ANALYTIC ERDÖS-TURÁN CONJECTURES AND ERDÖS-FUCHS THEOREM

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We consider and study formal power series, that we call supported series, with real coefficients which are either zero or bounded below by some positive constant. The sequences of such coefficients have a lot of similarity with sequences of natural numbers considered in additive number theory. It is this analogy that we pursue, thus establishing many properties and giving equivalent statements to the well-known Erdös-Turán conjectures in terms of supported series and extending to them a version of Erdös-Fuchs theorem.

1. Introduction

In a seminal paper of 1941, Erdös and Turán [4] made two conjectures in additive number theory, which have had an important impact on the field. They concern the number $r(A,n)$ of representations of a natural number n as a sum of two elements of a subset A of the set $\mathbb{N}$ of natural numbers. One of them, the so-called Erdös-Turán conjecture, still a notorious open question, can be formulated as follows.

(ET) If A is a basis of $\mathbb{N}$, if every natural number is the sum of two elements of A, then the number $r(A,n)$ of such representations is unbounded for $n \in \mathbb{N}$.

The other one predicted that $r(A,n)$ cannot be asymptotically too well approximated by its average value; more precisely, it is impossible to have $\sum_{m=0}^{n} r(A,m) = cn + O(1)$ for any positive real number c. Fifteen years later, in another very influential paper, Erdös and Fuchs [3] proved even more than that, namely that $\sum_{m=0}^{n} r(A,m) = cn + o(n^{1/4}\log^{-1/2} n)$ is impossible. This surprising result stirred a lot of interest since it was almost as good as a classical estimate of its kind for the number of lattice points in a circle, specific to the set A of the squares in $\mathbb{N}$ and obtained via difficult analytic techniques, while this one was valid for any subset A of $\mathbb{N}$ and with a simpler proof. Consequently, several authors presented various versions of the Erdös-Fuchs theorem [1, 7, 9, 10, 11]. Moreover, the Erdös-Fuchs paper contained the statement of a more general conjecture than (ET), namely the following.

(GET) If $A = \{a_1 < a_2 < \cdots < a_n < \cdots\}$ is an infinite subset of $\mathbb{N}$ such that $a_n \le dn^2$, for some constant $d > 0$ and all n in $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, then the number $r(A,n)$ of representations of n as a sum of two elements of A is unbounded for $n \in \mathbb{N}$.

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Indeed, (GET) implies (ET) because of the well-known fact [7] that if A is a basis of $\mathbb{N}$, then its elements a_n, taken in strictly increasing order, verify the condition $a_n \leq dn^2$ for some constant $d > 0$ and all $n \in \mathbb{N}^*$ (it also follows from [5, Lemma 3.15] that if A is a basis of $\mathbb{N}$, then we may take $d = 9/16$). In addition, Erdös and Fuchs remarked in their paper that their "[3, Theorem 1] remains true for sequences of nonnegative numbers $\{a_k\}$, not necessarily integers." Similarly, Halberstam and Roth [7, page 98] noted about their statement of the Erdös-Fuchs theorem: "Here we do not assume that the integers in A are distinct. In fact even the assumption that the elements of A are integers is superfluous (cf. Erdös-Fuchs, loc. cit., page 68)." In the light of such facts and remarks, it is natural to explore the extent to which some concepts and questions in additive number theory are independent of the specific nature of the subsets of $\mathbb{N}$ under consideration. The quest for a broader context, in which the above conjectures and results still make sense or remain valid, leads to the consideration and study of what we call "supported series." These are formal power series whose coefficients form sequences of nonnegative real numbers resembling the subsets of $\mathbb{N}$ in that their nonzero terms are bounded below by some positive constant. We thus establish various equivalent statements to the conjectures (ET) and (GET) in the realm of the supported series, and we extend to them a version of the Erdös-Fuchs theorem, due to Newman [12]. More precisely, we prove that for any supported series $f = \sum_{n=0}^{\infty} a_n X^n$, with $f^2 = \sum_{n=0}^{\infty} r_n X^n$, if we have $\sum_{k=0}^{n}(r_k - c) = O(n^t)$ for some real numbers $c > 0$ and $t \geq 0$, then $t \geq 1/4$. Our point is that most of the concepts, questions, and techniques pertaining to the additive representation of integers by subsets of $\mathbb{N}$ are not just about integers, but have a more general scope and can be naturally extended to the context of the supported series. The definitions, notions, and results presented in the sequel are all aimed at determining the essential features of the underlying ideas and problems, in a general setting, thus shedding more light and allowing for a more direct approach.

It is to be noted that special cases of supported series have already been considered in [1, 9, 10]. Thus in [1], we find: "In this paper, we consider sequences $c_0, c_1, c_2, \ldots,$ of real numbers satisfying the following two conditions: $c_n^2 \geq c_n \geq 0$ $(n = 0, 1, 2, \ldots)$ and $\sum_{n=0}^{\infty} c_n r^n < \infty$ for every r in $(0,1)$." And in [9], we read: "Let $\{r_1(n)\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers such that if $r_1(n) \neq 0$, then $r_1(n) \geq 1$. (The lower bound 1 is chosen for convenience; any positive lower bound would suffice.)"

2. Definitions and simple properties

The sets of natural numbers, rational integers, rational numbers, real numbers, and complex numbers are, respectively, denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. If $\mathbb{E}$ is anyone of these sets, then $\mathbb{E}^* = \mathbb{E} \setminus \{0\}$. For $\mathbb{E} = \mathbb{Q}$ or $\mathbb{R}$, we write $\mathbb{E}^+ = \{x \in \mathbb{E} : x \geq 0\}$ and $\mathbb{E}^{+*} = \{x \in \mathbb{E} : x > 0\}$. Moreover, $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, while $\overline{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{\infty\}$.

Definition 2.1. Let $f = \sum_{n=0}^{\infty} a_n X^n \in \mathbb{R}[[X]]$ be a power series with real coefficients. The *mass function* of f is the function F of a real variable x defined by $F(x) = \sum_{n \leq x} a_n$. In particular, $F(x) = 0$ if $x < 0$. The *support* of f is the set $\mathrm{Supp}(f) = \{n \in \mathbb{N} : a_n \neq 0\}$. The *norm* of f is the element $\|f\| = \sup\{|a_n| : n \in \mathbb{N}\}$ of $\overline{\mathbb{R}}^+$. The *size* of f is $s(f) = \|f^2\|$. We write $f^2 = \sum_{n=0}^{\infty} r(f,n) X^n$, where $r(f,n) = \sum_{i+j=n} a_i a_j$ for all $n \in \mathbb{N}$.

Lemma 2.2. *Let $f = \sum_{n=0}^{\infty} a_n X^n$ and $g = \sum_{n=0}^{\infty} b_n X^n$ in $\mathbb{R}[[X]]$, with mass functions F and G, respectively.*

(1) $f(X) = (1-X)(\sum_{n=0}^{\infty} F(n)X^n)$.

(2) *The mass function of $f+g$ is $F+G$. Moreover, $\|f+g\| \le \|f\| + \|g\|$ and $s(f+g) \le s(f)+s(g)+2\|fg\|$.*

(3) *The mass function of fg is given by $H(x) = \sum_{j\le x} a_j G(x-j) = \sum_{k\le x} b_k F(x-k)$, where the summations can also be extended to all $j,k \in \mathbb{N}$.*

(4) *For any $a \in \mathbb{R}$, the mass function of af is aF. Also, $\|af\| = |a| \cdot \|f\|$ and $s(af) = a^2 s(f)$. Moreover, if $a \neq 0$, then $\operatorname{Supp}(af) = \operatorname{Supp}(f)$.*

(5) *For any $m \in \mathbb{N}$, the mass function of $X^m f$ is $F_m(x) = F(x-m)$, where $x \in \mathbb{R}$. Moreover, $\|X^m f\| = \|f\|$ and $s(X^m f) = s(f)$.*

Proof. The proofs are mostly straightforward. Note that (1) amounts to $a_0 = F(0)$ and $a_n = F(n) - F(n-1)$ for $n \ge 1$. As to (3), the mass function of fg is by definition $H(x) = \sum_{n\le x}\sum_{j+k=n} a_j b_k = \sum_{j+k\le x} a_j b_k = \sum_{j\le x} a_j \sum_{k\le x-j} b_k = \sum_{j\le x} a_j G(x-j)$, with a similar relation exchanging F and G; moreover, the summations can be extended to all $j,k \in \mathbb{N}$, since $F(x) = G(x) = 0$ for $x < 0$. □

Definition 2.3. The set of all subsets of $\mathbb{N}$ will be written as $\mathcal{S}(\mathbb{N})$. Let $P \in \mathcal{S}(\mathbb{N})$. The *characteristic function* of P is the function χ_P defined on $\mathbb{N}$ by $\chi_P(n) = 1$ or 0 according as $n \in P$ or $n \notin P$. The *companion series* of P is $f_P = \sum_{p\in P} X^p = \sum_{n=0}^{\infty} \chi_P(n) X^n$. The *counting function* of P is defined for $x \in \mathbb{R}^+$ by $P(x) = |P \cap [0,x]|$, where $|E|$ denotes the cardinality of the set E.

