0 > 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}, \\ u_0 &\in C^2(\mathbb{R}^n \setminus \{0\}), \quad \lim_{x \rightarrow 0} u_0(x) = +\infty, \end{aligned} \tag{1.3}$$

such that

$$u(x) = u_0(x)(1 + o(1)) \tag{1.4}$$

as  $x \rightarrow 0$ . A consequence of (1.4) is that for any solution  $u$  of (1.3), there exist positive constants  $c_1, c_2$  such that

$$c_1|x|^{(2-n)/2} \leq u(x) \leq c_2|x|^{(2-n)/2}. \tag{1.5}$$

In case  $k(x)$  is a nonconstant function, Chen and Lin extended these results in [4] by assuming that

$$c_1|x|^{l-1} \leq |\nabla k(x)| \leq c_2|x|^{l-1}, \tag{1.6}$$

where  $l, c_1, c_2$  are positive constants. They showed in [3] that if the previous inequalities hold for some  $l \geq (n - 2)/2$ , then  $u$  satisfies the upper bound of (1.5),

$$u(x) \leq c|x|^{(2-n)/2}, \tag{1.7}$$

for any solution of (1.1) in a neighborhood of the origin, where  $c$  is a positive constant. They established in [4] that the asymptotic symmetry for solutions of (1.1) follows from (1.7).

In this paper, we are interested in studying singular solutions of an elliptic problem related to conformal deformation of metrics in  $\mathbb{R}_+^n$ . We will use the methods of Chen and Lin in [4] to study the positive singular solutions  $u \in C^2(B_1^+) \cap C^1(\partial B_1^+ \setminus \{0\})$  of the scalar and mean curvature equations

$$\begin{aligned} \Delta u + k(x)u^{(n+2)/(n-2)} &= 0 \quad \text{in } B_1^+, \\ \frac{\partial u}{\partial \eta} &= 0 \quad \text{in } \partial' B_1^+ \setminus \{0\}, \end{aligned} \tag{1.8}$$

where  $n \geq 3, B_1^+ = B_1 \cap \mathbb{R}_+^n, \partial' B_1^+ = \partial B_1^+ \cap \partial \mathbb{R}_+^n$ , and  $\eta$  denotes the normal outward unit vector in the boundary.

We assume that  $k \in C^1(\bar{B}_1^+)$  is bounded between two positive constants and satisfies, in a neighborhood of the origin,

$$c_1|x|^{l-1} \leq |\nabla k(x)| \leq c_2|x|^{l-1}, \tag{1.9}$$

where  $l, c_1, c_2$  are positive constants.

A motivation in studying problem (1.8) arises from the problem of finding a metric, conformal to the Euclidean  $\mathbb{R}_+^n$ , such that  $k(x)$  is the scalar curvature and zero is the mean curvature of the new metric. Our first observation is that if  $\partial k/\partial \eta = 0$  in  $\partial' B_1^+$ , then  $k$  can be extended to a function  $\tilde{k} \in C^1(\bar{B}_1)$ . Moreover, any solution  $u$  of problem (1.8) can be extended to a solution of problem (1.1) in the punctured ball  $B_1 \setminus \{0\}$ . If (1.9) holds for some  $l \geq (n - 2)/2$ , then  $u$  satisfies (1.7).

According to the previous discussion, we are going to assume condition (1.7) on the solutions  $u$  of problem (1.8). We point out that Chen and Lin in [5] conjectured that if  $k$  is Hölder continuous, then any solution of (1.1) satisfies (1.7).

In order to present the results, we start by defining the numbers  $P(r, u)$  and  $D(u)$  associated with a solution  $u$  of problem (1.8), as follows:

$$P(r, u) = \int_{\partial'' B_r^+} \left( \frac{n-2}{2} u \frac{\partial u}{\partial \eta} - \frac{1}{2} (x \cdot \eta) |\nabla u|^2 + \frac{\partial u}{\partial \eta} (x \cdot \nabla u) + \frac{n-2}{2n} (x \cdot \eta) k u^{2n/(n-2)} \right) ds_x, \tag{1.10}$$

where  $B_r^+$  is the upper half-ball of radius  $r$ ,  $\partial'' B_r^+ = \partial B_r^+ \cap \mathbb{R}_+^n$ , and  $D(u) = \lim_{r \rightarrow 0} P(r, u)$ . Under assumptions (1.7) on the growth of  $u$  and (1.9) on the growth of  $k$ , we have the following characterization of the singular solutions of (1.1).

**THEOREM 1.1.** *If  $u$  is a positive solution of (1.8), then  $D(u) \leq 0$ . Moreover,  $D(u) = 0$  if and only if zero is a removable singularity of  $u$ .*

As a consequence of the characterization obtained in the last theorem, we are able to establish a result about the nonexistence of singular solutions for an elliptic problem with Dirichlet and Neumann conditions on the boundary (Theorem 1.2) and a result which is related to the asymptotic symmetry in a neighborhood of the origin (Theorem 1.3).

Now, under the same assumptions on the growth of  $u$  and  $k$  as in Theorem 1.1, we prove the following two results.

**THEOREM 1.2.** *Assume that  $k(tx)$  is nonincreasing in  $t$  for any unit vector  $x \in \mathbb{R}^n$ , and in a neighborhood of the origin either  $k(x) \equiv 1$  or (1.9) holds for  $l \geq (n - 2)/2$ . Let  $\Omega$  be a smooth bounded domain in  $\mathbb{R}^n$ , such that  $\Lambda \equiv \Omega \cap \mathbb{R}_+^n$  and  $\partial' \Lambda = \partial \Lambda \cap \partial \mathbb{R}_+^n$  are nonempty, and  $x \cdot \eta > 0$  for all  $x \in \partial \Omega \cap \mathbb{R}_+^n$ . Then, there are no positive smooth solutions of*

$$\begin{aligned} \Delta u + k(x) u^{(n+2)/(n-2)} &= 0 \quad \text{in } \Lambda, \\ u &= 0 \quad \text{in } \partial' \Lambda, \\ \frac{\partial u}{\partial \eta} &= 0 \quad \text{in } \partial' \Lambda \setminus \{0\}, \end{aligned} \tag{1.11}$$

where  $\partial'' \Lambda = \partial \Lambda \cap \mathbb{R}_+^n$ .

**THEOREM 1.3.** *If  $u$  is a positive solution of problem (1.8), then*

$$u(x) = \bar{u}(|x|) (1 + o(1)) \tag{1.12}$$

as  $x \rightarrow 0$ , where  $\bar{u}(r) = \int_{\partial'' B_r^+} u(x) ds_x$  denotes the integral average of  $u$  on  $\partial'' B_r^+$ .

We want to mention that if  $v$  is a positive smooth solution of

$$\begin{aligned} \Delta v + \bar{k}(x)v^{(n+2)/(n-2)} &= 0 \quad \text{in } \mathbb{R}_+^n, \\ \frac{\partial v}{\partial \eta} &= 0 \quad \text{in } \partial\mathbb{R}_+^n, \end{aligned} \tag{1.13}$$

such that  $v(x) = O(|x|^{2-n})$  at infinity, then  $v$  satisfies the following Kazdan-Warner-Escobar-type identity:

$$\int_{\mathbb{R}_+^n} (x \cdot \nabla \bar{k}) v^{2n/(n-2)} dx = 0. \tag{1.14}$$

Conversely, Theorem 1.1 implies the following result of conformal geometry.

