A NOTE ON SURFACES WITH PRESCRIBED ORIENTED EUCLIDEAN GAUSS MAP

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We present another proof of a theorem due to Hoffman and Osserman in Euclidean space concerning the determination of a conformal immersion by its Gauss map. Our approach depends on geometric quantities, that is, the hyperbolic Gauss map G and formulae obtained in hyperbolic space. We use the idea that the Euclidean Gauss map and the hyperbolic Gauss map with some compatibility relation determine a conformal immersion, proved in a previous paper.

1. Introduction

Throughout this paper we will consider surfaces immersed in Euclidean space thinking that, *locally*, the surface is immersed into the upper half-space $\mathbb{R}^3_+ = \{(u,v,w),\ w>0\}$. Of course there are two important metrics in $\mathbb{R}^3_+$: the standard Euclidean metric and the hyperbolic metric given by $\mathrm{d}s^2 = (1/w^2)\cdot(\mathrm{d}u^2 + \mathrm{d}v^2 + \mathrm{d}w^2)$. Notice that $\mathbb{R}^3_+$ endowed with the hyperbolic metric is the upper half-space model of hyperbolic space $\mathbb{H}^3$.

Our main goal is to show how certain geometric quantities relating to the oriented Euclidean Gauss map, the hyperbolic Gauss map, and the coordinate functions for conformal immersions of a planar domain into the upper half-space model of hyperbolic space, can be used to infer that the oriented Euclidean Gauss map determines locally a conformal immersion in Euclidean space with nonvanishing mean curvature, up to a homothety and Euclidean translation. This is a theorem due to Hoffman and Osserman [8] on account of a theorem by Kenmotsu [9]. Their proof specializes to dimension-3 results for the Euclidean Gauss map of conformal immersions in $\mathbb{R}^n$. The main idea of our proof is that both Euclidean Gauss map $\mathbf{E}$ and hyperbolic Gauss map G together with some compatibility relation determine a conformal immersion into upper half-space. Thus, we use in the proof the relation between Euclidean and hyperbolic geometry. We hope that this approach gives a significant insight into the theory.

Let $X : U \subset \mathbb{C} \looparrowright \mathbb{R}^3_+$, $z = x+iy \mapsto X(z)$ be an oriented conformal immersion of a simply connected domain U into the upper half-space $\mathbb{R}^3_+$. Let N be the Euclidean Gauss map of X such that $(X_x, X_y, N)(z)$ is a positively oriented basis of $\mathbb{R}^3$ for each $z \in U$,

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where $X_x = \partial X/\partial x$ and $X_y = \partial X/\partial y$. That is,

$$N = \frac{X_x \wedge X_y}{|X_x \wedge X_y|}, \tag{1.1}$$

where $|\cdot|$ stands for the Euclidean norm and $\wedge$ for the Euclidean vector product. We call $N = (N_1, N_2, N_3)$ the *oriented Euclidean Gauss map* of X, or more briefly the *Euclidean Gauss map* of X.

Let $\Pi : \mathbb{S}^2 \to \mathbb{C} \cup \{\infty\}$ be the standard stereographic projection. We set

$$E = \Pi \circ N = \frac{N_1 + iN_2}{1 - N_3}, \tag{1.2}$$

so that

$$N = \frac{(2\Re \mathbf{E}, 2\Im \mathbf{E}, \mathbf{E}\overline{\mathbf{E}} - 1)}{\mathbf{E}\overline{\mathbf{E}} + 1}. \tag{1.3}$$

We call $\mathbf{E}$ the *oriented Euclidean Gauss map* of X.

Given a point $p = X(z)$, the geodesic ray lying in hyperbolic space $\mathbb{H}^3$, issuing from $X(z)$ in the direction of N, fits the asymptotic boundary $\mathbb{C} \cup \{\infty\}$ at a point $G(z)$. This defines an application $G : U \to \mathbb{C} \cup \{\infty\}$, called the hyperbolic Gauss map.

For every C^1-function $f : U \to \mathbb{C} \cup \{\infty\}$, the notation f_z (resp., $f_{\overline{z}}$) stands for the derivative of f with respect to z (resp., $\overline{z}$), that is,

$$f_z = \frac{1}{2}(f_x - if_y), \qquad f_{\overline{z}} = \frac{1}{2}(f_x + if_y). \tag{1.4}$$

We will prove the following theorem.