For $P,Q \in \mathcal{S}(\mathbb{N})$ and $n \in \mathbb{N}$, we set $r(P,Q;n) = |\{(p,q) \in P \times Q : p+q = n\}|$; we further set $s(P,Q) = \sup\{r(P,Q;n) : n \in \mathbb{N}\}$ in $\overline{\mathbb{N}}$. The *sumset* of P and Q is $P+Q = \{p+q : p \in P,\ q \in Q\}$. In particular, if $P = Q$, we write $r(P,n) = r(P,P;n)$ and $s(P) = s(P,P)$. We say that P is a *basis* of $\mathbb{N}$ if $P+P = \mathbb{N}$.

For two subsets $P = \{p_1 < p_2 < \cdots < p_n < \cdots\}$ and $Q = \{q_1 < q_2 < \cdots < q_n < \cdots\}$ of $\mathbb{N}$, we set $P \ll Q$ if $|Q| \le |P|$ and $p_n \le q_n$ for all positive integers n not exceeding $|Q|$ (here, $|P|$ and $|Q|$ may be finite or infinite).

Lemma 2.4. *Let P, Q be subsets of $\mathbb{N}$.*

(1) *The mass function of the companion series f_P coincides with the counting function $P(x) = \sum_{n\le x} \chi_P(n)$ for $x \in \mathbb{R}^+$. Moreover, $\operatorname{Supp}(f_P) = P$.*

(2) *$f_P f_Q = \sum_{n=0}^{\infty} r(P,Q;n)\ X^n$, and $r(P,Q;n) \le \min(|P|,|Q|)$, for all $n \in \mathbb{N}$, so that $\|f_P f_Q\| = s(P,Q) \le \min(|P|,|Q|)$ is finite if P or Q is finite. Moreover, $\operatorname{Supp}(f_P f_Q) = P+Q$. In particular, $f_P^2 = \sum_{n=0}^{\infty} r(P,n) X^n$ and $s(f_P) = s(P)$. Moreover, P is a basis of $\mathbb{N}$ if and only if $\operatorname{Supp}(f_P^2) = \mathbb{N}$.*

Remark 2.5. To every subset P of $\mathbb{N}$ corresponds its companion series f_P. In a certain sense, this embeds $\mathcal{S}(\mathbb{N})$ into $\mathbb{R}[[X]]$. There are already two partial orders defined in $\mathcal{S}(\mathbb{N})$, namely the set inclusion $\subset$ and the relation $\ll$. Both relations can be extended to $\mathbb{R}[[X]]$.

Definition 2.6. For $f = \sum_{n=0}^{\infty} a_n X^n$ and $g = \sum_{n=0}^{\infty} b_n X^n$ in $\mathbb{R}[[X]]$, with mass functions F and G, respectively, we say that f is *contained by* g and write $f \sqsubset g$ if $a_n \le b_n$ for all $n \in \mathbb{N}$.

We also say that f is *subordinated* to g and write $f \ll g$ if $F(x) \geq G(x)$ for all $x \in \mathbb{R}$. This defines two partial orders in $\mathbb{R}[[X]]$.

Lemma 2.7. *Let $f, g, h \in \mathbb{R}[[X]]$.*

(1) *$f \ll g$ if and only if $g(X)/(1-X) \sqsubset f(X)/(1-X)$.*
(2) *If $f \sqsubset g$, then $g \ll f$. The converse is false, as shown by the example where $f = \sum_{n=0}^{\infty} X^{2n+1}$ and $g = \sum_{n=0}^{\infty} X^{2n}$.*
(3) *For $P, Q \in \mathcal{S}(\mathbb{N})$, the relation $f_P \sqsubset f_Q$ is equivalent to $P \subset Q$, while $f_P \ll f_Q$ is equivalent to $P \ll Q$.*
(4) *If $0 \sqsubset f$, that is, if all the coefficients of f are ≥ 0, then $g \sqsubset f+g$ and $f+g \ll g$.*
(5) *If $0 \sqsubset f \sqsubset g$, then $\|f\| \leq \|g\|$ and $0 \sqsubset f^2 \sqsubset fg \sqsubset g^2$, so that $s(f) \leq \|fg\| \leq s(g)$.*
(6) *If $\|f\| < \infty$ and $P = \operatorname{Supp}(f)$, then $f \sqsubset \|f\| f_P$ and therefore $\|f\| f_P \ll f$, that is, $F(x) \leq \|f\| P(x)$ for all $x \in \mathbb{R}^+$.*
(7) *If $f \sqsubset g$ (resp., $f \ll g$), then $f+h \sqsubset g+h$ (resp., $f+h \ll g+h$).*
(8) *If $f \sqsubset g$ (resp., $f \ll g$) and $0 \sqsubset h$, then $fh \sqsubset gh$ (resp., $fh \ll gh$) and, in particular, $af \sqsubset ag$ (resp., $af \ll ag$) for all $a \in \mathbb{R}^{+*}$.*

Proof. Property (1) follows from the relation $f(X)/(1-X) = \sum_{n=0}^{\infty} F(n) X^n$ given in Lemma 2.2. For (3), note that $f_P \sqsubset f_Q$ if and only if $\chi_P(n) \leq \chi_Q(n)$, that is, $n \in P$ implies $n \in Q$ for all $n \in \mathbb{N}$, that is, $P \subset Q$; while $f_P \ll f_Q$ if and only if $P(x) \geq Q(x)$ for all $x \in \mathbb{R}^+$, which is equivalent to $P(q_n) \geq Q(q_n) = n$, that is, $p_n \leq q_n$ for all possible n, that is, $P \ll Q$. For (8), note that if $f = \sum_{n=0}^{\infty} a_n X^n$, $g = \sum_{n=0}^{\infty} b_n X^n$, and $h = \sum_{n=0}^{\infty} c_n X^n$ with respective mass functions F, G, and H, then, by Lemma 2.2, the mass functions of fh and gh are $U(x) = \sum_n c_n F(x-n)$ and $V(x) = \sum_n c_n G(x-n)$, respectively. Everything else is straightforward. □

Example 2.8. The set of squares in $\mathbb{N}^*$ will be written as $\mathbb{S} = \{n^2 : n \in \mathbb{N}^*\}$. We have $\mathbb{S}(x) = [\sqrt{x}\,]$ for all $x \in \mathbb{R}^+$, where $[y]$ is the integer part of the real number y. For an infinite subset P of $\mathbb{N}$, we have $P \ll \mathbb{S}$, that is, $p_n \leq n^2$ for all $n \in \mathbb{N}^*$, if and only if $P(x^2) \geq \mathbb{S}(x^2) = [x]$ for all $x \in \mathbb{R}^+$, that is, $P(x^2) > x-1$ for all $x \in \mathbb{R}^+$.

Definition 2.9. For $f = \sum_{n=0}^{\infty} a_n X^n$ in $\mathbb{R}[[X]]$ and $c \in \mathbb{R}^{+*}$, we say that f is *c-supported* if, for any $n \in \mathbb{N}$, the condition $a_n \neq 0$ implies that $a_n \geq c$. We denote by $\mathcal{F}_c$ the set of all c-supported power series in $\mathbb{R}[[X]]$. We say that the series f is *supported* if it is c-supported for some $c \in \mathbb{R}^{+*}$. We denote by $\mathcal{F}$ the set of all supported series in $\mathbb{R}[[X]]$.

Lemma 2.10. *Let $f, g \in \mathbb{R}[[X]]$, with $f = \sum_{n=0}^{\infty} a_n X^n$, $P = \operatorname{Supp}(f)$, and $c, d \in \mathbb{R}^{+*}$.*

(1) *The series f is c-supported if and only if $a_n \geq c\chi_P(n)$ for all $n \in \mathbb{N}$, that is, $cf_P \sqsubset f$. So, if f is c-supported, then $f \ll cf_P$.*
(2) *The set $\mathcal{F}$ is closed under addition and multiplication.*
(3) *The series f lies in $\mathcal{F}$ if and only if $\inf\{a_n : n \in P\} > 0$ or $f = 0$. In particular, a constant a lies in $\mathcal{F}$ if and only if $a \geq 0$.*
(4) *If A is a nonempty subset of $\mathbb{N}$, then $f_A \in \mathcal{F}_1$.*

Remark 2.11. The following construction affords a better grasp of some features of the order relation $\ll$. Let $P = \{p_1 < p_2 < \cdots < p_n < \cdots\}$ be an infinite subset of $\mathbb{N}$ and let

$f(X)=\sum_{n=1}^{\infty}c_nX^{p_n}$, with $c_n\in\mathbb{R}^+$ (for $n\in\mathbb{N}$), and $g(X)=f_P(X)=\sum_{n=1}^{\infty}X^{p_n}$ in $\mathbb{R}[[X]]$, with mass functions F and G, respectively. For a real number $p_n\le x<p_{n+1}$, we clearly have $F(x)=F(p_n)=\sum_{j=1}^{n}c_j$ and $G(x)=n$, while for $x<p_1$, we have $F(x)=G(x)=0$. Note also that $\mathrm{Supp}(f)\subset P$. Then the condition $f_P\ll f$ is equivalent to the following one:

$(*)$ $\sum_{j=1}^{n}c_j\le n$ for all $n\in\mathbb{N}^*$.