**THEOREM 1.4.** *Suppose that  $\bar{k}(x)$  satisfies  $\bar{k}(\infty) = \lim_{|x| \rightarrow +\infty} \bar{k}(x) > 0$ , and*

$$c_1|x|^{-(l+1)} \leq |\nabla \bar{k}(x)| \leq c_2|x|^{-(l+1)} \tag{1.15}$$

for large  $|x|$ ,  $l \geq (n - 2)/2$ , and  $c_1, c_2$  positive constants. Let  $v$  be a positive solution of problem (1.13) satisfying the inequality

$$v(x) \leq c|x|^{(2-n)/2}, \tag{1.16}$$

for large  $|x|$  and some positive constant  $c$ . If the identity (1.14) holds then the conformal metric  $\bar{g} = v^{4/(n-2)} dx^2$  can be realized as a smooth metric in  $\mathbb{S}_+^n$ . Furthermore, if  $v$  is such that the conformal metric cannot be realized as a smooth metric on  $\mathbb{S}_+^n$ , then there exist positive constants  $c_1$  and  $c_2$  such that

$$c_1|x|^{(2-n)/2} \leq v(x) \leq c_2|x|^{(2-n)/2} \tag{1.17}$$

for large  $|x|$ ,  $x_n > 0$ .

Observe that Theorem 1.1 says that if  $k(x)$  is a positive continuous function which satisfies (1.9) for some  $l \geq (n - 2)/2$ , then, for any positive solution  $u$  of (1.8) which satisfies (1.7), we have  $D(u) = 0$  if and only if 0 is a removable singularity of  $u$ . We note that the hypothesis  $l \geq (n - 2)/2$  in Theorem 1.1 is optimal.

In fact, assume that  $k(x) = k(|x|)$  is radially symmetric, bounded between two positive constants and satisfies

$$k(r) = 1 - Ar^l + R(r) \tag{1.18}$$

in a neighborhood of zero for some  $A > 0, l > 0, R(r) = o(r^l)$ , and  $R'(r) = o(r^{l-1})$  when  $r \rightarrow 0$ . Let  $u(r, \alpha)$  be the unique solution of the initial value problem

$$\begin{aligned} u''(r) + \frac{n-1}{r}u'(r) + k(r)u^{(n+2)/(n-2)} &= 0 \quad \text{in } [0,1], \\ u(0) &= \alpha, \quad u'(0) = 0. \end{aligned} \tag{1.19}$$

Then, as a consequence of [4, Theorem 1.6], we obtain the following theorem.

**THEOREM 1.5.** *Assume that (1.18) holds for  $l < (n - 2)/2$  and  $k(r)$  is nonincreasing in  $r$  for  $0 \leq r \leq 1$ . Then, there exists a sequence  $\alpha_j \rightarrow +\infty$  such that  $u(r, \alpha_j)$  converges in  $C_{loc}^2(B_1^+)$  to a positive and singular solution  $u$  of*

$$\begin{aligned} \Delta u + k(x)u^{(n+2)/(n-2)} &= 0 \quad \text{in } B_1^+, \\ \frac{\partial u}{\partial \eta} &= 0 \quad \text{in } \partial' B_1^+ \setminus \{0\}, \end{aligned} \tag{1.20}$$

such that  $D(u) = 0$ .

This paper is organized as follows. In Section 2, we will study some properties of the smooth solutions of problem (1.8) and, when  $k$  is a constant, we discuss the behavior of solutions  $u$  of problem (1.8) or (1.13) with a nonremovable singularity at the origin. In Section 3, we will prove the theorem of characterization of removable singularities of solutions of problem (1.8). In Section 4, we will prove the theorem of asymptotic symmetry of solutions of problem (1.8) in a neighborhood of the origin and the other applications of Theorem 1.1.

### 2. Preliminaries

First we recall without proof a Pohozaev-type identity derived in [6], and two theorems which follow from the results of Caffarelli et al. in [2].

**THEOREM 2.1 (Pohozaev identity).** *Let  $\Omega \subset \mathbb{R}^n$  ( $n \geq 3$ ) be a bounded domain with smooth boundary;  $u : \Omega \rightarrow \mathbb{R} \in C^2(\Omega)$ ;  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  continuous in  $x, u$ ;  $F : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  defined by  $F(x, u) = \int_0^u f(x, s)ds$ . Suppose that  $u$  is a solution of the equation  $-\Delta u = f(x, u)$ ,  $x \in \Omega$ . Then*

$$\begin{aligned} \int_{\Omega} \left( nF - \frac{n-2}{2}uf + \sum_i x^i F_{x^i} \right) dx \\ = \int_{\partial\Omega} \left( \frac{n-2}{2}u \frac{\partial u}{\partial \eta} + \sum_i x^i \eta^i \left( F - \frac{1}{2}|\nabla u|^2 \right) + \frac{\partial u}{\partial \eta} \sum_i x^i u_i \right) ds_x, \end{aligned} \tag{2.1}$$

where  $\eta$  denotes the normal outward unit vector in the boundary.

If  $f(x, u) = k(x)u^{(n+2)/(n-2)}$ ,  $u > 0$ , then the identity transforms into

$$\begin{aligned} \frac{n-2}{2n} \int_{\Omega} (x \cdot \nabla k)u^{2n/(n-2)} dx \\ = \int_{\partial\Omega} \left( \frac{n-2}{2}u \frac{\partial u}{\partial \eta} - \frac{1}{2}(x \cdot \eta)|\nabla u|^2 + \frac{\partial u}{\partial \eta}(x \cdot \nabla u) + \frac{n-2}{2n}(x \cdot \eta)ku^{2n/(n-2)} \right) ds_x. \end{aligned} \tag{2.2}$$

**THEOREM 2.2.** *Let  $u$  be a positive solution of (1.1) with a nonremovable isolated singularity. If  $k \equiv 1$ , then there exists a unique asymptotic constant  $-(2/n)((n - 2)/n)^n \leq D_{\infty} < 0$  and a radial singular solution  $h(r) = h_D(\log r)/r^{(n-2)/2}$  such that*

$$u(x) = h(|x|)(1 + o(1)) \quad \text{as } x \rightarrow 0. \tag{2.3}$$

THEOREM 2.3. Let  $\lambda$  be a constant and let  $u$  be a positive  $C^2$  solution of

$$\Delta u + \lambda u^{(n+2)/(n-2)} = 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}. \tag{2.4}$$

If the origin is an isolated nonremovable singularity, then  $u$  is radially symmetric with respect to the origin.

Now, we will prove the results of this section. First observe the following proposition.

PROPOSITION 2.4. If  $u(x)$  is a smooth solution of (1.8) in a neighborhood of the origin in  $\overline{\mathbb{R}^n_+}$ , then

$$\int_{\partial' B_r^+} \left( \frac{n-2}{2} u \frac{\partial u}{\partial \eta} - \frac{1}{2} (x \cdot \eta) |\nabla u|^2 + \frac{\partial u}{\partial \eta} (x \cdot \nabla u) + \frac{n-2}{2n} (x \cdot \eta) k u^{2n/(n-2)} \right) ds_x = 0, \tag{2.5}$$

where  $\partial' B_r^+ = \partial B_r^+ \cap \partial \mathbb{R}^n_+$ .

A straightforward calculation shows the following proposition.

PROPOSITION 2.5. If  $k(x) \equiv 1$ , the function

$$u_0(x) = \left( \frac{n-2}{2} \right)^{(n-2)/2} |x|^{(2-n)/2} \tag{2.6}$$

is a solution of problem (1.8).

Set  $\Omega = B_r^+$ . The Pohozaev identity implies that

$$\begin{aligned} \frac{n-2}{2n} \int_{B_r^+} (x \cdot \nabla k) u^{2n/(n-2)} dx &= P(r, u) + \int_{\partial' B_r^+} \left( \frac{n-2}{2} u \frac{\partial u}{\partial \eta} - \frac{1}{2} (x \cdot \eta) |\nabla u|^2 \right. \\ &\quad \left. + \frac{\partial u}{\partial \eta} (x \cdot \nabla u) + \frac{n-2}{2n} (x \cdot \eta) k u^{2n/(n-2)} \right) ds_x, \end{aligned} \tag{2.7}$$

where  $P(r, u)$  was defined in (1.10).

