RATE OF CONVERGENCE OF BETA OPERATORS OF SECOND KIND FOR FUNCTIONS WITH DERIVATIVES OF BOUNDED VARIATION

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We study the approximation properties of beta operators of second kind. We obtain the rate of convergence of these operators for absolutely continuous functions having a derivative equivalent to a function of bounded variation.

1. Introduction

For Lebesgue integrable functions f on the interval $I = (0,\infty)$, beta operators L_n of second kind are given by

$$\left(L_n f\right)(x) = \frac{1}{B(nx, n+1)} \int_0^\infty \frac{t^{nx-1}}{(1+t)^{nx+n+1}} f(t)\,dt. \tag{1.1}$$

Obviously the operators L_n are positive linear operators on the space of locally integrable functions on I of polynomial growth as $t \to \infty$, provided that n is sufficiently large.

In 1995, Stancu [10] gave a derivation of these operators and investigated their approximation properties. We mention that similar operators arise in the work by Adell et al. [3, 4] by taking the probability density of the inverse beta distribution with parameters nx and n.

Recently, Abel [1] derived the complete asymptotic expansion for the sequence of operators (1.1). In [2], Abel and Gupta studied the rate of convergence for functions of bounded variation.

In the present paper, the study of operators (1.1) will be continued. We estimate their rate of convergence by the decomposition technique for absolutely continuous functions f of polynomial growth as $t \to +\infty$, having a derivative f' coinciding a.e. with a function which is of bounded variation on each finite subinterval of I.

Several researchers have studied the rate of approximation for functions with derivatives of bounded variation. We mention the work of Bojanić and Chêng (see [5, 6]) who estimated the rate of convergence with derivatives of bounded variation for Bernstein and Hermite-Fejer polynomials by using different methods. Further papers on the subject were written by Bojanić and Khan [7] and by Pych-Taberska [9]. See also the very recent paper by Gupta et al. [8] on general class of summation-integral type operators.

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For the sake of convenient notation in the proofs we rewrite operators (1.1) as

$$(L_n f)(x) = \int_0^\infty K_n(x,t) f(t)\,dt, \tag{1.2}$$

where the kernel function K_n is given by

$$K_n(x,t) = \frac{1}{B(nx, n+1)} \frac{t^{nx-1}}{(1+t)^{nx+n+1}}. \tag{1.3}$$

Moreover, we put

$$\lambda_n(x,y) = \int_0^y K_n(x,t)\,dt \quad (y \geq 0). \tag{1.4}$$

Note that $0 \leq \lambda_n(x,y) \leq 1$ $(y \geq 0)$.

Our main result is contained in Section 3, while the next section contains some auxiliary results.

2. Auxiliary results

For fixed $x \in I$, define the function ψ_x, by $\psi_x(t) = t - x$. The first central moments for the operators L_n are given by

$$(L_n \psi_x^0)(x) = 1, \qquad (L_n \psi_x^1)(x) = 0, \qquad (L_n \psi_x^2)(x) = \frac{x(1+x)}{n-1} \tag{2.1}$$

(see [1, Proposition 2]). In general, we have the following result.

Lemma 2.1 [1, Proposition 2]. *Let $x \in I$ be fixed. For $r = 0,1,2,\ldots$ and $n \in \mathbb{N}$, the central moments for the operators L_n satisfy*

$$(L_n \psi_x^r)(x) = O\left(n^{-\lfloor (r+1)/2 \rfloor}\right) \quad (n \longrightarrow \infty). \tag{2.2}$$

In view of (1.2), an application of the Schwarz inequality, for $r = 0,1,2,\ldots$, yields

$$(L_n |\psi_x^r|)(x) \leq \sqrt{(L_n \psi_x^{2r})(x)} = O\left(n^{-r/2}\right) \quad (n \longrightarrow \infty). \tag{2.3}$$

In particular, by (2.1) we have

$$(L_n |\psi_x|)(x) \leq \sqrt{\frac{x(1+x)}{(n-1)}}. \tag{2.4}$$

Lemma 2.2 [2, Proposition 2]. *Let $x \in I$ be fixed and $K_n(x,t)$ be defined by (1.3). Then, for $n \geq 2$,*

$$\begin{aligned} \lambda_n(x,y) &= \int_0^y K_n(x,t)\,dt \leq \frac{x(1+x)}{(n-1)(x-y)^2} \quad (0 \leq y < x), \\ 1-\lambda_n(x,z) &= \int_z^\infty K_n(x,t)\,dt \leq \frac{x(1+x)}{(n-1)(z-x)^2} \quad (x < z < \infty). \end{aligned} \tag{2.5}$$

3. The main result

Throughout this paper, for each function g of bounded variation on I and fixed $x \in I$, we define the auxiliary function g_x, which is given by

$$g_x(t) = \begin{cases} g(t) - g(x-) & (0 \leq t < x), \\ 0 & (t = x), \\ g(t) - g(x+) & (x < t < \infty). \end{cases} \tag{3.1}$$

Furthermore, $\bigvee_a^b(g)$ denotes the total variation of g on $[a,b]$. For $r \geq 0$, let $DB_r(I)$ be the class of all absolutely continuous functions f defined on I,

(i) having on I a derivative f' coinciding a.e. with a function which is of bounded variation on each finite subinterval of I,

(ii) satisfying $f(t) = O(t^r)$ as $t \to +\infty$.

Note that all functions $f \in DB_r(I)$ possess, for each $a > 0$, a representation

$$f(x) = f(a) + \int_a^x \psi(t)\,dt \quad (x \geq a) \tag{3.2}$$

with a function ψ of bounded variation on each finite subinterval of I.