Theorem 1.1 (Hoffman and Osserman). *Let $\mathbf{E}$ be a C^3 function in a simply connected domain Ω. Assume $\mathbf{E}_{\overline{z}} \neq 0$ everywhere. If $\mathbf{E}$ satisfies*

$$\left(\frac{\mathbf{E}_{z\overline{z}}}{\mathbf{E}_{\overline{z}}} - 2\overline{\mathbf{E}}\frac{\mathbf{E}_z}{\mathbf{E}\overline{\mathbf{E}} + 1}\right)_z = \left(\frac{(\overline{\mathbf{E}})_{z\overline{z}}}{(\overline{\mathbf{E}})_z} - 2\mathbf{E}\frac{(\overline{\mathbf{E}})_{\overline{z}}}{\mathbf{E}\overline{\mathbf{E}} + 1}\right)_{\overline{z}}, \tag{1.5}$$

then there is a conformal immersion $X : \Omega \hookrightarrow \mathbb{R}^3$ of Ω into $\mathbb{R}^3$ with nonvanishing mean curvature H. Furthermore, $\mathbf{E}$ is the oriented Euclidean Gauss map and any geometric quantity related to the immersion X can be determined in terms of $\mathbf{E}$. The mean curvature H is determined up to a positive multiplicative constant and is given by

$$H\left(\mathbf{E}_{z\overline{z}} - 2\mathbf{E}\frac{\mathbf{E}_z\mathbf{E}_{\overline{z}}}{\mathbf{E}\overline{\mathbf{E}} + 1}\right) = H_z\mathbf{E}_{\overline{z}} \quad \textit{(Kenmotsu's equation).} \tag{1.6}$$

If X and $\widetilde{X}$ are two conformal immersions of a simply connected domain U into $\mathbb{R}^3$ with the same oriented Euclidean Gauss map $\mathbf{E}$ and nonvanishing mean curvature, then $X = \widetilde{X}$ up to a homothety and Euclidean translation.

Remark 1.2. The authors have deduced a Kenmotsu-type theorem in hyperbolic space [14]. In fact, given a C^2 function $\mathbf{E}$ and a C^1 function $\mathcal{H}$ on a simply connected domain Ω, we have stated a necessary and sufficient compatibility equation to obtain a conformal (possibly branched) immersion of Ω into hyperbolic space with oriented Euclidean Gauss map $\mathbf{E}$ and hyperbolic mean curvature $\mathcal{H}$. Namely, the equation in hyperbolic space corresponding to (1.6) is the following:

$$\big(2+(\mathcal{H}-1)(1+E\overline{E})\big)(1+E\overline{E})E_{z\overline{z}}-2\big(1+(\mathcal{H}-1)(1+E\overline{E})\big)\overline{E}E_zE_{\overline{z}}-(1+E\overline{E})^2E_{\overline{z}}\mathcal{H}_z=0. \tag{1.7}$$

Notice that Aiyama and Akutagawa proved a related result in the 3-sphere, see [1].

2. Formulae for conformal immersions in $\mathbb{R}^3_+$

We now recall formulae about the immersion $X(z)=(u(z)+iv(z),w(z))$, established by the authors in a previous work [16].

Proposition 2.1. *Let $X:U\subset\mathbb{C}\hookrightarrow\mathbb{R}^3_+$, $z=x+iy\mapsto X(z)$ be a conformal immersion of a simply connected domain U into the upper half-space $\mathbb{R}^3_+$. Let H be the Euclidean mean curvature. Assume $\mathbf{E}(z)\neq\infty$, then*

$$u+iv=G-w\mathbf{E}, \tag{2.1}$$

$$G_z=w\mathbf{E}_z, \tag{2.2}$$

$$ds^2=\big|G_{\overline{z}}-w\mathbf{E}_{\overline{z}}\big|^2|dz|^2 \quad (\textit{induced Euclidean metric}), \tag{2.3}$$

$$w_{\overline{z}}=\frac{\overline{\mathbf{E}}}{\mathbf{E}\overline{\mathbf{E}}+1}\big(G_{\overline{z}}-w\mathbf{E}_{\overline{z}}\big), \tag{2.4}$$

$$H\big(G_{\overline{z}}-w\mathbf{E}_{\overline{z}}\big)=-2\frac{\mathbf{E}_{\overline{z}}}{\mathbf{E}\overline{\mathbf{E}}+1}. \tag{2.5}$$

Remark 2.2. (1) Notice that from (2.5), it follows that X is a minimal conformal immersion into Euclidean space if and only if the Euclidean Gauss map is a meromorphic function.