Different choices for the sequence (c_n) provide various examples and counterexamples. Here, we just give two illustrations.

(1) For any infinite subset P of $\mathbb{N}$, there exists $f\in\mathcal{F}$ such that $\mathrm{Supp}(f)=P$ and its mass function F satisfies $F(x)\ge x^2$ for all large enough real numbers x. Indeed, just take $f(X)=\sum_{n=1}^{\infty}p_{n+1}^2X^{p_n}$, that is, $c_n=p_{n+1}^2$ for $n\in\mathbb{N}^*$. This implies, for $p_n\le x<p_{n+1}$, that $F(x)\ge c_n=p_{n+1}^2>x^2$.

(2) For any infinite subset P of $\mathbb{N}$, there exists $f\in\mathcal{F}$ such that $\|f\|=\infty$ and $f_P\ll f$. Indeed, just take $f(X)=\sum_{n=1}^{\infty}c_nX^{p_n}$ with $c_n=k$ if $n=k^2$ for some $k\in\mathbb{N}^*$, and $c_n=0$ otherwise. Clearly, $f\in\mathcal{F}$ and $\|f\|=\infty$. Moreover, for any $n\in\mathbb{N}^*$, there is a unique $m\in\mathbb{N}^*$ such that $m^2\le n<(m+1)^2$, and we have $\sum_{j=1}^{n}c_j=\sum_{k=1}^{m}k=m(m+1)/2\le m^2\le n$. Thus the condition $(*)$ is satisfied, that is, $f_P\ll f$.

3. The extended class of Erdös-Turán sets

Definition 3.1. We say that an element f of $\mathcal{F}$ belongs to the *extended class* $\mathscr{C}(\mathrm{EET})$ *of Erdös-Turán sets* if any supported power series which is subordinated to f has infinite size; that is, for any $g\in\mathcal{F}$ such that $g\ll f$, we have $s(g)=\infty$.

We say that an infinite subset P of $\mathbb{N}$ belongs to the *class* $\mathscr{C}(\mathrm{ET})$ *of Erdös-Turán sets* if for any infinite subset Q of $\mathbb{N}$ such that $Q\ll P$, we have $s(Q)=\infty$.

Remark 3.2. Note first that if $f\in\mathscr{C}(\mathrm{EET})$ and $P=\mathrm{Supp}(f)$, then, since $f\ll f$, we have $s(f)=\infty$ and thus $|P|=\infty$.

The class $\mathscr{C}(\mathrm{ET})$ was defined in [6]. Now let P be an infinite subset of $\mathbb{N}$. If $f_P\in\mathscr{C}(\mathrm{EET})$, then the relation $Q\ll P$, which by Lemma 2.7 means that $f_Q\ll f_P$, implies that $s(Q)=s(f_Q)=\infty$ (by Lemma 2.4). Thus, if $f_P\in\mathscr{C}(\mathrm{EET})$, then $P\in\mathscr{C}(\mathrm{ET})$. However, as indicated below, it is not known if the converse holds.

Proposition 3.3. *Let $f,g\in\mathbb{R}[[X]]$ be such that $0\sqsubset f$ and $0\sqsubset g$, with mass functions F and G, respectively. Let $P=\mathrm{Supp}(f)$ and $Q=\mathrm{Supp}(g)$. Then the following hold.*

(1) $\max(\|f\|,\|g\|)\le\|f+g\|\le\|f\|+\|g\|$.

(2) $\|f\|\cdot\|g\|\le\|fg\|\le\|f\|\cdot\|g\|\cdot s(P,Q)$.

(3) $\max(s(f),s(g))\le s(f+g)\le s(f)+s(g)+2\|fg\|$.

(4) $\mathrm{Supp}(f+g)=P\cup Q$ *and* $\mathrm{Supp}(fg)=P+Q$.

(5) $s(f)\cdot s(g)\le s(fg)\le s(f)\cdot s(g)\cdot s(P+P,\ Q+Q)$.

(6) *The mass function H of fg satisfies $H(x)\le F(x)G(x)\le H(2x)$ for all $x\in\mathbb{R}$.*

(7) $\|f\|^2\le s(f)\le\|f\|^2s(P)$ *and* $\mathrm{Supp}(f^2)=P+P=\mathrm{Supp}(f_P^2)$.

(8) $F(x)\le\|f\|P(x)\le\|f\|(x+1)$ *for all* $x\in\mathbb{R}^+$.

(9) *If $f\in\mathcal{F}_c$ for some $c\in\mathbb{R}^{+*}$, then $c^2s(P)\le s(f)\le\|f\|^2s(P)$ and $cP(x)\le F(x)\le\|f\|P(x)$ for all $x\in\mathbb{R}^+$.*

(10) *If* $\|g\| < \infty$, *then* $\|f+g\| = \infty$ *if and only if* $\|f\| = \infty$.

(11) *Assume that* g *is a nonzero polynomial with coefficients in* $\mathbb{R}^+$. *Then* $\|fg\| = \infty$ *if and only if* $\|f\| = \infty$. *Moreover, the following conditions are equivalent:* (i) $s(f+g) = \infty$; (ii) $s(f) = \infty$; (iii) $s(fg) = \infty$.

Proof. (1) and (3) follow from Lemmas 2.2 and 2.7. Also, (4) is straightforward, since the coefficients of f and g are ≥ 0. As to (5) and (7), they follow directly from (2) and (4). Also, (10) follows from (1).

Let $f = \sum_{n=0}^{\infty} a_n X^n$, $g = \sum_{n=0}^{\infty} b_n X^n$, $fg = \sum_{n=0}^{\infty} c_n X^n$, and $f_P f_Q = \sum_{n=0}^{\infty} d_n X^n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ and $d_n = \sum_{k=0}^{n} \chi_P(k)\chi_Q(n-k)$ for all $n \in \mathbb{N}$.

(2) We have $a_n \leq \|f\|\chi_P(n)$ and $b_n \leq \|g\|\chi_Q(n)$ for all $n \in \mathbb{N}$. Hence $c_n \leq \|f\| \cdot \|g\| d_n$ for all $n \in \mathbb{N}$. Therefore $\|fg\| \leq \|f\| \cdot \|g\| \cdot \|f_P f_Q\| = \|f\| \cdot \|g\| \cdot s(P,Q)$ by Lemma 2.4. Moreover, $a_m b_n \leq c_{m+n} \leq \|fg\|$ for all $m,n \in \mathbb{N}$. Hence $\|f\| \cdot \|g\| = \sup\{a_m b_n : m,n \in \mathbb{N}\} \leq \|fg\|$.

(6) By Lemma 2.2, we have $H(x) = \sum_{j+k\leq x} a_j b_k \leq (\sum_{j\leq x} a_j)(\sum_{k\leq x} b_k) = F(x)G(x) \leq \sum_{j+k\leq 2x} a_j b_k = H(2x)$ for all $x \in \mathbb{R}$.

(8) Since $a_n \leq \|f\|\chi_P(n)$ for all $n \in \mathbb{N}$, then $F(x) = \sum_{n\leq x} a_n \leq \|f\| \sum_{n\leq x} \chi_P(n) = \|f\| P(x)$, and $P(x) \leq |\mathbb{N} \cap [0,x]| \leq x+1$ for all $x \in \mathbb{R}^+$. Hence the inequalities.

(9) By Lemma 2.10, if $f \in \mathcal{F}_c$, then $0 \sqsubset c f_P \sqsubset f$, so that $0 \sqsubset c^2 f_P^2 \sqsubset f^2$ by Lemma 2.7. Hence $c^2 s(P) = \|c^2 f_P^2\| \leq \|f^2\| = s(f)$. Moreover, $c\chi_P(n) \leq a_n$ for all $n \in \mathbb{N}$. Hence $cP(x) \leq F(x)$ for all $x \in \mathbb{R}^+$. The other inequalities are contained in (7) and (8).

(11) Note that g being a nonzero polynomial, $|Q|$, $\|g\|$, and $s(g)$ are finite and positive. Also, by Lemma 2.4, $s(P,Q) \leq |Q| < \infty$ and $s(P+P,Q+Q) \leq |Q+Q| < \infty$. Now, the first equivalence results from (2). The equivalence of (i) and (ii) results from (3) and (7), since if $s(f+g) = \infty$, then $s(f) = \infty$ or $\|fg\| = \infty$; but if $\|fg\| = \infty$, then $\|f\| = \infty$ (previous case) and thus $s(f) = \infty$ by (7). The equivalence of (ii) and (iii) results from (5). □

Corollary 3.4. *Let* $f \in \mathcal{F}$ *and* $P = \mathrm{Supp}(f)$.

