ANOTHER VERSION OF "EXOTIC CHARACTERIZATION OF A COMMUTATIVE H^*-ALGEBRA"

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Commutative H^*-algebra is characterized in a somewhat unusual fashion without assuming either Hilbert space structure or commutativity. Existence of an involution is not postulated also.

1. Introduction and main result

About 5 years ago, the author wrote an article in which he characterized commutative H^*-algebras in a somewhat unusual way. Now we will show that a similar characterization can be achieved without assuming a Hilbert space structure on the algebra. More specifically we will prove the following theorem.

THEOREM 1.1. *Let A be a semisimple complex Banach algebra with the following properties:*

(i) *for every closed right ideal R in A, there exists a closed left ideal L such that $R \cap L = \{0\}$ and $R + L = A$ (each $a \in A$ can be written in the form $a = a_1 + a_2$ with $a_1 \in R$, $a_2 \in L$);*
(ii) *if a, b in A are such that $ab = ba = 0$, then $\|a + b\|^2 = \|a\|^2 + \|b\|^2$.*

Then A is a commutative proper H^*-algebra [1].

It is easy to see that each proper commutative H^*-algebra has properties (i) and (ii), stated in the theorem.

A proper H^*-algebra is a Banach algebra A, whose underlying Banach space is a Hilbert space, which has an involution $x \to x^*$ such that $(xy, z) = (y, x^*z) = (x, zy^*)$ for all $x, y \in A$. An idempotent is a member e of A such that $e^2 = e$; e is primitive if it cannot be written as a sum, $e = e_1 + e_2$, of two nonzero idempotents e_1, e_2 such that $e_1e_2 = e_2e_1 = 0$ ($e = e_1 + e_2$ implies either $e_1 = 0$ or $e_2 = 0$). A Banach algebra A is semisimple if its radical [2] (Jacobson radical) consists of 0 alone. One of the properties of radical [2, Theorem 16] is the following proposition: if R is a right ideal consisting of nilpotents ($x \in R$ implies $x^n = 0$ for some positive integer n), then R is included in the radical. This proposition is relevant to both the present note and [4].

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2. Relevant lemmas

We will establish the main result (Theorem 1.1) by proving a series of lemmas first.

Let A be a semisimple Banach algebra such that for each closed right ideal $R \subset A$, there is a closed left ideal L such that $R \cap L = \{0\}$ and $R + L = A$.

Lemma 2.1. *The ideal L (for each closed right ideal R) is a two-sided ideal. It coincides with both the right and the left annihilators of R, that is, $L = r(R) = l(R)$, where $r(R) = \{x \in A : Rx = \{0\}\}$ and $l(R) = \{x \in A : xR = \{0\}\}$.*

Proof. First note that $L \subset r(R)$ and that $r(R)$ is a two-sided ideal. Hence, $R \cap r(R)$ is also a right ideal. But $x^2 = 0$ for each $x \in R \cap r(R)$ and so $R \cap r(R)$ is included in the radical of A, that is, $R \cap r(R) = \{0\}$ since A is semisimple.

Now let $a \in r(R) \sim L(a \in r(R),\ a \notin L)$. Write $a = b + c$ with $b \in R$, $c \in L$. Then $b = a - c$ belongs to $r(R)$ also, and this means that $b = 0$, $a = c$. Thus $L = r(R)$, L is a two-sided ideal. □

Corollary 2.2. *Each closed right ideal R in A is a two-sided ideal and $R = r(L) = l(L)$ where L is as above.*

Lemma 2.3. *Each closed right ideal R in A contains a nonzero idempotent.*

Proof. One can use the argument, which is used in the first part of the proof of Lemma 2.6 in [4]. Let $x \in R$ be such that $x + y + xy \neq 0$ for all $y \in A$ (x has no right quasi-inverse; existence of $x \in R$ is guaranteed by semisimplicity of A). Let R^1 be the closure of $\{xy + y : y \in A\}$; write $-x = e + u$ with $e \in r(R^1)$, $u \in R^1$ (we use Lemma 2.1 here). Then it is easy to verify that $e \neq 0$, $e^2 = e$, and $e \in R$ (note that $r(R^1) \subset R$). □

Now assume that A has property (ii) above (if $a, b \in A$ and $ab = ba = 0$, then $\|a + b\|^2 = \|a\|^2 + \|b\|^2$).