PROPOSITION 2.6. If  $u(x)$  is a solution of (1.8) with  $k(x) \equiv 1$ , then  $P(r, u)$  is a constant independent of  $r$ .

Proof. Observe that  $\partial u / \partial \eta = 0$  in  $\partial' B_r^+ \setminus \{0\}$ ,  $(x \cdot \eta) = 0$  on  $\partial' B_r^+ = \partial B_r^+ \cap \partial \mathbb{R}^n_+$ , and  $(x \cdot \eta) = r$  on  $\partial'' B_r^+ = \partial B_r^+ \cap \mathbb{R}^n_+$ . Then for  $0 < s < r$ , we have

$$\frac{n-2}{2n} \int_{B_r^+ \setminus B_s^+} (x \cdot \nabla k) u^{2n/(n-2)} dx = P(r, u) - P(s, u), \tag{2.8}$$

and therefore  $P(r, u) = P(s, u)$ . □

PROPOSITION 2.7. Let  $u(x)$  be a smooth solution of (1.8) in a neighborhood of the origin in  $\overline{\mathbb{R}^n_+}$ , then

- (i) if  $k(x) \equiv 1$ ,  $P(r, u) = 0$ ;
- (ii) if  $k(x)$  is smooth and nonconstant, then  $D(u) = \lim_{r \rightarrow 0} P(r, u) = 0$ .

*Proof.* Item (i) is a consequence of the Pohozaev identity and Proposition 2.4. From (1.10), we have

$$P(r, u) = \int_{\partial' B_r^+} \left( \frac{n-2}{2} u \frac{\partial u}{\partial \eta} - \frac{1}{2} r |\nabla u|^2 + \frac{\partial u}{\partial \eta} (x \cdot \nabla u) + \frac{n-2}{2n} r k u^{2n/(n-2)} \right) ds_x. \tag{2.9}$$

By the assumption on  $u$  and  $k$ , we conclude that  $D(u) = \lim_{r \rightarrow 0} P(r, u) = 0$ . □

**PROPOSITION 2.8.** *Let  $k(x) \equiv 1$ . If  $u(x)$  is a positive solution of (1.8) and zero is a nonremovable singular point of  $u$ , then*

$$u(x) = u_0(x)(1 + o(1)) \tag{2.10}$$

as  $x \rightarrow 0$ ; furthermore  $P(r, u) < 0$ .

*Proof.* Since  $\partial u / \partial \eta = 0$  in  $\partial' B_r^+ \setminus \{0\}$ , we can extend  $u$  to a function  $\hat{u}$  defined in the ball  $B_r$  by

$$\hat{u}(y, t) = \begin{cases} u(y, t) & \text{if } t \geq 0, \\ u(y, -t) & \text{if } t < 0, \end{cases} \tag{2.11}$$

where  $y \in \mathbb{R}^{n-1}$ ,  $|(y, t)| < r$ . It follows from Theorem 2.2 that

$$\hat{u}(x) = u_0(x)(1 + o(1)) \tag{2.12}$$

as  $x \rightarrow 0$ . Consequently

$$u(x) = u_0(x)(1 + o(1)) \tag{2.13}$$

as  $x \rightarrow 0$ .

Using this equality and Proposition 2.5, we get

$$\begin{aligned} P(r, u) &= \int_{\partial' B_r^+} \left( \frac{n-2}{2} u \frac{\partial u}{\partial \eta} - \frac{1}{2} r |\nabla u|^2 + \frac{\partial u}{\partial \eta} (x \cdot \nabla u) + \frac{n-2}{2n} r u^{2n/(n-2)} \right) ds_x \\ &\leq c \int_{\partial' B_r^+} \left( -\frac{1}{2} \left( \frac{n-2}{2} \right)^n |x|^{1-n} + \frac{n-2}{2n} \left( \frac{n-2}{2} \right)^n |x|^{1-n} \right) ds_x \\ &= c \int_{\partial' B_r^+} \left( \left( \frac{n-2}{2} \right)^n |x|^{1-n} \left( -\frac{1}{n} \right) \right) ds_x < 0, \end{aligned} \tag{2.14}$$

where  $c$  is a positive constant. □

**PROPOSITION 2.9.** *Let  $\lambda$  be a constant and let  $u$  be a positive  $C^2$  solution of*

$$\begin{aligned} \Delta u + \lambda u^{(n+2)/(n-2)} &= 0 \quad \text{in } \mathbb{R}_+^n, \\ \frac{\partial u}{\partial \eta} &= 0 \quad \text{in } \partial \mathbb{R}_+^n \setminus \{0\}. \end{aligned} \tag{2.15}$$

*If the origin is a nonremovable isolated singularity, then  $u$  is radially symmetrical with respect to the origin.*

*Proof.* Using the extension of  $u$ , as in the previous proposition, the result is a consequence of Theorem 2.3. □

### 3. Removable singularities

In this section, we prove that the quantity  $D(u)$  characterizes the positive singular solutions of problem (1.8), which helps us to study the asymptotic behavior of such solutions and prove a result of nonexistence. An important application of this characterization to conformal geometry says that if  $u$  is a positive solution to problem (1.13) for which the identity (1.14) holds, then the conformal metric  $u^{4/(n-2)}|dx|^2$  is smooth in  $S_+^n$ .

From now on, assume that  $k \in C^1(B_r^+)$  is bounded between two positive constants and satisfies inequalities (1.9). Furthermore, we will suppose that if  $u$  is a positive solution of (1.8), then there exists a positive constant  $c$  such that  $u$  satisfies inequality (1.7) in a neighborhood of the origin.

Theorem 1.1 is a consequence of the following five lemmas which will be proved later in this section. We start by establishing a Harnack inequality on the upper half-sphere of radius  $r$ .

LEMMA 3.1. *Let  $u$  be a positive solution of problem (1.8). Then there exists a positive constant  $C$  such that there exists the estimates*

$$\max_{|x|=r} u \leq C \min_{|x|=r} u, \quad |\nabla u(x)| \leq C|x|^{-1}u(x), \tag{3.1}$$

for  $|x| \leq 1/2$  with  $x_n \geq 0$ .

Using inequalities (3.1), we prove the following estimates.

LEMMA 3.2. *Let  $u$  be a positive solution of (1.8) and let  $w(t) = \bar{u}(r)r^{(n-2)/2}$ , where  $t = \log r$ ,  $r \in (0, 1]$  and  $\bar{u}(r) = \int_{\partial' B_r^+} u$  is the integral average of  $u$  on  $\partial' B_r^+$ . Then  $w$  satisfies*

$$\left(\frac{n-2}{2}\right)^2 w - c_1 w^{(n+2)/(n-2)} \leq w_{tt} \leq \left(\frac{n-2}{2}\right)^2 w - c_2 w^{(n+2)/(n-2)}, \tag{3.2}$$

for  $t \leq 0$ , where  $c_1$  and  $c_2$  are two positive constants.

The following two lemmas are technical; they will be used in the proof of Lemma 3.5, which is one of the fundamental steps in the proof of Theorem 1.1.