(1) *If* $s(P) = \infty$, *then* $s(f) = \infty$.

(2) *If* $\|f\| < \infty$, *then* $s(f) = \infty$ *if and only if* $s(P) = \infty$.

(3) *If* $\|f\| = \infty$, *then* $s(f) = \infty$.

These follow from Proposition 3.3(9) *and* (7).

Remark 3.5. Let $f \in \mathcal{F}$. Assume that $\|f\| < \infty$. If $f \in \mathcal{C}(\mathrm{EET})$, then $\mathrm{Supp}(f) \in \mathcal{C}(\mathrm{ET})$. Indeed, let $P = \mathrm{Supp}(f)$. By Remark 3.2, we have $|P| = \infty$. Now, if Q is an infinite subset of $\mathbb{N}$ such that $Q \ll P$, then, by Lemma 2.7, we have $f_Q \ll f_P$, and $f \sqsubset \|f\| f_P$, so that $\|f\| f_Q \ll \|f\| f_P \ll f$. Since $f \in \mathcal{C}(\mathrm{EET})$, it follows that $s(\|f\| f_Q) = \infty$. But $s(\|f\| f_Q) = \|f\|^2 s(Q)$ by Lemmas 2.2 and 2.4, and since $\|f\| < \infty$, then $s(Q) = \infty$. Thus $P \in \mathcal{C}(\mathrm{ET})$.

However, if $\|f\| = \infty$, we may have $f \in \mathcal{C}(\mathrm{EET})$ with $\mathrm{Supp}(f) \notin \mathcal{C}(\mathrm{ET})$. Indeed, given any infinite subset P of $\mathbb{N}$, belonging to $\mathcal{C}(\mathrm{ET})$ or not, one can easily construct a supported series f belonging to $\mathcal{C}(\mathrm{EET})$ such that $\mathrm{Supp}(f) = P$. Indeed, by Remark 2.11(1), there exists $f \in \mathcal{F}$ whose support is P and whose mass function F satisfies $F(x) \geq x^2$ for large enough real x. Then, for any $g \in \mathcal{F}$ with mass function G such that $g \ll f$, we have $G(x) \geq F(x) \geq x^2$. Consequently, $\|f\| = \|g\| = \infty$ by Proposition 3.3(8), and therefore $s(g) = \infty$ by Corollary 3.4(3). Thus $f \in \mathcal{C}(\mathrm{EET})$.

Proposition 3.6. *For $f \in \mathscr{C}(\mathrm{EET})$, the following properties hold.*

(1) *For any $g \in \mathscr{F}$ such that $g \ll f$, $g \in \mathscr{C}(\mathrm{EET})$.*
(2) *For any $t \in \mathbb{R}^{+*}$, $tf \in \mathscr{C}(\mathrm{EET})$.*
(3) *For any $g \in \mathscr{F}$, $f + g \in \mathscr{C}(\mathrm{EET})$.*
(4) *For any $m \in \mathbb{N}$, $X^m f \in \mathscr{C}(\mathrm{EET})$.*
(5) *For any $g \in \mathscr{F}$, provided $g \neq 0$, $fg \in \mathscr{C}(\mathrm{EET})$.*
(6) *For $g \in \mathscr{F}$, if the mass functions F and G of f and g, respectively, satisfy $G(x) \geq F(x)$ for large enough $x \in \mathbb{R}$, then $s(g) = \infty$.*

Furthermore, for $f \in \mathscr{F}$, the following hold.

(7) *If g is a polynomial with coefficients in $\mathbb{R}^+$, then $f \in \mathscr{C}(\mathrm{EET})$ if and only if $f + g \in \mathscr{C}(\mathrm{EET})$.*
(8) *If g is a nonzero polynomial with coefficients in $\mathbb{R}^+$, then $f \in \mathscr{C}(\mathrm{EET})$ if and only if $fg \in \mathscr{C}(\mathrm{EET})$.*

Proof. (1) follows directly from the definitions.

(2) If $g \in \mathscr{F}$ is such that $g \ll tf$, then $t^{-1}g \ll f$ by Lemma 2.7. Since $f \in \mathscr{C}(\mathrm{EET})$, then $s(t^{-1}g) = \infty$. But $s(t^{-1}g) = t^{-2}s(g)$ by Lemma 2.2. Hence $s(g) = \infty$. Thus $tf \in \mathscr{C}(\mathrm{EET})$.

(3) This follows from (1), since $f + g \in \mathscr{F}$ and $f + g \ll f$.

(4) By Lemma 2.2, the mass function of $X^m f$ is $F_m(x) = F(x - m)$ for all $x \in \mathbb{R}$. Let $g = \sum b_n X^n$ in $\mathscr{F}$ be such that $g \ll X^m f$, that is, the mass function G of g satisfies $F(x) \leq G(x + m)$ for all $x \in \mathbb{R}^+$. Let $h = \sum_{n=0}^{\infty} c_n X^n$, with $c_0 = \sum_{n=0}^{m} b_n$ and $c_n = b_{n+m}$ for $n \in \mathbb{N}^*$. Then h lies in $\mathscr{F}$, and its mass function H is given, for $x \in \mathbb{R}^+$, by $H(x) = \sum_{n \leq x} c_n = \sum_{n \leq x+m} b_n = G(x + m)$. Therefore $F(x) \leq H(x)$ for all $x \in \mathbb{R}^+$, that is, $h \ll f$. Since $f \in \mathscr{C}(\mathrm{EET})$, then $s(h) = \infty$. Now, $g + c_0 X^m = X^m h + u$, where u is a polynomial with coefficients in $\mathbb{R}^+$. Moreover, by Lemma 2.2, $s(X^m h) = s(h) = \infty$. Hence $s(g + c_0 X^m) = s(X^m h + u) = \infty$ and therefore $s(g) = \infty$ by Proposition 3.3(11). Thus $X^m f \in \mathscr{C}(\mathrm{EET})$.

(5) Let $f = \sum_{n=0}^{\infty} a_n X^n$ and $g = \sum_{n=0}^{\infty} b_n X^n$ in $\mathscr{F}$, and $h = fg$ (also in $\mathscr{F}$ by Lemma 2.10), with mass functions F, G, and H respectively. Since $g \neq 0$, there exists $m \in \mathbb{N}$ such that $b_m > 0$. By Lemma 2.2, $H(x) = \sum_{n \leq x} b_n F(x - n) \geq b_m F(x - m)$, for $x \geq m$—an inequality which also holds trivially for $x < m$. Thus, if $u \in \mathscr{F}$ is such that $u \ll h$, then its mass function U satisfies $U(x) \geq H(x) \geq b_m F(x - m)$ for all $x \in \mathbb{R}^+$. But, by Lemma 2.2, $x \mapsto b_m F(x - m)$ is the mass function of $b_m X^m f$, so that $u \ll b_m X^m f$. Now, since $f \in \mathscr{C}(\mathrm{EET})$, then by (4) and (2), $b_m X^m f \in \mathscr{C}(\mathrm{EET})$. Hence $s(u) = \infty$. Thus $h \in \mathscr{C}(\mathrm{EET})$.

(6) Let $f = \sum_{n=0}^{\infty} a_n X^n$ and $g = \sum_{n=0}^{\infty} b_n X^n$ in $\mathscr{F}$ be such that $G(x) \geq F(x)$ for $x \geq x_0$, where $x_0 \in \mathbb{R}^+$. Define $h = \sum_{n=0}^{\infty} c_n X^n$ by $c_n = \max(a_n, b_n)$ if $n < x_0$, and $c_n = b_n$ if $n \geq x_0$ ($n \in \mathbb{N}$). Then its mass function H satisfies $H(x) \geq F(x)$ for $x < x_0$ and $H(x) \geq G(x) \geq F(x)$ for $x \geq x_0$ ($x \in \mathbb{R}$), so that $h \ll f$, and since $f \in \mathscr{C}(\mathrm{EET})$, we get $s(h) = \infty$. Moreover, $h = g + u$, where $u = \sum_{n < x_0} (c_n - b_n) X^n$ is a polynomial with coefficients in $\mathbb{R}^+$, and therefore $s(g) = \infty$ by Proposition 3.3(11).

(7) Assume that $f + g \in \mathscr{C}(\mathrm{EET})$. Then, for any $h \in \mathscr{F}$ such that $h \ll f$, we have $h + g \ll f + g$, by Lemma 2.7, so that $s(h + g) = \infty$, and therefore, by Proposition 3.3(11), since $0 \sqsubset g$ and g is a polynomial, $s(h) = \infty$. Thus $f \in \mathscr{C}(\mathrm{EET})$. The converse follows from (3) above.