(2) The following formula has important significance for surfaces theory in hyperbolic space. Let $\mathcal{H}$ be the mean curvature of X in hyperbolic space

$$(1-\mathcal{H})\big(G_{\overline{z}}-w\mathbf{E}_{\overline{z}}\big)=2\frac{G_{\overline{z}}}{\mathbf{E}\overline{\mathbf{E}}+1}. \tag{2.6}$$

Thus, X has mean curvature 1 in hyperbolic space if and only if the hyperbolic Gauss map is a meromorphic function. This is a result proved by Bryant [4] and seems to be known by Bianchi, see [3, 6]. Mean curvature one surfaces in hyperbolic space has been intensively studied after Bryant's work. See, for example, [2, 5, 7, 11, 12, 15, 13, 18].

(3) We remark that in the context where we allow branch point, that is, the metric $ds^2=0$ at some points, then from (2.3) and (2.5), we conclude that z_0 is a branch point of an immersion with nonvanishing mean curvature H if and only if $\mathbf{E}_{\overline{z}}(z_0)=0$.

(4) When the mean curvature H is a real constant, we say $H = \text{cte}$, we infer from (1.6), Kenmotsu's equation,

$$\mathbf{E}_{z\bar{z}} = 2\overline{\mathbf{E}}\frac{\mathbf{E}_z\mathbf{E}_{\bar{z}}}{\mathbf{E}\overline{\mathbf{E}}+1}. \tag{2.7}$$

The above equation shows that $\mathbf{E}:\Omega\subset\mathbb{C}\to\mathbb{C}\cup\{\infty\}$, with the sphere $\mathbb{C}\cup\{\infty\}$ equipped with the standard metric $d\sigma^2 = (4/(1+\zeta\bar{\zeta})^2)|d\zeta|^2$, is a harmonic map [17]. We observe that if $H = \text{cte} \neq 0$, then a point z such that $\mathbf{E}_{\bar{z}} = 0$ is a zero of the Hopf function ϕ related to the harmonic map $\mathbf{E}$, since ϕ is holomorphic, we deduce that the set of points $\{\mathbf{E}_{\bar{z}} = 0\}$ is discrete.

(5) An equation relating the mean curvature H in Euclidean space and the mean curvature $H_{\mathbb{H}^3}$ in hyperbolic space can also be deduced, that is,

$$(H_{\mathbb{H}^3}-1)\mathbf{E}_{\bar{z}} = HG_{\bar{z}}. \tag{2.8}$$

(6) An important equation in hyperbolic space is given by

$$\mathbf{E}_{z\bar{z}} = \overline{\mathbf{E}}\frac{\mathbf{E}_z\mathbf{E}_{\bar{z}}}{\mathbf{E}\overline{\mathbf{E}}+1}. \tag{2.9}$$

Indeed, this equation is satisfied for mean curvature one surfaces in hyperbolic space, taking into account (1.7). Any solution of this equation gives rise to mean curvature one conformal immersion into hyperbolic space [16]. All solutions of this equation can be explicitly given by meromorphic data (h,T) according to a quite simple formula [16]. It can be considered as a harmonic map, taking a metric in $\mathbb{C}$ given by $d\sigma^2 = (2/(1+\zeta\bar{\zeta}))|d\zeta|^2$. As far as we know, the solutions of (2.7) cannot be expressed in a simple way. However, there are results of Ritoré [10].

3. Proof of the theorem

The idea of the proof to obtain existence is the following. We outline it locally. Using (1.5), it is straightforward to infer a candidate H for the mean curvature, up to a multiplicative positive constant. Then, using the equations established for hyperbolic space, see Proposition 2.1, we obtain a candidate for the height $w > 0$. Now we recall that the Euclidean Gauss map $\mathbf{E}$ and hyperbolic Gauss map G, with some compatibility relation, determine a conformal immersion into upper half-space [15]. Thus we expect to obtain G with the aid of (1.5). This happens to be true, and using $\mathbf{E}$, w, and G, we are able to find the horizontal coordinates u, v and to finally get the desired immersion.