LEMMA 3.3. *Let  $u$  be a positive solution of (1.8). Assume that  $D(u) \geq 0$ . Then*

$$\lim_{x \rightarrow 0} u(x)|x|^{(n-2)/2} = 0. \tag{3.3}$$

LEMMA 3.4. *Let  $u$  and  $w$  be as in Lemma 3.2. Let  $r_i = e^{t_i}$ , where  $t_i$  is a sequence of negative numbers such that  $t_i \rightarrow -\infty$ , and  $w(t_i) \rightarrow 0$  as  $i \rightarrow +\infty$ , with  $w'(t_i) = 0$ . Then  $v_i(y) = r_i^{(n-2)/2} u(r_i y)$  is such that  $(v_i(e))^{-1} v_i(y)$ , with  $e = (1, 0, \dots, 0)$ , converges uniformly to the harmonic function  $h(y) = (1/2)|y|^{2-n} + 1/2$ .*

LEMMA 3.5. Suppose that  $k(x)$  satisfies (1.9) for some  $l \geq (n - 2)/2$ . Let  $u$  and  $w$  be as in Lemma 3.2. If  $D(u) \geq 0$ , then  $\lim_{t \rightarrow -\infty} w(t) = 0$  and there exists  $T$  such that  $w'(t) > 0$  for  $t \leq T$ .

*Proof of Theorem 1.1.* Let  $u$  be a positive solution of (1.8). Assuming that  $D(u) \geq 0$ , Lemma 3.5 implies that  $w(t) \rightarrow 0$  as  $t \rightarrow -\infty$ , and there exists  $T$  such that  $w'(t) > 0$  for  $t \leq T$ . For any  $\epsilon > 0$ ,  $|\epsilon| < 1$ ,  $T$  can be chosen such that

$$w_{tt} - \left(\frac{n-2}{2} - \epsilon\right)^2 w \geq 0 \quad \text{for } t \leq T. \tag{3.4}$$

Integrating this differential inequality, we have for  $t \leq T$ ,

$$w(t) \leq w(T) \exp\left[\left(\frac{n-2}{2} - \epsilon\right)(t - T)\right]. \tag{3.5}$$

Since  $\bar{u}(r) = w(t)r^{(2-n)/2}$ , where  $r = e^t$ , applying the Harnack inequality (3.1), we have that for any  $\epsilon > 0$ ,  $|\epsilon| < 1$ , there exists  $r_0 = e^T$  such that

$$\begin{aligned} u(x) &\leq C\bar{u}|x| = Cw(t)|x|^{(2-n)/2} \\ &\leq Cw(T) \exp\left[\left(\frac{n-2}{2} - \epsilon\right)(t - T)\right] |x|^{(2-n)/2} \\ &= Cw(T)r^{-\epsilon} \exp\left[-\left(\frac{n-2}{2} - \epsilon\right)T\right] \\ &\leq c(\epsilon)r^{-\epsilon}, \end{aligned} \tag{3.6}$$

for  $r = |x| < r_0$ . Hence  $u \in L^p(B_{r_0})$  for  $p$  large. By estimates of linear elliptic equations (see [7]), the function  $u(x)$  is smooth at the origin. Therefore, zero is a removable singularity and so, Theorem 1.1 follows. □

Now, we will prove the lemmas used in the proof of Theorem 1.1.

*Proof of Lemma 3.1.* Let  $v_r(x) = u(rx)$  for each  $r \in (0, 3/4]$  and for  $1/2 \leq |x| \leq 1$ . A straightforward calculation shows that  $v_r$  satisfies

$$\begin{aligned} \Delta v_r(x) + b_r(x)v_r(x) &= 0 \quad \text{in } \frac{1}{2} \leq |x| \leq 1, \quad x_n \geq 0, \\ \frac{\partial v_r}{\partial \eta} &= 0, \quad \frac{1}{2} \leq |x| \leq 1, \quad x_n = 0, \end{aligned} \tag{3.7}$$

where  $b_r(x) = r^2k(rx)(u(rx))^{4/(n-2)}$ . Due to inequality (1.7),  $b_r(x)$  is bounded for  $1/2 \leq |x| \leq 1$ .

Set

$$\hat{v}_r = \begin{cases} u(r(y, t)) & \text{if } t \geq 0, \\ u(r(y, -t)) & \text{if } t < 0, \end{cases} \tag{3.8}$$

where  $y \in \mathbb{R}^{n-1}$ ,  $|(y, t)| < 1$ .

After applying the Harnack inequality and the estimate of the gradient for  $\hat{v}_r$ , (see [4]), the conclusion of Lemma 3.1 follows. □

*Proof of Lemma 3.2.*

$$w(t) = \frac{2r^{(n-2)/2}}{n\alpha_n r^{n-1}} \int_{\partial' B_r^+} u(x) ds_x = \frac{2r^{-n/2}}{n\alpha_n} \int_{\partial' B_r^+} u(x) ds_x, \tag{3.9}$$

where  $\alpha_n$  is the volume of the unit ball  $B_1$  in  $\mathbb{R}^n$ . Hence

$$\begin{aligned} w_t &= \left(\frac{2r}{n\alpha_n}\right) \left(-\frac{n}{2} r^{-(n+2)/2} \int_{\partial' B_r^+} u(x) ds_x + r^{-n/2} \frac{d}{dr} \int_{\partial' B_r^+} u(ry) r^{n-1} ds_y\right) \\ &= -\frac{nr^{(n-2)/2}}{2} \int_{\partial' B_r^+} u(x) ds_x + \frac{2r^{(2-n)/2}}{n\alpha_n} \left[ \int_{\partial' B_r^+} (\nabla u(ry) \cdot yr^{n-1}) ds_y \right] \\ &\quad + \frac{2(n-1)r^{(n-2)/2}}{n\alpha_n} \left[ \int_{\partial' B_r^+} u(ry) ds_y \right] \\ &= \frac{n-2}{2} w + \frac{2r^{(2-n)/2}}{n\alpha_n} \left( \int_{B_r^+} \Delta u(x) dx - \int_{\partial' B_r^+} \nabla u(x) \cdot \eta ds_x \right) \\ &= \frac{n-2}{2} w + \frac{2r^{(2-n)/2}}{n\alpha_n} \int_{B_r^+} \Delta u(x) dx. \end{aligned} \tag{3.10}$$

Taking again the derivative with respect to  $t$ , we get

$$\begin{aligned} w_{tt} &= \frac{n-2}{2} w_t + \frac{(2-n)r^{(2-n)/2}}{2n\alpha_n} \int_{B_r^+} \Delta u(x) dx \\ &\quad + \frac{2r^{(2-n)/2}}{n\alpha_n} \left( \frac{d}{dr} \int_{B_r^+} \Delta u(x) dx \right) \frac{dr}{dt} \\ &= \left(\frac{n-2}{2}\right)^2 w - r^{(n+2)/2} \int_{\partial' B_r^+} k(x) u^{(n+2)/(n-2)} ds_x. \end{aligned} \tag{3.11}$$

By the assumption  $k(x)$  is bounded between two positive constants; furthermore (3.1) implies the existence of two positive constants  $c_3$  and  $c_4$  such that

$$c_3 (\bar{u}(r))^{(n+2)/(n-2)} \leq u(x)_{|x|=r}^{(n+2)/(n-2)} \leq c_4 (\bar{u}(r))^{(n+2)/(n-2)}. \tag{3.12}$$

Applying these estimates in (3.11), the conclusion of Lemma 3.2 follows. □

*Proof of Lemma 3.3.* Suppose that there exists a positive constant  $c_0$  such that

$$c_0 \leq u(x) |x|^{(n-2)/2} \leq c \tag{3.13}$$

for  $0 < |x| \leq 1/2$ ,  $x_n > 0$ . Let  $r_j$  be a sequence of positive numbers such that  $r_j \rightarrow 0$  as  $j \rightarrow +\infty$ . Set  $v_j(x) = u(r_j x) r_j^{(n-2)/2}$ . By (3.13), we have

$$c_0 \leq v_j(x) |x|^{(n-2)/2} \leq c, \tag{3.14}$$

and  $v_j$  satisfies

$$\begin{aligned} \Delta v_j + k(r_j x) v_j^{(n+2)/(n-2)} &= 0 \quad \text{in } \mathbb{R}_+^n, \\ \frac{\partial v_j}{\partial \eta} &= 0 \quad \text{in } \partial \mathbb{R}_+^n \setminus \{0\}. \end{aligned} \tag{3.15}$$