(8) Assume that $fg \in \mathscr{C}(\mathrm{EET})$. Then, for any $h \in \mathscr{F}$ such that $h \ll f$, we have $hg \ll fg$, by Lemma 2.7, so that $s(hg) = \infty$, and therefore, by Proposition 3.3(11), since $0 \sqsubset g$,

$g \neq 0$ and g is a polynomial, $s(h) = \infty$. Thus $f \in \mathscr{C}(\text{EET})$. The converse follows from (5) above. □

Definition 3.7. For $f \in \mathbb{R}[[X]]$ and $m \in \mathbb{N}$, the *truncated series* of f at m is $f \downharpoonleft m = \sum_{n=m}^{\infty} a_n X^n$.

For f and g in $\mathbb{R}[[X]]$, with mass functions F and G, respectively, we say that g is *asymptotically subordinated* to f if $G(x) \geq F(x)$ for all large enough real numbers x.

Corollary 3.8. *For $f \in \mathscr{F}$, the following properties hold.*

(1) *For any $m \in \mathbb{N}$, the series f lies in $\mathscr{C}(\text{EET})$ if and only if $f \downharpoonleft m$ lies in $\mathscr{C}(\text{EET})$.*
(2) *The series f lies in $\mathscr{C}(\text{EET})$ if and only if, for any g asymptotically subordinated to f in $\mathscr{F}$, we have $s(g) = \infty$.*

Proof. The first property follows from Proposition 3.6(7) since $f \downharpoonleft m$ differs from f by the polynomial consisting of the first m terms of f. The second property follows from Proposition 3.6(6). □

Definition 3.9. Let $P = \{p_1 < p_2 < \cdots < p_n < \cdots\}$ be a subset of $\mathbb{N}$, identified to the sequence (p_n) of its elements indexed in strictly increasing order. For every $k \in \mathbb{N}^*$, the *kth ray* of P is the set $P_k = \{p_k < p_{2k} < \cdots < p_{nk} < \cdots\}$ consisting of the elements of P whose index is a multiple of k.

Remark 3.10. Simple properties for an infinite subset $P = \{p_1 < p_2 < \cdots\}$ of $\mathbb{N}$ are the following.

(1) For $x \in \mathbb{R}^+$ and $k, n \in \mathbb{N}^*$, we have $P_k(x) = n$ if and only if $p_{kn} \leq x < p_{k(n+1)}$. Thus if $P_k(x) = n$, then $kn \leq P(x) < k(n+1)$.
(2) For $x \in \mathbb{R}^+$, we have $kP_k(x) \leq P(x) < kP_k(x) + k$.
(3) For $x \in \mathbb{R}^+$ and $k, n \in \mathbb{N}^*$, we have $P_k(x) \geq n$ if and only if $p_{kn} \leq x$.

Lemma 3.11. *Let $P = \{p_1 < p_2 < \cdots < p_n < \cdots\}$ and $Q = \{q_1 < q_2 < \cdots < q_n < \cdots\}$ be two infinite subsets of $\mathbb{N}$, and $k \in \mathbb{N}^*$.*

(1) *If $P(x) \leq kQ(x)$ (resp., $P(x) < kQ(x)$, resp., $P(x) = kQ(x)$) for all $x \in \mathbb{R}^+$, then $q_n \leq p_{kn}$ (resp., $q_n < p_{kn}$, resp., $q_n = p_{kn}$) for all $n \in \mathbb{N}^*$.*
(2) *If $q_n \leq p_{kn}$ for all $n \in \mathbb{N}^*$, then $P(x) < kQ(x) + k$ for all $x \in \mathbb{R}^+$.*
(3) *If $kQ(x) \leq P(x) < kQ(x) + k$ for all $x \in \mathbb{R}^+$, then $p_{kn} \leq q_n < p_{kn+k}$ for all $n \in \mathbb{N}^*$.*
(4) *Let $d \in \mathbb{R}^{+*}$. The inequality $p_n \leq dn^2$ holds for large enough n in $\mathbb{N}^*$ if and only if $P(dx^2) > x - 1$ for large enough x in $\mathbb{R}^+$.*
(5) *There exists $d \in \mathbb{R}^{+*}$ such that $p_n \leq dn^2$ for large enough n in $\mathbb{N}^*$ if and only if there exists $e \in \mathbb{R}^{+*}$ such that $P(x) \geq e\sqrt{x}$ for large enough x in $\mathbb{R}^+$.*

Proof. (1) If $P(x) \leq kQ(x)$ for $x \in \mathbb{R}^+$, then $kn = P(p_{kn}) \leq kQ(p_{kn})$, that is, $Q(p_{kn}) \geq n$, that is, $q_n \leq p_{kn}$, for $n \in \mathbb{N}^*$. Moreover, if we have $q_n = p_{kn}$ for some n, then taking $x = p_{kn} = q_n$, we get $P(x) = kn = kQ(x)$. Thus, if the inequality in the assumption is strict, then so it is in the conclusion. Similarly, if we have $q_n < p_{kn}$ for some n, then taking $q_n \leq x < p_{kn}$, we get $P(x) < kn \leq kQ(x)$. Thus if there is equality in the assumption, then so it is in the conclusion. Hence the results.

(2) If $x \geq p_k$, there is a unique $n \in \mathbb{N}^*$ such that $p_{kn} \leq x < p_{k(n+1)}$, and we have $P(x) < k(n+1) \leq k(Q(x) + 1)$, since $q_n \leq p_{kn} \leq x$, by the assumption. If $0 \leq x < p_k$, then $P(x) < k \leq kQ(x) + k$. Hence the result in all cases.

(3) By the assumption, $kn = kQ(q_n) \leq P(q_n) < kQ(q_n) + k = kn + k$, that is, $p_{kn} \leq q_n < p_{kn+k}$, for $n \in \mathbb{N}^*$.

(4) The first condition means that $P(dn^2) \geq n$ for large enough $n \in \mathbb{N}^*$, while the second one means that $P(dx^2) \geq [x]$ for large enough $x \in \mathbb{R}^+$. The first one implies the second upon taking $n = [x]$; and the converse implication is trivial.

(5) The first condition means that the sequence (p_n/n^2) is bounded, while the second one means that the function $x/P(x)^2$ is bounded for $x \geq p_1$. Now, for $p_n \leq x < p_{n+1}$, one has $P(x) = n$, so that $p_n/n^2 \leq x/P(x)^2 < p_{n+1}/n^2$. Hence the equivalence of the two conditions. $\square$

Theorem 3.12. *Let $f \in \mathcal{F}$ with* $\operatorname{Supp}(f) = P$. *Also, let A be a nonempty subset of* $\mathbb{N}$. *For every $k \in \mathbb{N}^*$, let P_k (resp., A_k) denote the kth ray of P (resp., A).*

(1) *If $P_k \in \mathcal{C}(\mathrm{ET})$ for all $k \in \mathbb{N}^*$, then $f \in \mathcal{C}(\mathrm{EET})$.*

(2) *When $\|f\| < \infty$, $f \in \mathcal{C}(\mathrm{EET})$ if and only if $P_k \in \mathcal{C}(\mathrm{ET})$ for all $k \in \mathbb{N}^*$.*

(3) *$f_A \in \mathcal{C}(\mathrm{EET})$ if and only if $A_k \in \mathcal{C}(\mathrm{ET})$ for all $k \in \mathbb{N}^*$.*

(4) *When $\|f\| < \infty$, $f \in \mathcal{C}(\mathrm{EET})$ if and only if $f_P \in \mathcal{C}(\mathrm{EET})$.*

Proof. (1) Let $g \in \mathcal{F}$ be such that $g \ll f$, with $Q = \operatorname{Supp}(g) = \{q_1 < q_2 < \cdots\}$, and let F and G be the mass functions of f and g, respectively, so that $F(x) \leq G(x)$ for $x \in \mathbb{R}^+$. There exists $c \in \mathbb{R}^{+*}$ such that $f \in \mathcal{F}_c$, and therefore $cP(x) \leq F(x) \leq G(x) \leq \|g\|Q(x)$ for $x \in \mathbb{R}^+$ by Proposition 3.3. Now, if $\|g\| = \infty$, then $s(g) = \infty$ by Corollary 3.4. Otherwise, let k be a positive integer $\geq \|g\|/c$. Then $P(x) \leq kQ(x)$ for $x \in \mathbb{R}^+$, and therefore $q_n \leq p_{kn}$ for $n \in \mathbb{N}^*$, that is, $Q \ll P_k$ by Lemma 3.11. Since $P_k \in \mathcal{C}(\mathrm{ET})$, it follows that $s(Q) = \infty$ and therefore $s(g) = \infty$ by Corollary 3.4. Thus $f \in \mathcal{C}(\mathrm{EET})$.