We will now proceed with the details of the proof. Note first that from integrability condition (1.5), we deduce the existence of a real function F defined in the whole simply connected domain Ω such that

$$\mathbf{E}_{z\bar{z}} - 2\overline{\mathbf{E}}\frac{\mathbf{E}_z\mathbf{E}_{\bar{z}}}{\mathbf{E}\overline{\mathbf{E}}+1} = F_z\mathbf{E}_{\bar{z}}. \tag{3.1}$$

We define a real C^1 positive function H on Ω setting $H := e^F$. Notice that H is defined up to a multiplicative positive constant, since F is well defined up to additive constant.

We infer therefore Kenmotsu's equation (1.6). We fix now the (height) w on the whole Ω by choosing a solution of the following equation:

$$w_{\bar{z}} = \frac{\overline{\mathbf{E}}}{\mathbf{E}\overline{\mathbf{E}}+1} \cdot \frac{-2\mathbf{E}_{\bar{z}}}{H(\mathbf{E}\overline{\mathbf{E}}+1)}. \tag{3.2}$$

This can be done since a computation shows that (1.6) is the compatibility equation for (3.2).

Let $\cup_{n\geq 1} U_n = \Omega$ be an exhaustion of Ω by open simply connected subdomains $U_n \subset\subset \Omega$, that is, $U_n \subset U_{n+1} \subset \Omega$ and $\overline{U_n}$ is compact.

We will now proceed with the construction of the coordinate functions $(u+iv, w)$ of the desired immersion by constructing an auxiliary conformal immersion $X^{(n)}$ of U_n into $\mathbb{R}^3_+$ by recurrence such that the height of $X^{(n)}$ is equal to $w+\alpha_n > 0$.

We will see that $X^{(n+1)} - (0,0,\alpha_{n+1}) = X^{(n)} - (0,0,\alpha_n)$ on U_n, $n = 1,2,\ldots$. This will provide the desired immersion. Notice that (3.2) for a conformal immersion X of Ω into $\mathbb{R}^3_+$ can be deduced with the aid of (2.4) and (2.5). Of course this choice is done up to an additive constant.

We will first work in U_1. We choose a positive constant α_1 such that $w_1 := w+\alpha_1 > 0$. We define $G = G_1$ on U_1 by

$$G_{\bar{z}} = \left(w_1 - \frac{2}{H(\mathbf{E}\overline{\mathbf{E}}+1)}\right)\mathbf{E}_{\bar{z}}, \qquad G_z = w_1 \mathbf{E}_z. \tag{3.3}$$

With a bit of surprise, we find that the compatibility of (3.3) is still Kenmotsu's equation (1.6). We observe that (3.3) for a given conformal immersion follows again from Proposition 2.1. Of course this choice is done up to an additive complex constant. Now using (2.1), we obtain the horizontal coordinates $u_1 + iv_1$, that is, $u_1 + iv_1 := G_1 - w_1\mathbf{E}$. Hence we get a C^2 map $X^{(1)} : U_1 \to \mathbb{R}^3_+$, by $X^{(1)}(z) := (u_1(z)+iv_1(z), w_1(z))$, $z = x+iy$. On account of the above construction, we get (we write $w = w_1$, $G = G_1$ for simplicity)

$$\begin{aligned} w_{\bar{z}} &= \frac{\overline{\mathbf{E}}}{1+\mathbf{E}\overline{\mathbf{E}}}\left(G_{\bar{z}} - w\mathbf{E}_{\bar{z}} + G_z - w\mathbf{E}_z\right) = \frac{\overline{\mathbf{E}}}{1+\mathbf{E}\overline{\mathbf{E}}}\left(G_x - w\mathbf{E}_x\right) \\ &= \frac{\overline{\mathbf{E}}}{1+\mathbf{E}\overline{\mathbf{E}}}\left(G_{\bar{z}} - w\mathbf{E}_{\bar{z}} - \left(G_z - w\mathbf{E}_z\right)\right) = \frac{\overline{\mathbf{E}}}{1+\mathbf{E}\overline{\mathbf{E}}} i\left(G_y - w\mathbf{E}_y\right). \end{aligned} \tag{3.4}$$