Since  $v_j$  is uniformly bounded in any compact subset of  $\mathbb{R}_+^n$ , by elliptic estimates there exists a subsequence of  $v_j$  (still denoted by  $v_j$ ) such that  $v_j$  converges in  $C_{\text{loc}}^2(\mathbb{R}_+^n)$  to a solution  $v$  of

$$\begin{aligned} \Delta v + k(0) v^{(n+2)/(n-2)} &= 0 \quad \text{in } \mathbb{R}_+^n, \\ \frac{\partial v}{\partial \eta} &= 0 \quad \text{in } \partial \mathbb{R}_+^n \setminus \{0\}, \\ v(x) &\geq c_1 |x|^{-(n-2)/2}. \end{aligned} \tag{3.16}$$

Proposition 2.8 implies that  $P(1, v) < 0$ ; furthermore, by (1.10), we get

$$\begin{aligned} P(r_j, u) &= \int_{\partial' B_{r_j}^+} \left( \frac{n-2}{2} u(x) \frac{\partial u}{\partial \eta} - \frac{1}{2} r_j |\nabla u|^2 + r_j \left| \frac{\partial u}{\partial \eta} \right|^2 + \frac{n-2}{2n} r_j k(x) u^{2n/(n-2)} \right) ds_x \\ &= \int_{\partial' B_{r_j}^+} \left( \frac{n-2}{2} u(r_j y) \frac{\partial u}{\partial \eta} r_j^{n-1} - \frac{1}{2} r_j^n |\nabla u|^2 \right. \\ &\quad \left. + r_j^n \left| \frac{\partial u}{\partial \eta} \right|^2 + \frac{n-2}{2n} r_j^n k(r_j y) u^{2n/(n-2)} \right) ds_y \\ &= P(1, v_j). \end{aligned} \tag{3.17}$$

Then

$$0 \leq D(u) = \lim_{r_j \rightarrow 0} P(r_j, u) = \lim_{j \rightarrow +\infty} P(1, v_j) = P(1, v) < 0 \tag{3.18}$$

yields a contradiction. This finishes the proof of Lemma 3.3. □

*Proof of Lemma 3.4.* Let  $x = r_i y$ . The Harnack inequality on any compact set of  $\mathbb{R}_+^n$  implies that

$$\begin{aligned} v_i(y) = r_i^{(n-2)/2} u(x) &\leq r_i^{(n-2)/2} \max u(x) \leq C r_i^{(n-2)/2} \min u(x) \\ &\leq C r_i^{(n-2)/2} \min_{|x|=r_i} u(x) \leq C r_i^{(n-2)/2} \bar{u}(r_i) = C w(t_i). \end{aligned} \tag{3.19}$$

Therefore,  $v_i(y)$  uniformly converges to 0 in any compact subset of  $\mathbb{R}_+^n$ . Therefore,  $z_i(y) = (v_i(e))^{-1} v_i(y)$ , with  $e = (1, 0, \dots, 0)$ , uniformly converges to a harmonic function  $h(y)$  with a possible singularity at 0. By Liouville's theorem (see [1]),  $h(y) = a|y|^{2-n} + b$ . Since  $h(e) = 1$ , we have  $a + b = 1$ .

Set  $\bar{g}_i(r) = r^{(n-2)/2} \bar{z}_i(y) = c(rr_i)^{(n-2)/2} \bar{u}(r;r)$ , where  $c = (v_i(e))^{-1}$  and  $r = |y|$ . Then  $\bar{g}_i(r) = cw(t)$  and  $\bar{g}'_i(r) = cw'(t)(1/r)$ , with  $t = \log(rr_i)$ . Then  $\bar{g}'_i(1) = cw'(t_i) = 0$ . Hence,  $(d/dr)(h(r)r^{(n-2)/2}) = 0$  in  $r = 1$ , where  $g(r) = h(r)r^{(n-2)/2} = \lim_{i \rightarrow +\infty} \bar{g}_i(r)$ . Calculating  $g'(1)$ , we obtain that  $a = b$ , which completes the proof of Lemma 3.4.  $\square$

*Proof of Lemma 3.5.* We will prove by contradiction that  $\lim_{t \rightarrow -\infty} w(t) = 0$ . Otherwise, by Lemma 3.3, there exists a sequence  $\{t_i\}$  of local minima of  $w$  with  $\lim_{i \rightarrow +\infty} t_i = -\infty$  and  $\lim_{i \rightarrow +\infty} w(t_i) = 0$ . The first inequality of (3.2) implies that there exists  $\epsilon_0 > 0$  such that  $w_{tt} > 0$  if  $w \leq \epsilon_0$ . Then there exists  $t_i^* < t_i < \bar{t}_i$  such that  $w(t_i^*) = w(\bar{t}_i) = \epsilon_0$ ;  $w_t < 0$  in  $[t_i^*, t_i]$  and  $w_t > 0$  in  $(t_i, \bar{t}_i]$ . Since  $\bar{t}_i < t_{i-1}$ , then  $\lim_{i \rightarrow +\infty} \bar{t}_i = -\infty$ . Under these conditions, we will prove the following inequalities:

$$\frac{2}{n-2} \log \frac{w(t)}{w(t_i)} - c_1 \leq t_i - t \leq \frac{2}{n-2} \log \frac{w(t)}{w(t_i)} + c_2 \tag{3.20}$$

for  $t \in [t_i^*, t_i]$ , and

$$\frac{2}{n-2} \log \frac{w(t)}{w(t_i)} - c_1 \leq t - t_i \leq \frac{2}{n-2} \log \frac{w(t)}{w(t_i)} + c_2 \tag{3.21}$$

for  $t \in [t_i, \bar{t}_i]$ .

First, consider the function

$$h(t) = w_t^2 - \left(\frac{n-2}{2}\right)^2 w^2 + cw^{2n/(n-2)} \tag{3.22}$$

for  $t \in [t_i^*, t_i]$ , where  $c = ((n-2)/n)c_1$  and  $c_1$  is the same constant of inequalities (3.2). Then

$$h'(t) = 2w_t \left[ w_{tt} - \left(\frac{n-2}{2}\right)^2 w + c_1 w^{(n+2)/(n-2)} \right] \leq 0 \tag{3.23}$$

and  $h(t) \geq h(t_i)$ , that is,

$$w_t^2 - g(w) \geq -g(w(t_i)), \tag{3.24}$$

where  $g(w) = [((n-2)/2)^2 w^2 - cw^{2n/(n-2)}]$  and we use  $w_t(t_i) = 0$ . Integrating the previous inequality, we find that

$$t_i - t \leq \int_{w(t_i)}^{w(t)} \frac{dw}{\sqrt{g(w) - g(w(t_i))}}. \tag{3.25}$$