(2) Assume that $\|f\| < \infty$ and $f \in \mathcal{C}(\mathrm{EET})$. Let $k \in \mathbb{N}^*$ and let $Q = \{q_1 < q_2 < \cdots\}$ be an infinite subset of $\mathbb{N}$ such that $Q \ll P_k$. Then $P(x) < kQ(x) + k$ for $x \in \mathbb{R}^+$ by Lemma 3.11. Thus, by Proposition 3.3, the mass function F of f satisfies $F(x) \leq \|f\|P(x) \leq k\|f\|Q(x) + k\|f\|$ for $x \in \mathbb{R}^+$. Let $t = k\|f\|$ (in $\mathbb{R}^{+*}$) and $h = tf_Q + t$. Then $h \in \mathcal{F}$ and the mass function of h is given by $H(x) = tQ(x) + t$ (by Lemmas 2.2 and 2.4), so it satisfies $F(x) \leq H(x)$ for $x \in \mathbb{R}^+$. Hence $h \ll f$ and since $f \in \mathcal{C}(\mathrm{EET})$, therefore $s(h) = \infty$. But $s(h) = t^2 s(f_Q + 1) \leq t^2(s(f_Q) + s(1) + 2\|f_Q\|) = t^2(s(Q) + 3)$ by Lemmas 2.2 and 2.4. It follows that $s(Q) = \infty$. Thus $P_k \in \mathcal{C}(\mathrm{ET})$. This shows that if $f \in \mathcal{C}(\mathrm{EET})$ then $P_k \in \mathcal{C}(\mathrm{ET})$ for all k. The converse follows from (1).

Finally, (3) follows from (2), and (4) follows from (2) and (3). $\square$

Remark 3.13. The results in Theorem 3.12 raise the question of determining the infinite subsets P of $\mathbb{N}$ all of whose rays P_k lie in $\mathcal{C}(\mathrm{ET})$. In particular, one may ask whether if $P \in \mathcal{C}(\mathrm{ET})$, then $P_k \in \mathcal{C}(\mathrm{ET})$ for all $k \in \mathbb{N}^*$. A partial answer is provided in what follows.

Definition 3.14. Let $P = \{p_1 < p_2 < \cdots < p_n < \cdots\}$ be an infinite subset of $\mathbb{N}$. The *caliber* of P is $\operatorname{cal}(P) = \liminf_{n\to\infty}(p_n/n^2)$ in $\overline{\mathbb{R}}^+$. We say that P belongs to the *restricted class* $\mathcal{C}(\mathrm{RET})$ *of Erdös-Turán sets* if $\operatorname{cal}(P) = 0$.

Remark 3.15. In [6], we showed that $\mathcal{C}(\mathrm{RET})$ is a subset of $\mathcal{C}(\mathrm{ET})$ and that the conjecture (GET) is equivalent to the assertion that $\mathcal{C}(\mathrm{RET}) \subsetneq \mathcal{C}(\mathrm{ET})$.

Lemma 3.16. *Let $P = \{p_1 < p_2 < \cdots < p_n < \cdots\}$ be an infinite subset of $\mathbb{N}$. For any $k \in \mathbb{N}^*$, the caliber of the kth ray P_k of P is given by* $\operatorname{cal}(P_k) = k^2 \operatorname{cal}(P)$.

Proof. Given $k \in \mathbb{N}^*$, for every integer $0 \le i \le k$, let $c_i = \liminf_{n\to\infty}(p_{kn+i}/(kn+i)^2)$. Since, for all $m \in \mathbb{N}^*$, $\inf\{p_n/n^2 : n \ge km\}$ is the minimum of $\inf\{p_{kn+i}/(kn+i)^2 : n \ge m\}$, as $0 \le i \le k-1$, then $\mathrm{cal}(P) = \min\{c_i : 0 \le i < k\}$. Also $\mathrm{cal}(P_k) = \liminf_{n\to\infty}(p_{kn}/n^2) = k^2 c_0$. Moreover, for any $0 \le i \le k$ and $n \in \mathbb{N}^*$, we have $p_{kn} \le p_{kn+i} \le p_{k(n+1)}$. Dividing these inequalities by $(kn+i)^2$ and passing to the limit, we get $\liminf_{n\to\infty}(p_{kn}/(kn+i)^2) \le c_i \le \liminf_{n\to\infty}(p_{k(n+1)}/(kn+i)^2)$. But, since $(kn+i)^2 \sim (kn)^2$, asymptotically as $n \to \infty$, we have $\liminf_{n\to\infty}(p_{kn}/(kn+i)^2) = \liminf_{n\to\infty}(p_{kn}/(kn)^2) = c_0$, and similarly $\liminf_{n\to\infty}(p_{k(n+1)}/(kn+i)^2) = c_0$. Therefore $c_0 \le c_i \le c_0$ for all i. Hence $\mathrm{cal}(P) = \min\{c_i : 0 \le i < k\} = c_0$ and $\mathrm{cal}(P_k) = k^2 c_0 = k^2\,\mathrm{cal}(P)$. □

Corollary 3.17. *If $P \in \mathscr{C}(\mathrm{RET})$, then $P_k \in \mathscr{C}(\mathrm{RET})$ for all $k \in \mathbb{N}^*$.*

Remark 3.18. If there exist some $P \in \mathscr{C}(\mathrm{ET})$ and some $k \in \mathbb{N}^*$ such that $P_k \notin \mathscr{C}(\mathrm{ET})$ (answering a question in Remark 3.13), then $P \notin \mathscr{C}(\mathrm{RET})$, by Corollary 3.17, so that $\mathscr{C}(\mathrm{RET}) \subsetneq \mathscr{C}(\mathrm{ET})$, and therefore, as noted in Remark 3.15, the conjecture (GET) would be true.

4. The conjectures

Remark 4.1. We can restate (ET) and (GET) as follows.

(ET) If P is a basis of $\mathbb{N}$, then $s(P) = \infty$.

(GET) For any infinite subset $P = \{p_1 < p_2 < \cdots < p_n < \cdots\}$ of $\mathbb{N}$, if $p_n \le dn^2$ for some $d \in \mathbb{R}^{+*}$ and all $n \in \mathbb{N}^*$, then $s(P) = \infty$.

Moreover, since in the condition $p_n \le dn^2$ we can assume that $d \in \mathbb{N}^*$ (upon replacing d by any integer $\ge d$), then, in view of Definition 3.1, (GET) can be restated as follows.

(GET) For any $d \in \mathbb{N}^*$, the set $d\mathbb{S}$ lies in $\mathscr{C}(\mathrm{ET})$.

Definition 4.2. Let $f = \sum_{n=0}^{\infty} a_n X^n$. We call f a *supported basis* of $\mathbb{N}$ if f is a supported series such that $f^2 = \sum_{n=0}^{\infty} r(f,n)X^n$ has all its coefficients $r(f,n) > 0$; that is, if $f \in \mathscr{F}$ and $\mathrm{Supp}(f)$ is a basis of $\mathbb{N}$ by Proposition 3.3.

We consider the following analytic versions of the conjectures (ET) and (GET).

(AET) If f is a supported basis of $\mathbb{N}$, then $s(f) = \infty$.

(GAET) The series $f_{\mathbb{S}} = \sum_{n=1}^{\infty} X^{n^2}$ belongs to the class $\mathscr{C}(\mathrm{EET})$.

Theorem 4.3. (1) *The conjectures* (AET) *and* (ET) *are equivalent.*

(2) *The conjectures* (GAET) *and* (GET) *are also equivalent.*

Proof. (1) Assume first that (AET) holds. Let P be a basis of $\mathbb{N}$. Then $f_P = \sum_{n=0}^{\infty} \chi_P(n)X^n$ lies in $\mathscr{F}_1$ and $\mathrm{Supp}(f_P) = P$, so that f_P is a supported basis. Therefore, by the assumption, $s(f_P) = \infty$, that is, $s(P) = \infty$ by Lemma 2.4. Thus (ET) holds.

Conversely, assume that (ET) holds. Let f be a supported basis of $\mathbb{N}$. Then $P = \mathrm{Supp}(f)$ is a basis of $\mathbb{N}$, so that, by the assumption, $s(P) = \infty$. Moreover, $f \in \mathscr{F}_c$, for some $c \in \mathbb{R}^{+*}$. Therefore $s(f) \ge c^2 s(P)$ by Proposition 3.3. Hence $s(f) = \infty$. Thus (AET) holds.