Thus

$$w_x = 2\Re\left(\frac{\overline{\mathbf{E}}(G_x - wE_x)}{\mathbf{E}\overline{\mathbf{E}}+1}\right), \qquad w_y = 2\Re\left(\frac{\overline{\mathbf{E}}(G_y - wE_y)}{\mathbf{E}\overline{\mathbf{E}}+1}\right). \tag{3.5}$$

We infer from (3.5) that $X^{(1)}_x \cdot X^{(1)}_x = |G_x - w\mathbf{E}_x|^2 = X^{(1)}_y \cdot X^{(1)}_y = |G_y - w\mathbf{E}_y|^2 = |G_{\bar{z}} - w\mathbf{E}_{\bar{z}}|^2$ and $X^{(1)}_x \cdot X^{(1)}_y = 0$ in U_1, where $\cdot$ stands for the inner product in Euclidean space. We conclude therefore that $X^{(1)}$ is indeed a conformal immersion. Now computations show that the unit normal N to $X^{(1)}$ is given by $N = (2\Re\mathbf{E}, 2\Im\mathbf{E}, \mathbf{E}\overline{\mathbf{E}} - 1)/(\mathbf{E}\overline{\mathbf{E}}+1)$, hence $\mathbf{E}$ is the oriented Euclidean Gauss map and H (nonvanishing) is the mean curvature of the immersion $X^{(1)}$, see [16] for further details. We point out that fixing w and H, the immersion $X^{(1)}$ is uniquely defined up to a horizontal translation. We define the immersion X on U_1 setting $X := X^{(1)} - (0,0,\alpha_1)$. Of course the function w is the height, as desired. Now we will work in U_2. If $w + \alpha_1 > 0$ on U_2 we are done, if not we first choose a positive constant $\alpha_2 > \alpha_1$ such that $w_2 := w + \alpha_2$ is positive on U_2. Now making again the same construction as before, we can find a conformal immersion $X^{(2)}$ of U_2 into $\mathbb{R}^3_+$ whose oriented Gauss map is $\mathbf{E}$ and the mean curvature is H such that $u_2 + iv_2 = u_1 + iv_1$ on U_1. In fact, on account of (3.3), we deduce $(G_2)_{\overline{z}} = (G_1 + (\alpha_2 - \alpha_1)\mathbf{E})_{\overline{z}}$ and $(G_2)_z = (G_1 + (\alpha_2 - \alpha_1)\mathbf{E})_z$. Hence, there is a complex constant $a + ib$ such that $G_2 = G_1 + (\alpha_2 - \alpha_1)\mathbf{E} + a + ib$ in U_1. Thus doing a horizontal translation, if necessary, we may assume that $u_2 + iv_2 = u_1 + iv_1$ on U_1. This implies $X^{(2)} - (0,0,\alpha_2)\,|_{U_1} = X^{(1)} - (0,0,\alpha_1)$. We then define $X := X^{(2)} - (0,0,\alpha_2)$ on U_2. Now we can infer by recurrence that there exist positive constants α_n, $n \in \mathbb{N}^*$, and conformal immersions $X^{(n)}$ of U_n into $\mathbb{R}^3_+$ such that $X^{(n+1)} - (0,0,\alpha_{n+1}) = X^{(n)} - (0,0,\alpha_n)$ on U_n, $n = 1,2,\ldots$. We therefore inductively define a conformal immersion X of Ω into $\mathbb{R}^3$, setting $X := X^{(n)} - (0,0,\alpha_n)$ on U_n with nonvanishing mean curvature, whose Euclidean Gauss map is $\mathbf{E}$. The uniqueness part of the statement uses the same ideas and we will outline it to the reader. Firstly, notice that from (1.6), it follows that the mean curvature of a given conformal immersion with Euclidean Gauss map $\mathbf{E}$ is determined up to a positive multiplicative constant. Secondly, using (3.2), we see that the height (3.2) is determined up to an additive real constant and the same multiplicative constant. Finally, we conclude therefore that two given immersions with nonvanishing mean curvature are the same up to a homothety and Euclidean translation.

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