By scaling  $w(t) \equiv w = \eta w(t_i)$ ,  $\eta \in [1, w(t)/w(t_i)]$ , we get

$$\int_{w(t_i)}^{w(t)} \frac{dw}{\sqrt{g(w) - g(w(t_i))}} = \int_1^{w(t)/w(t_i)} \frac{d\eta}{\sqrt{\bar{g}(\eta) - \bar{g}(1)}}, \tag{3.26}$$

where  $\bar{g}(\eta) = ((n - 2)/2)^2 \eta^2 - cw(t_i)^{4/(n-2)} \eta^{2n/(n-2)}$ . Hence, for  $\epsilon_0$  sufficiently small, we have

$$\begin{aligned} \bar{g}(\eta) - \bar{g}(1) &= (\eta^2 - 1) \left[ \left( \frac{n-2}{2} \right)^2 - cw(t_i)^{4/(n-2)} \left( \frac{\eta^{2n/(n-2)} - 1}{\eta^2 - 1} \right) \right] \\ &\geq (\eta^2 - 1) \left[ \left( \frac{n-2}{2} \right)^2 - c_3 w^{4/(n-2)} \right] > 0 \end{aligned} \tag{3.27}$$

for  $w \leq \epsilon_0$ . Therefore

$$\begin{aligned} \int_1^{w(t)/w(t_i)} \frac{d\eta}{\sqrt{\bar{g}(\eta) - \bar{g}(1)}} &\leq \int_1^{w(t)/w(t_i)} \frac{d\eta}{\{(\eta^2 - 1) [((n - 2)/2)^2 - c_3 w^{4/n-2}]\}^{1/2}} \\ &= \int_1^{w(t)/w(t_i)} \frac{d\eta}{((n - 2)/2) \{(\eta^2 - 1) [1 - c_4 w^{4/n-2}]\}^{1/2}} \\ &\leq \frac{2}{n - 2} \int_1^{w(t)/w(t_i)} \frac{d\eta}{\sqrt{\eta^2 - 1}} + c_5 w(t_i)^{4/(n-2)} \int_1^{w(t)/w(t_i)} \frac{\eta^{4/(n-2)}}{\sqrt{\eta^2 - 1}} d\eta, \end{aligned} \tag{3.28}$$

where we used

$$\alpha = c_4 w^{4/(n-2)} < 1 \quad (\epsilon_0 \text{ sufficiently small}), \quad \sqrt{\frac{1}{1 - \alpha}} \leq 1 + \alpha \left( \lim_{n \rightarrow \infty} \sum_{q=1}^n \alpha^q \right). \tag{3.29}$$

By the last integral in the previous inequality, we get  $\eta^{4/n-2} \leq (w(t)/w(t_i))^{4/(n-2)-1} \eta$  in  $[1, w(t)/w(t_i)]$ . Hence

$$\begin{aligned} w(t_i)^{4/(n-2)} \int_1^{w(t)/w(t_i)} \frac{\eta^{4/(n-2)}}{\sqrt{\eta^2 - 1}} d\eta &\leq w(t_i)^{4/(n-2)} \left( \frac{w(t)}{w(t_i)} \right)^{4/(n-2)-1} \int_1^{w(t)/w(t_i)} \frac{\eta}{\sqrt{\eta^2 - 1}} d\eta \\ &= w(t_i)^{4/(n-2)} \left( \frac{w(t)}{w(t_i)} \right)^{4/(n-2)-1} \sqrt{\left( \frac{w(t)}{w(t_i)} \right)^2 - 1} \leq \epsilon_0, \end{aligned} \tag{3.30}$$

and therefore

$$\int_1^{w(t)/w(t_i)} \frac{d\eta}{\sqrt{\bar{g}(\eta) - \bar{g}(1)}} \leq \frac{2}{n - 2} \log \frac{w(t)}{w(t_i)} + c_6 \tag{3.31}$$

for some positive constant  $c_6$ ; this proves the first part of (3.20). To prove the second part, we consider the function

$$\psi(t) = w_t^2 - \left( \frac{n - 2}{2} \right)^2 w^2(t) \tag{3.32}$$

for  $t \in [t_i^*, t_i]$ . By inequalities (3.2), we have

$$\psi'(t) = 2w_t \left[ w_{tt} - \left( \frac{n-2}{2} \right)^2 w(t) \right] \geq 0, \tag{3.33}$$

and therefore

$$w_t^2 - \left( \frac{n-2}{2} \right)^2 w^2(t) \leq - \left( \frac{n-2}{2} \right)^2 w^2(t_i), \tag{3.34}$$

where we used  $w_t(t_i) = 0$ . Integrating the previous inequality, we get

$$\frac{n-2}{2} (t_i - t) \geq \int_{w(t_i)}^{w(t)} \frac{dw}{\sqrt{w^2(t) - w^2(t_i)}}. \tag{3.35}$$

By scaling  $w = \eta w(t_i)$ ,  $\eta \in [1, w(t)/w(t_i)]$ , we have

$$\begin{aligned} & \int_{w(t_i)}^{w(t)} \frac{dw}{\sqrt{w^2(t) - w^2(t_i)}} \\ &= \int_1^{w(t)/w(t_i)} \frac{d\eta}{\sqrt{\eta^2 - 1}} = \log \left( \sqrt{\left( \frac{w(t)}{w(t_i)} \right)^2 - 1} + \frac{w(t)}{w(t_i)} \right) \geq \log \frac{w(t)}{w(t_i)}, \end{aligned} \tag{3.36}$$

which completes the proof of inequalities (3.20). The proof of inequalities (3.21) is analogous and we skipped it. Evaluating these two expressions for  $t_i^*$  and  $\bar{t}_i$ , respectively, we have

$$\begin{aligned} \frac{2}{n-2} \log \frac{w(t_i^*)}{w(t_i)} - c_1 &\leq t_i - t_i^* \leq \frac{2}{n-2} \log \frac{w(t_i^*)}{w(t_i)} + c_2, \\ \frac{2}{n-2} \log \frac{w(\bar{t}_i)}{w(t_i)} - c_1 &\leq \bar{t}_i - t_i \leq \frac{2}{n-2} \log \frac{w(\bar{t}_i)}{w(t_i)} + c_2. \end{aligned} \tag{3.37}$$

Hence,

$$\frac{4}{n-2} \log \left( \frac{\epsilon_0}{w(t_i)} \right) - c \leq \bar{t}_i - t_i^*. \tag{3.38}$$

Let  $r_i = e^{t_i}$  and  $v_i(y) = r_i^{(n-2)/2} u(r_i y)$ . By Lemma 3.4, we have that if  $|y| = 1$  then  $v_i(y) = v_i(e)(1 + o(1))$ . Consequently,

$$\begin{aligned} u(x) &= \bar{u}(r)(1 + o(1)), \\ |\nabla u(x)| &= -\bar{u}'(r)(1 + o(1)) \end{aligned} \tag{3.39}$$

in  $x = r_i y$ ,  $|x| = r = r_i$ . Using this result together with (3.17), we get

$$\begin{aligned}
 P(r, u) &= \int_{\partial' B_1^+} \left( \frac{n-2}{2} u(r y) \frac{\partial u}{\partial \eta} r^{n-1} - \frac{1}{2} r^n |\nabla u|^2 + r^n \left| \frac{\partial u}{\partial \eta} \right|^2 + \frac{n-2}{2n} r^n k(r y) u^{2n/(n-2)} \right) ds_y \\
 &= \int_{\partial' B_1^+} \left( \frac{n-2}{2} \bar{u}(r) \bar{u}'(r) r^{n-1} (1 + o(1))^2 + \frac{1}{2} r^n (\bar{u}'(r))^2 (1 + o(1))^2 \right) ds_y \\
 &\quad + \int_{\partial' B_1^+} \left( \frac{n-2}{2n} k(r y) (r^{(n-2)/2} \bar{u}(r) (1 + o(1)))^{2n/(n-2)} \right) ds_y \\
 &= \int_{\partial' B_1^+} \left[ \left( \frac{1}{2} w'^2(t_i) - \frac{1}{2} \left( \frac{n-2}{2} \right)^2 w^2(t_i) \right) (1 + o(1)) \right] ds_y \\
 &\quad + \int_{\partial' B_1^+} \left[ \left( \frac{n-2}{2n} k(r_i y) w^{2n/(n-2)}(t_i) \right) (1 + o(1)) \right] ds_y \\
 &= \frac{\sigma_{n-1}}{2} \left[ \frac{1}{2} w'^2(t_i) - \frac{1}{2} \left( \frac{n-2}{2} \right)^2 w^2(t_i) + \frac{n-2}{2n} \bar{k}(t_i) w^{2n/(n-2)}(t_i) \right] (1 + o(1)),
 \end{aligned} \tag{3.40}$$

where  $\sigma_{n-1} \equiv n\alpha_n$  is the volume of  $\partial B_1$  in  $\mathbb{R}^n$  and  $\bar{k}(t) = \int_{\partial' B_r^+} k$  with  $r = e^t$ . By the above equation, we obtain the following conclusions.