(2) By Theorem 3.12, (GAET) holds if and only if $\mathbb{S}_k$, which is equal to $k^2\mathbb{S}$, lies in $\in \mathscr{C}(\mathrm{ET})$ for all $k \in \mathbb{N}^*$. On the other hand, by Remark 4.1, (GET) holds if and only if $d\mathbb{S}$ lies in $\mathscr{C}(\mathrm{ET})$ for all $d \in \mathbb{N}^*$. Thus (GET) trivially implies (GAET). But the latter also implies the former in view of the fact that if $P \in \mathscr{C}(\mathrm{ET})$ and if Q is an infinite subset of $\mathbb{N}$

such that $Q \ll P$, then $Q \in \mathscr{C}(\mathrm{ET})$, as can be easily seen from Definition 3.1. Therefore for any $d \in \mathbb{N}^*$, taking $k \in \mathbb{N}^*$ such that $k^2 \geq d$, if $k^2\mathbb{S} \in \mathscr{C}(\mathrm{ET})$, in virtue of (GAET), then $d\mathbb{S} \in \mathscr{C}(\mathrm{ET})$ since $d\mathbb{S} \ll k^2\mathbb{S}$. Thus (GAET) is equivalent to (GET). □

Remark 4.4. There is an asymptotic version (ETa) of (ET), which is equivalent to (ET) (cf. [7]). First, we note that a subset P of $\mathbb{N}$ is called an *asymptotic basis* of $\mathbb{N}$ if $r(P,n) > 0$ for all large enough n in $\mathbb{N}$. Then, we state the following.

(ETa) If P is an asymptotic basis of $\mathbb{N}$, then $s(P) = \infty$.

We can similarly state an asymptotic form of (AET), namely, (AETa).

(AETa) If f is an *asymptotic supported basis*, that is, if $f \in \mathscr{F}$ and $r(f,n) > 0$ for all large enough n, then $s(f) = \infty$.

By the same argument as in the proof of Theorem 4.3(1), we see that (AETa) is equivalent to (ETa), and since (ETa) is equivalent to (ET), then all four statements (ET), (ETa), (AET), and (AETa) are equivalent.

5. A version of the Erdös-Fuchs theorem

In this section, we present a version of the Erdös-Fuchs theorem for supported series, along the lines in Newman [12], trying to be as explicit and complete as possible in the proofs. Namely, we establish the following result.

Theorem 5.1. *Let $f = \sum_{n=0}^{\infty} a_n X^n$ be any supported series in $\mathbb{R}[[X]]$; and let*

$$f^2 = \sum_{n=0}^{\infty} r(f,n)X^n, \tag{5.1}$$

where $r(f,n) = \sum_{i+j=n} a_i a_j$ $(n \in \mathbb{N})$. For any $c \in \mathbb{R}^{+*}$, if $\sum_{k=0}^{n}(r(f,k) - c) = O(n^t)$ for some $t \in \mathbb{R}^+$, then $t \geq 1/4$.

Remark 5.2. The version of the Erdös-Fuchs theorem given by Newman [12] reads: if A is a subset of $\mathbb{N}$, and if $c \in \mathbb{R}^{+*}$ is such that $\sum_{k=0}^{n}(r(A,k) - c) = O(n^t)$ for some $t \in \mathbb{R}^+$, then $t \geq 1/4$. Theorem 5.1 generalizes this result by extending it to all sets (or sequences), $A = \{a_n : n \in \mathbb{N}\}$, of nonnegative real numbers whose nonzero elements are bounded below by a positive constant. This is done by introducing and studying the corresponding formal power series $f = \sum_{n=0}^{\infty} a_n X^n$, having such sequences as coefficients, that we here call the supported series. The point of this generalization is that such properties, as the Erdös-Fuchs theorem, are not exclusively characteristic of sequences of natural numbers, but belong to a much broader class of sequences of real numbers. It is to be further noted that the version of Newman that we here extend is slightly weaker than the original one by Erdös and Fuchs [3], which asserts that the relation $\sum_{k=0}^{n}(r(A,k) - c) = o(n^{1/4}\log^{-1/2} n)$ is impossible. However, the truly far-reaching generalization of the latter result is the one presented by Montgomery and Vaughan [11], credited by them to an unpublished manuscript of Jurkat and described as first appearing in the Ph.D. thesis of Hayashi [8], namely: for any subset A of $\mathbb{N}$ and any $c \in \mathbb{R}^{+*}$, the relation $\sum_{k=0}^{n}(r(A,k) - c) = o(n^{1/4})$ is impossible. Its extension to our context states that for any supported series $f = \sum_{n=0}^{\infty} a_n X^n$ in $\mathbb{R}[[X]]$, with $f^2 = \sum_{n=0}^{\infty} r(f,n)X^n$, and any $c \in \mathbb{R}^{+*}$, the relation $\sum_{k=0}^{n}(r(f,k) - c) = o(n^{1/4})$ is impossible. This is a natural and more difficult generalization, for another occasion.

Clearly, if Theorem 5.1 holds for some $f \in \mathcal{F}$ and all $c \in \mathbb{R}^{+*}$, then it also holds for all df, with $d \in \mathbb{R}^{+*}$. Thus, it is enough to establish it for $f \in \mathcal{F}_1$, that is, we may assume that $f = \sum_{n=0}^{\infty} a_n X^n$ satisfies the condition $a_n = 0$ or $a_n \geq 1$ for all $n \in \mathbb{N}$. Throughout this section, we fix such a series f as well as a constant $c \in \mathbb{R}^{+*}$. We set $A_n = \sum_{k=0}^{n}(r(f,k) - c)$ for all $n \in \mathbb{N}$, $h = \sum_{n=0}^{\infty} A_n X^n$, and $w = \sum_{n=0}^{\infty} A_n^2 X^n$. We also introduce the series $u_s(X) = \sum_{n=0}^{\infty} n^s X^n$ for $s \in \mathbb{R}^{+*}$, whose radius of convergence in $\mathbb{C}$ is obviously 1, and we use the polynomials $p_m(X) = \sum_{k=0}^{m-1} X^k$ for $m \in \mathbb{N}^*$. Several technical results will be needed for the proof of Theorem 5.1. Some are unconditional general properties, while others require the hypothesis made in the theorem, namely, for a given $t \in \mathbb{R}^+$,

(H_t) $A_n = O(n^t)$.

We start with a lemma listing two identities and an inequality, which are the analogues of formulas (1), (2), and (3) in [12], with similar proofs, but in a more general setting.

Lemma 5.3. *The following properties hold:*

(1) $f(X)^2 = (1-X)h(X) + c/(1-X)$, *in* $\mathbb{R}[[X]]$;

(2) $f(X)^2 p_m(X)^2 = c p_m(X)^2/(1-X) + (1-X^m)h(X)p_m(X)$, *in* $\mathbb{R}[[X]]$;

(3) *Under the hypothesis* (H_t), *the radii of convergence of* f, f^2, h, *and* w *are* ≥ 1, *and* $|f(z)p_m(z)|^2 \leq cm^2/|1-z| + 2|h(z)p_m(z)|$ *for all* $z \in \mathbb{C}$ *such that* $|z| < 1$.

Lemma 5.4. *For any* $s \in \mathbb{R}^{+*}$, $u_s(r) = O((1-r)^{-s-1})$ *for real* $0 < r < 1$.

Proof. For $0 < r < 1$, we have the binomial series expansion $(1-r)^{-s-1} = \sum_{n=0}^{\infty} b_n r^n$, with $b_n = (s+1)(s+2)\cdots(s+n)/n!$ for all $n \in \mathbb{N}$. By a classical formula for the Γ-function [2], $\lim_{n\to\infty}(n! \cdot n^s/s(s+1)\cdots(s+n)) = \Gamma(s)$. Therefore the sequence (n^s/b_n) is convergent to $s\Gamma(s)$ and is thus bounded; that is, there exists a constant $C > 0$ such that $n^s \leq Cb_n$ for all $n \in \mathbb{N}$. Hence, $u_s(r) = \sum_{n=0}^{\infty} n^s r^n \leq C\sum_{n=0}^{\infty} b_n r^n = C(1-r)^{-s-1}$, for $0 < r < 1$. □

As in [12], we integrate in the complex plane, over a circle $C_r = \{z \in \mathbb{C} : |z| = r\}$ with $0 < r < 1$, relative to the measure $\mu = |dz|/2\pi r$. Thus, for a complex function $v(z)$, integrable in the open unit disk, we set $\int_{C_r} v(z)d\mu = (1/2\pi)\int_0^{2\pi} |v(re^{it})|dt$. The next lemma is a sharpening of formula (6) in [12]. Similarly, the remaining formulas in [12] have been adapted or modified to be used in Lemmas 5.7 and 5.8, and in the proof of Theorem 5.1.

Lemma 5.5. *For any real* $0 < r < 1$, $\int_{C_r}(1/(1-z))d\mu \leq -(1/r^2)\log(1-r^2)$.