Lemma 2.4. *Every closed right ideal R contains a primitive idempotent.*

Proof. We already know that R contains an idempotent e. If e is not primitive, then we can write $e = e_1 + e_2$, where e_1, e_2 are some nonzero idempotents such that $e_1 e_2 = e_2 e_1 = 0$. From $e_i^2 = e_i$, it follows that $\|e_i\| \geq 1$ for $i = 1, 2$ and from (ii), it follows that $\|e_1\|^2 \leq \|e\|^2 - 1$. One can use the argument of Ambrose in [1, Theorem 3.2] that e is a sum, $e = \sum_{i=1}^{n} e_i$ of primitive idempotents $e_1, \ldots, e_n$ such that $e_1 e_j = 0$ for $i \neq j$, $(i, j \in \{1, \ldots, n\})$. From $e_1 = e e_i$, it follows that $e_i \in R$ for $i = 1, \ldots, n$. □

Lemma 2.5. *An idempotent $e \in A$ is primitive if and only if the closed right ideal $R = eA$ is minimal.*

Proof. The proof follows by direct verification. □

Lemma 2.6. *If $e \in A$ is a primitive idempotent, then $R = eA$ is 1-dimensional, that is, $R = \{\lambda e : \lambda$ is a complex number$\}$.*

Proof. One can use an obvious modification of the proof of [4, Lemma 2.8]. (In the present case, we use $l(R) = r(R)$ instead of R^p, the present primitive idempotent corresponds to primitive left projection in [4].) First we show that e is also a right identity of R and then we prove that each $x \in R$ has both the right and the left inverse (which do coincide). Then we apply Gelfand-Mazur theorem which states that the only complex Banach algebra, which is a division algebra, is the complex number system. □

LEMMA 2.7. *The product of any two distinct primitive idempotents* e_1, e_2 *in* A *is zero,* $e_1e_2 = e_2e_1 = 0$ *if* $e_1 \neq e_2$.

The proof is the same as the proof of [4, Lemma 2.9].

COROLLARY 2.8. *If* e_1, e_2 *are primitive idempotents,* $e_1 \neq e_2$ *and* $a_1 \in e_1A$, $a_2 \in e_2A$ *then* $\|a_1 + a_2\|^2 = \|a_1\|^2 + \|a_2\|^2$.

Proof. Note that $a_1a_2 = a_2a_1 = 0$ since $a_1 = \lambda_1 e_1$, $a_2 = \lambda_2 e_2$ for some complex numbers λ_1 and λ_2. □

COROLLARY 2.9. *If* $e_1, e_2, \ldots, e_n$ *are primitive idempotents and* $x \in \sum_{i=1}^n e_iA$, *then* $\|x\|^2 = \sum_{i=1}^n \|e_ix\|^2$.

Proof. The proof follows by induction on n. □

LEMMA 2.10. *Let* $e_1, \ldots, e_n$ *be primitive idempotents and let* x, y *be members of* $\sum_{i=1}^n e_iA$, *then* $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

Proof. It follows from Lemma 2.6 above that there are complex numbers $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ such that $x = \sum_{i=1}^n \lambda_i e_i$, $y = \sum_{i=1}^n \mu_i e_i$. Lemma 2.7 implies that $\|x\|^2 = \sum_{i=1}^n |\lambda_i|^2 \|e_i\|^2$, $\|y\|^2 = \sum_{i=1}^n |\mu_i|^2 \|e_i\|^2$, and $\|x \pm y\|^2 = \sum_{i=1}^n (|\lambda_i \pm \mu_i|^2)\|e_i\|^2$. Then $\|x + y\|^2 + \|x - y\|^2 = \sum_{i=1}^n (|\lambda_i + \mu_i|^2 + |\lambda_i - \mu_i|^2)\|e_i\|^2 = 2\sum_{i=1}^n (|\lambda_i|^2 + |\mu_i|^2)\|e_i\|^2 = 2(\|x\|^2 + \|y\|^2)$. □

3. Proof Theorem 1.1

Let $\wedge$ be the set of all primitive idempotents in A and let R_0 be the set of all finite sums $x = \sum_{i=1}^n x_i$ of members of ideals $e_iA : x_i \in e_iA$, $i = 1, 2, \ldots, n$, where $e_1, \ldots, e_n$ are some members of $\wedge$. Then R_0 is dense in A, otherwise one could find a nonzero primitive idempotent $e_0 \in l(R) = r(R)$, and this would lead to a contradiction.

It follows from Lemma 2.10 that $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for all $x, y \in R_0$. Since R_0 is dense in A, we may conclude that this relation holds for all $x, y \in A$. It follows from [3, Section 10A] that A is a Hilbert space with respect to the inner product $(x, y) = (1/4)\{\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2\}$.

We show that $L = l(R) = r(R)$ coincides with $R^p = \{x \in A : (x, y) = 0 \text{ for all } y \in R\}$ for each closed right ideal R in A. If $x \in R$, $y \in L$, then $xy = yx = 0$ and it follows that $\|x + y\|^2 = \|x\|^2 + \|y\|^2 = \|x - y\|^2 = \|x + iy\|^2 = \|x - iy\|^2$ from which one can conclude that $(x, y) = 0$. This simply means that $L \subset R^p$. Now let $a \in R^p$. Then $a = b + c$ with $b \in R$, $c \in L$. This means that $c \in R^p$ and so $b = a - c$ is also member of R^p. It follows that $b \in R^p \cap R = \{0\}$. Thus $b = 0$ and so $a = c$ is a member of L.

The theorem now follows from [4, Theorem 2.1].

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