(1) Since  $w'(t_i) = 0$  and  $w(t_i) \rightarrow 0$  as  $i \rightarrow +\infty$ , then

$$D(u) = \lim_{i \rightarrow +\infty} P(r_i, u) = 0. \tag{3.41}$$

(2)  $P(r_i, u) \leq c_1(n)w^2(t_i) < 0$  for  $\epsilon_0$  sufficiently small.

Hence,

$$\begin{aligned}
 w^2(t_i) &\leq c_n |P(r_i, u)| \\
 &\leq c_n \left[ \int_{B_{r_i}^+ \setminus B_{r_i^*}^+} |x \cdot \nabla k(x)| u^{2n/n-2} dx + \int_{B_{r_i^*}^+} |x \cdot \nabla k(x)| u^{2n/n-2} dx \right] \equiv I_1 + I_2,
 \end{aligned} \tag{3.42}$$

where  $r_i^* = e^{t_i^*}$ . By the assumption, we have  $|x \cdot \nabla k(x)| \leq c_0 |x|^{(n-2)/2}$ . Since  $u$  satisfies inequality (1.7), we obtain the following estimate:

$$|I_2| = \left| c_n \int_{B_{r_i^*}^+} |x \cdot \nabla k(x)| u^{2n/(n-2)} dx \right| \leq c r_i^{*(n-2)/2} = c \exp\left(\frac{n-2}{2} t_i^*\right). \tag{3.43}$$

To calculate  $I_1$ , by the first inequality of (3.20),

$$w(t) \leq c w(t_i) \exp\left[\frac{n-2}{2}(t_i - t)\right]. \tag{3.44}$$

Setting  $\Lambda \equiv B_{r_i}^+ \setminus B_{r_i^*}^+$ , we have

$$\begin{aligned}
 |I_1| &= \left| c_n \int_{\Lambda} |x \cdot \nabla k(x)| u^{2n/(n-2)} dx \right| \\
 &\leq c_0 \int_{\Lambda} |x|^{(n-2)/2} u^{2n/(n-2)} dx \\
 &= c_0 \int_{r_i^*}^{r_i} \left( \int_{\partial' B_r^+} r^{(n-2)/2} u(x)^{2n/(n-2)} ds_x \right) dr \\
 &\leq \tilde{c}_0 \int_{r_i}^{r_i^*} w(t)^{2n/(n-2)} r^{(n-4)/2} dr \\
 &\leq \tilde{c} w^{2n/(n-2)}(t_i) \exp(nt_i) \int_{t_i^*}^{t_i} \exp\left(-\frac{n+2}{2}t\right) dt \\
 &\leq \hat{c} w^{2n/(n-2)}(t_i) \exp(nt_i) \exp\left(-\frac{n+2}{2}t_i^*\right).
 \end{aligned}
 \tag{3.45}$$

By the second inequality of (3.20), we get

$$w(t_i) \leq \bar{c} w(t_i^*) \exp\left[\frac{n-2}{2}(t_i^* - t_i)\right]. \tag{3.46}$$

Putting these two inequalities together, we have

$$|I_1| \leq \zeta \epsilon_0^{2n/(n-2)} \exp\left(\frac{n-2}{2}t_i^*\right). \tag{3.47}$$

Then, by (3.42), we get

$$w^2(t_i) \leq C \exp\left(\frac{n-2}{2}t_i^*\right), \tag{3.48}$$

which implies  $-\log w(t_i) \geq -C_2 + ((n-2)/4)(-t_i^*)$ ; applying (3.38),

$$\begin{aligned}
 \bar{t}_i - t_i^* &\geq \frac{4}{n-2} \log \epsilon_0 - \frac{4}{n-2} \log w(t_i) - c \\
 &\geq \frac{4}{n-2} \left[ -C_2 + \frac{n-2}{4}(-t_i^*) \right] - \frac{4}{n-2} \log\left(\frac{1}{\epsilon_0}\right) - c \\
 &= \frac{4}{n-2} \frac{n-2}{4}(-t_i^*) - c_3 \log\left(\frac{1}{\epsilon_0}\right),
 \end{aligned}
 \tag{3.49}$$

that is,

$$\bar{t}_i \geq -c_3 \log\left(\frac{1}{\epsilon_0}\right), \tag{3.50}$$

which yields a contradiction with the fact that  $\lim_{i \rightarrow +\infty} \bar{t}_i = -\infty$ . Lemma 3.5 has been proved.  $\square$

### 4. Applications

The theorems in this section are consequences of Theorem 1.1. Recall that Theorem 1.2 is a result of nonexistence of singular solutions, Theorem 1.3 is a result of asymptotic symmetry in a neighborhood of the origin, and Theorem 1.4 is a result of conformal geometry.

*Proof of Theorem 1.2.* Suppose that  $u > 0$  is a solution of (1.11). Let  $s > 0$ ; for (2.8) and the assumption  $u = 0$  in  $\partial''\Lambda$ , we have

$$\frac{n-2}{2n} \int_{\Lambda \setminus B_s^+} (x \cdot \nabla k) u^{2n/(n-2)} dx = \int_{\partial''\Lambda} \frac{1}{2} (x \cdot \eta) |\nabla u|^2 ds_x - P(s, u); \tag{4.1}$$

taking the limit when  $s \rightarrow 0$ , we obtain

$$\frac{1}{2} \int_{\partial''\Lambda} (x \cdot \eta) |\nabla u|^2 ds_x = \frac{n-2}{2n} \int_{\Lambda} (x \cdot \nabla k) u^{2n/(n-2)} dx + D(u) \leq 0 \tag{4.2}$$

by the assumption for  $k$  and Theorem 1.1. Since  $x \cdot \eta > 0$  in  $\partial''\Lambda$ , we conclude that  $|\nabla u| \equiv 0$  and  $\partial u / \partial \eta = 0$  in  $\partial''\Lambda$ . This implies  $u \equiv 0$  in  $\Lambda$ , which yields a contradiction.  $\square$

*Proof of Theorem 1.3.* Suppose that 0 is a nonremovable singularity of  $u$ , otherwise the result is obvious. By Theorem 1.1,  $D(u) = \lim_{r \rightarrow 0} P(r, u) < 0$ . We claim that there exists a constant  $c_0$  such that

$$u(x) \geq c_0 |x|^{(2-n)/2} \tag{4.3}$$

for  $0 < |x| \leq 1/2$ . In fact, let  $w(t) = \bar{u}(r)r^{(n-2)/2}$ ,  $t = \log r$ . If the claim is false then  $\lim_{t \rightarrow -\infty} w(t) = 0$ . Therefore,  $\overline{\lim}_{t \rightarrow -\infty} w(t) > 0$ , otherwise,  $\lim_{t \rightarrow -\infty} w(t) = 0$  and 0 must be a removable singularity of  $u$ . Then there exists a sequence of  $t_i \rightarrow -\infty$  such that  $w'(t_i) = 0$ . Using the same argument as in Lemma 3.5, we have

$$P(r_i, u) = \frac{\sigma_{n-1}}{2} \left[ \frac{1}{2} w'^2(t_i) - \frac{1}{2} \left( \frac{n-2}{2} \right)^2 w^2(t_i) + \frac{n-2}{2n} \bar{k}(t_i) w^{2n/(n-2)}(t_i) \right] (1 + o(1)). \tag{4.4}$$

Then  $D(u) = \lim_{i \rightarrow +\infty} P(r_i, u) = 0$ , which is a contradiction. Consequently (4.3) has been proved. Next, we prove that

$$u(x) = \bar{u}(|x|)(1 + o(1)), \tag{4.5}$$

$$|\nabla u(x)| = -\frac{\partial \bar{u}}{\partial r}(|x|)(1 + o(1)) \tag{4.6}$$

as  $x \rightarrow 0$ .