Proof. We have $1/(1-z) = q(z)^2$, where $q(z) = (1-z)^{-1/2}$ has the binomial series expansion $q(z) = \sum_{n=0}^{\infty}\binom{-1/2}{n}(-z)^n = \sum_{n=0}^{\infty} b_n z^n$ for $|z| < 1$, with $b_n = \prod_{k=1}^{n}(2k-1)/2k$ for all $n \in \mathbb{N}$. Hence $\int_{C_r}(1/(1-z))d\mu = (1/2\pi)\int_0^{2\pi}|q(re^{it})|^2 dt = \sum_{n=0}^{\infty} b_n^2 r^{2n}$ by Parseval's identity [2]. Moreover, by a simple induction, we get $b_n^2 \leq 1/(n+1)$ for all $n \in \mathbb{N}$. Hence $\int_{C_r}(1/(1-z))d\mu \leq \sum_{n=0}^{\infty}(r^{2n}/(n+1)) = (1/r^2)\sum_{n=1}^{\infty}(r^{2n}/n) = -(1/r^2)\log(1-r^2)$. □

Lemma 5.6. *Let* m *be an integer* ≥ 2. *Under the hypothesis* (H_t), $|f(z)p_m(z)|^2 \leq cm^2/|1-z| + 2|h(z)p_m(z)|$ *for all* $z \in \mathbb{C}$ *such that* $|z| < 1$.

Proof. (H_t) secures the convergence of f and h for $|z| < 1$, and then, by Lemma 5.3(2), we have $|f(z)p_m(z)|^2 = |cp_m(z)^2/(1-z) + (1-z^m)h(z)p_m(z)| \leq c|p_m(z)|^2/|1-z| + |1-z^m| \cdot |h(z)p_m(z)|$. Moreover, $|p_m(z)| \leq \sum_{k=0}^{m-1}|z|^k < m$ and $|1-z^m| \leq 1 + |z|^m < 2$ for $|z| < 1$. Hence the inequality. □

LEMMA 5.7. *Under the hypothesis* (H_t), *for any integer* $m \geq 2$ *and any real* $0 < r < 1$, $f(r^2)p_m(r^2) \leq -(cm^2/r^2)\log(1-r^2) + 2\sqrt{w(r^2)p_m(r^2)}$.

Proof. In view of Lemma 5.6,

$$\int_{C_r} \left(f(z)p_m(z)\right)^2 d\mu \leq cm^2 \int_{C_r} \frac{1}{1-z} d\mu + 2\int_{C_r} h(z)p_m(z)d\mu. \tag{5.2}$$

Clearly, $fp_m = \sum_{n=0}^{\infty} c_n X^n$, where the coefficients c_n, being sums of coefficients a_k of f, satisfy likewise the condition $c_n = 0$ or $c_n \geq 1$, so that $c_n^2 \geq c_n$ for all $n \in \mathbb{N}$. Now, by Parseval's identity, $\int_{C_r} (f(z)p_m(z))^2 d\mu = (1/2\pi)\int_0^{2\pi} |(fp_m)(re^{it})|^2 dt = \sum_{n=0}^{\infty} c_n^2 r^{2n} \geq \sum_{n=0}^{\infty} c_n r^{2n} = f(r^2)p_m(r^2)$. Also, by the Cauchy-Schwarz inequality [2] applied to the real integrals over $[0,2\pi]$, we get $(\int_{C_r} h(z)p_m(z)d\mu)^2 \leq \int_{C_r} h(z)^2 d\mu \cdot \int_{C_r} p_m(z)^2 d\mu$. Moreover, by Parseval's identity, $\int_{C_r} h(z)^2 d\mu = \sum_{n=0}^{\infty} A_n^2 r^{2n} = w(r^2)$ and $\int_{C_r} p_m(z)^2 d\mu = \sum_{n=0}^{m-1} r^{2n} = p_m(r^2)$. Therefore $\int_{C_r} h(z)p_m(z)d\mu \leq \sqrt{w(r^2)p_m(r^2)}$. Finally, by Lemma 5.5, $\int_{C_r}(1/(1-z))d\mu \leq -(1/r^2)\log(1-r^2)$. Putting together all these inequalities yields the desired result. □

LEMMA 5.8. *Under the hypothesis* (H_t), *the following properties hold.*

(1) *There exists* $b \in \mathbb{R}^{+*}$ *such that* $w(r^2) \leq b/(1-r^2)^{2t+1}$ *for all reals* $0 < r < 1$.

(2) *If* $t < 1$, *then there exist real numbers* $d > 0$ *and* $0 < r_0 < 1$ *such that* $f(r^2) \geq d/\sqrt{1-r^2}$ *for all reals* $r_0 < r < 1$.

(3) *If* $t < 1$, *then there exist real numbers* $b, d > 0$ *and* $0 < r_0 < 1$ *such that* $dmr^{2m}/\sqrt{1-r^2} \leq -(cm^2\log(1-r^2)/r^2) + 2\sqrt{bm}/(1-r^2)^{t+1/2}$ *for all integers* $m \geq 2$ *and all reals* $r_0 < r < 1$.

Proof. By (H_t), there exists a real constant $C > 0$ such that $|A_n| \leq Cn^t$ for all $n \in \mathbb{N}^*$.

(1) For $0 < r < 1$, we have $w(r) = \sum_{n=0}^{\infty} A_n^2 r^n \leq A_0^2 + C^2 \sum_{n=0}^{\infty} n^{2t} r^n = A_0^2 + C^2 u_{2t}(r)$, so that $w(r) = O(u_{2t}(r))$. But, by Lemma 5.4, $u_{2t}(r) = O((1-r)^{-2t-1})$. So $w(r^2) = O((1-r^2)^{-2t-1})$ for $0 < r < 1$, which gives the desired result.

(2) By Lemma 5.3, $(1-r)f(r)^2 = c + (1-r)^2 h(r)$. Now, for $0 < r < 1$, we have $|h(r)| \leq \sum_{n=0}^{\infty} |A_n| r^n \leq |A_0| + C\sum_{n=0}^{\infty} n^t r^n = A_0 + Cu_t(r)$, so that $h(r) = O(u_t(r))$. Thus, by Lemma 5.4, $h(r) = O((1-r)^{-t-1})$, and then $(1-r)^2 h(r) = O((1-r)^{1-t})$. Since $t < 1$, it follows that $\lim_{r\to 1^-}(1-r)^2 h(r) = 0$. Consequently, $\lim_{r\to 1^-}\sqrt{1-r^2}f(r^2) = \sqrt{c} > 0$, so that $\sqrt{1-r^2}f(r^2)$ is bounded below in some left neighborhood of 1. Hence the result.

(3) By Lemma 5.7 and (1), (2) above, we have $dp_m(r^2)/\sqrt{1-r^2} \leq f(r^2)p_m(r^2) \leq -(cm^2/r^2)\log(1-r^2) + 2\sqrt{w(r^2)p_m(r^2)} \leq -(cm^2/r^2)\log(1-r^2) + 2\sqrt{bp_m(r^2)}/(1-r^2)^{t+1/2}$ for $r_0 < r < 1$. Moreover, $mr^{2m} < p_m(r^2) = \sum_{k=0}^{m-1} r^{2k} < m$ for $0 < r < 1$. The desired inequality follows immediately. □

Proof of Theorem 5.1. We proceed by contradiction. Assume that the hypothesis (H_t) holds with $0 \leq t < 1/4$. Then there exists a real number $2 < p < 1/2t$. For every integer $m \geq 2$, let $r_m = (1-1/m^p)^{1/2}$, so that $0 < r_m < 1$ and $r_m \to 1^-$ as $m \to \infty$. Thus, given $1 < r_0 < 1$ satisfying Lemma 5.8(3), there exists an integer $m_0 \geq 2$ such that for all $m \geq m_0$, we have $r_0 < r_m < 1$, and therefore $dmr_m^{2m}/\sqrt{1-r_m^2} \leq -(cm^2\log(1-r_m^2)/r_m^2) + 2\sqrt{bm}/(1-r_m^2)^{t+1/2}$. But $\sqrt{1-r_m^2} = m^{-p/2}$ and $\log(1-r_m^2) = -p\log m$, so that $dm^{1+p/2}r_m^{2m} \leq cpm^2\log m/r_m^2 + 2\sqrt{b}m^{pt+p/2+1/2}$, that is, $dr_m^{2m} \leq cpm^{1-p/2}\log m/r_m^2 + 2\sqrt{b}m^{pt-1/2}$. Moreover, by a simple induction, we get $(1-x)^m \geq 1-mx$ for $0<x<1$ and all $m \in \mathbb{N}^*$, so that $r_m^{2m} = (1-m^{-p})^m \geq 1-m^{1-p} > 1-m^{-1} \geq 1/2$ since $p>2$ and $m \geq 2$. Thus $d/2 \leq cpm^{1-p/2}\log m/r_m^2 + 2\sqrt{b}m^{pt-1/2}$, for all $m \geq m_0$. But since $2<p<1/2t$ implies that $1-p/2<0$ and $pt-1/2<0$, and since $r_m^2 \to 1$ as $m \to \infty$, then both $cpm^{1-p/2}\log m/r_m^2$ and $2\sqrt{b}m^{pt-1/2}$ approach 0 as $m \to \infty$, which contradicts the latter inequality for large enough m. Thus the assumption that $t<1/4$ leads to a contradiction, and therefore we must have $t \geq 1/4$. $\square$

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