Suppose that there exists  $\epsilon_0 > 0$  such that

$$u(x_i) \geq \bar{u}(r_i)(1 + \epsilon_0) \tag{4.7}$$

for a sequence of points  $x_i$ , where  $r_i = |x_i| \rightarrow 0$  as  $i \rightarrow +\infty$ . Let  $v_i(x) = u(r_i x)r_i^{(n-2)/2}$ . Then, as in the proof of Lemma 3.3, there exists a subsequence (still denoted by  $v_i$ ) that converges in  $C_{loc}^2(\mathbb{R}_+^n)$  to a solution  $v$  of

$$\begin{aligned} \Delta v + k(0)v^{(n+2)/(n-2)} &= 0 \quad \text{in } \mathbb{R}_+^n, \\ \frac{\partial v}{\partial \eta} &= 0 \quad \text{in } \partial\mathbb{R}_+^n \setminus \{0\}. \end{aligned} \tag{4.8}$$

Now, by (4.3),  $v(x) \geq c_0|x|^{(2-n)/2}$ , then  $v(x) = v(|x|)$  (see Proposition 2.9). But,

$$\bar{v}_j(1) = \int_{\partial' B_1^+} v_j(x) ds_x = r_j^{(n-2)/2} \bar{u}(r_j). \tag{4.9}$$

Hence, by (4.7), if  $|x| = 1$ ,

$$v_j(x) = u(r_j x)r_j^{(n-2)/2} \geq \bar{u}(r_j)r_j^{(n-2)/2}(1 + \epsilon_0) = \bar{v}_j(1)(1 + \epsilon_0). \tag{4.10}$$

Therefore, if  $|x| = 1$ ,

$$v(x) = \lim_{j \rightarrow +\infty} v_j(x) \geq \lim_{j \rightarrow +\infty} \bar{v}_j(1)(1 + \epsilon_0) = v(x)(1 + \epsilon_0) \tag{4.11}$$

yields a contradiction. This proves the asymptotic symmetry of  $u$ . Arguing similarly, suppose that there exists  $\epsilon_0 > 0$  such that

$$|\nabla u(x_i)| \geq -\bar{u}'(r_i)(1 + \epsilon_0) \tag{4.12}$$

for a sequence of points  $x_i$ , where  $r_i = |x_i| \rightarrow 0$  as  $i \rightarrow +\infty$ , we arrive at

$$\begin{aligned} |\nabla v(x)| &= \lim_{j \rightarrow +\infty} |\nabla v_j(x)| \\ &\geq -\lim_{j \rightarrow +\infty} r_j^{n/2} \bar{u}'(r_j)(1 + \epsilon_0) \\ &= -\lim_{j \rightarrow +\infty} r_j^{(n-2)/2} \int_{\partial B_{r_j}^+} (\nabla u(x) \cdot x) ds_x (1 + \epsilon_0) \\ &= \lim_{j \rightarrow +\infty} \int_{\partial B_1^+} |\nabla v_j(y)| ds_y (1 + \epsilon_0) \\ &= |\nabla v(x)| (1 + \epsilon_0), \end{aligned} \tag{4.13}$$

where  $|x| = 1$ . From this contradiction, (4.6) follows. □

*Proof of Theorem 1.4.* Suppose that  $v$  is a solution of (1.13) such that (1.14) holds. We want to prove that  $v(x) = O(|x|^{2-n})$  at infinity, which implies that the conformal metric  $\bar{g}$  can be realized as a smooth metric in  $\mathbb{S}_+^n$ . Let

$$u(y) = |y|^{2-n} v\left(\frac{y}{|y|^2}\right) \tag{4.14}$$

be the Kelvin transformation of  $v$ . Since (1.13) is invariant under the Kelvin transformation,  $u(y)$  satisfies

$$\begin{aligned} \Delta u(y) + k(y)u^{(n+2)/(n-2)} &= 0 \quad \text{in } \mathbb{R}_+^n, \\ \frac{\partial u}{\partial \eta} &= 0 \quad \text{in } \partial \mathbb{R}_+^n \setminus \{0\}, \\ u(y) &= O(|y|^{2-n}) \quad \text{at infinity,} \end{aligned} \tag{4.15}$$

where  $k(y) = \bar{k}(y/|y|^2)$ .

Taking  $x = y/|y|^2$ , we have  $|\nabla k(y)| = |\nabla \bar{k}(x)||y|^{-2}$ . By (1.15), we see that

$$c_1|y|^{l-1} \leq |\nabla k(y)| \leq c_2|y|^{l-1} \tag{4.16}$$

in a neighborhood of zero. Due to inequality (1.16),  $u(y)$  is bounded by  $c|y|^{(2-n)/2}$  in a neighborhood of the origin, where  $c$  is a positive constant. Hence,  $D(u) = \lim_{r \rightarrow 0} P(r, u)$  exists and  $D(u) \leq 0$ .

To apply the Pohozaev identity in  $\mathbb{R}_+^n \setminus B_r^+$ , observe that

$$\int_{\mathbb{R}_+^n \setminus B_r^+} (y \cdot \nabla \bar{k}) u^{2n/(n-2)} dy = \lim_{\gamma \rightarrow +\infty} \int_{B_\gamma^+ \setminus B_r^+} (y \cdot \nabla \bar{k}) u^{2n/(n-2)} dy = \lim_{\gamma \rightarrow +\infty} P(\gamma, u) - P(r, u). \tag{4.17}$$

Now, since  $u(y) = O(|y|^{2-n})$  at infinity, a straightforward calculation shows that  $P(\gamma, u) \rightarrow 0$  as  $\gamma \rightarrow +\infty$ . Therefore

$$\int_{\mathbb{R}_+^n \setminus B_r^+} (y \cdot \nabla k) u^{2n/(n-2)} dy = -P(r, u). \tag{4.18}$$

Making the change of variables  $x = (y/|y|^2)$ , with Jacobian determinant  $-|x|^{-2n}$ , we observe that  $\nabla k(y) \cdot y = -\nabla \bar{k}(x) \cdot x$ ; therefore

$$\int_{\mathbb{R}_+^n \setminus B_r^+} (y \cdot \nabla k(y)) u^{2n/(n-2)} dy = - \int_{|x| \leq 1/r} (x \cdot \nabla \bar{k}(x)) v^{2n/(n-2)}(x) dx, \tag{4.19}$$

which implies, by (1.14) and (4.18), that

$$D(u) = \int_{\mathbb{R}_+^n} (x \cdot \nabla \bar{k}(x)) v^{2n/(n-2)}(x) dx = 0. \tag{4.20}$$

Then, by Theorem 1.1,  $u$  has a removable singularity at zero; since  $v(x) = u(x/|x|^2)|x|^{2-n}$ , we conclude that  $v(x) = O(|x|^{2-n})$  at infinity. Now suppose that  $v$  is such that the conformal metric cannot be realized as a smooth metric in  $S_+^n$ . Let  $u$  be the Kelvin transformation of  $v$ . Then  $D(u) < 0$  and  $u$  has a nonremovable singularity at zero. Therefore there exists two positive constants  $c_1$  and  $c_2$  such that

$$c_1|y|^{2-n} \leq u(y) \leq c_2|y|^{2-n} \tag{4.21}$$

in a neighborhood of zero. Consequently,

$$c_1|x|^{(2-n)/2} \leq v(x) \leq c_2|x|^{(2-n)/2} \quad (4.22)$$

for large  $|x|$ ,  $x_n > 0$ . This finishes the proof of Theorem 1.4.  $\square$

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