The following theorem is our main result.

Theorem 3.1. *Let $r \in \mathbb{N}$, $x \in I$, and $f \in DB_r(I)$. Then there holds*

$$\begin{aligned} |(L_nf)(x) - f(x)| \leq{}& \frac{1}{2}\sqrt{\frac{x(1+x)}{n-1}}\,|f'(x+) - f'(x-)| + \frac{x}{\sqrt{n}} V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((f')_x) \\ &+ \frac{1+x}{n-1}\left(\sum_{k=1}^{\lfloor\sqrt{n}\rfloor} V_{x-x/k}^{x+x/k}((f')_x) + x^{-1}|f(2x) - f(x)| + 2|f'(x+)|\right) \\ &+ \frac{c_{r,x} \cdot M_{r,x}(f)}{n^{r/2}}, \end{aligned} \tag{3.3}$$

where the constants $c_{r,x}$ and $M_{r,x}(f)$ are given by

$$\begin{aligned} c_{r,x} &= \sup_{n \in \mathbb{N}} \sqrt{n^r (L_n\psi_x^{2r})(x)}, \\ M_{r,x}(f) &= 2^r \sup_{t \geq 2x} t^{-r}|f(t) - f(x)|. \end{aligned} \tag{3.4}$$

Remark 3.2. Note that, for each $f \in DB_r(I)$, we have $M_{r,x}(f) < +\infty$. Furthermore, Lemma 2.1 implies that $c_{r,x} < +\infty$.

Proof. For $x \in I$, we have

$$(L_n f)(x) - f(x) = \int_0^\infty K_n(x,t)(f(t) - f(x))dt = \int_0^\infty K_n(x,t)\int_x^t f'(u)du\,dt. \tag{3.5}$$

Now we take advantage of the identity

$$\begin{aligned} f'(u) &= (f')_x(u) + \frac{1}{2}(f'(x+) + f'(x-)) + \frac{1}{2}(f'(x+) - f'(x-))\operatorname{sign}(u-x) \\ &\quad + \left(f'(x) - \frac{1}{2}(f'(x+) + f'(x-))\right)\chi_x(u), \end{aligned} \tag{3.6}$$

where $\chi_x(u) = 1$ $(u = x)$ and $\chi_x(u) = 0$ $(u \neq x)$. Obviously, we have

$$\int_0^\infty K_n(x,t)\int_x^t \left(f'(x) - \frac{1}{2}(f'(x+) + f'(x-))\right)\chi_x(u)du\,dt = 0. \tag{3.7}$$

Furthermore, by (2.1) and (2.4), respectively, we have

$$\begin{aligned} \int_0^\infty K_n(x,t)\int_x^t \frac{1}{2}(f'(x+) + f'(x-))du\,dt &= \frac{1}{2}(f'(x+) + f'(x-))\int_0^\infty K_n(x,t)(t-x)dt = 0, \\ \left|\int_0^\infty K_n(x,t)\int_x^t \frac{1}{2}(f'(x+) - f'(x-))\operatorname{sign}(u-x)du\,dt\right| &\leq \frac{1}{2}\left|f'(x+) - f'(x-)\right|\int_0^\infty K_n(x,t)|t-x|dt \\ &\leq \frac{1}{2}\sqrt{\frac{x(1+x)}{n-1}}\left|f'(x+) - f'(x-)\right|. \end{aligned} \tag{3.8}$$

Collecting the latter relations, we obtain the estimate

$$\left|(L_n f)(x) - f(x)\right| \leq \left|A_n(f,x) + B_n(f,x) + C_n(f,x)\right| + \frac{1}{2}\sqrt{\frac{x(1+x)}{n-1}}\left|f'(x+) - f'(x-)\right| \tag{3.9}$$

with the denotations

$$\begin{aligned} A_n(f,x) &= \int_0^x K_n(x,t)\int_x^t (f')_x(u)du\,dt, \\ B_n(f,x) &= \int_x^{2x} K_n(x,t)\int_x^t (f')_x(u)du\,dt, \\ C_n(f,x) &= \int_{2x}^\infty K_n(x,t)\int_x^t (f')_x(u)du\,dt. \end{aligned} \tag{3.10}$$

In order to complete the proof, it is sufficient to estimate the terms $A_n(f,x)$, $B_n(f,x)$, and $C_n(f,x)$.

Using integration by parts, and application of Lemma 2.2 yields

$$\begin{aligned}
|A_n(f,x)| &= \left|\int_0^x \int_x^t (f^{'})_x(u)du\, d_t\lambda_n(x,t)\right| = \left|\int_0^x \lambda_n(x,t)(f^{'})_x(t)dt\right| \\
&\leq \left(\int_0^{x-x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^x\right) |\lambda_n(x,t)|\, V_t^x((f^{'})_x)dt \\
&\leq \frac{x(1+x)}{n-1}\int_0^{x-x/\sqrt{n}} (x-t)^{-2} V_t^x((f^{'})_x)dt + \frac{x}{\sqrt{n}} V_{x-x/\sqrt{n}}^x((f^{'})_x).
\end{aligned} \tag{3.11}$$

By the substitution of $u = x/(x-t)$, we obtain

$$\begin{aligned}
\int_0^{x-x/\sqrt{n}} (x-t)^{-2} V_t^x((f^{'})_x)dt &= x^{-1}\int_1^{\sqrt{n}} V_{x-x/u}^x((f^{'})_x)du \\
&\leq x^{-1}\sum_{k=1}^{\lfloor\sqrt{n}\rfloor}\int_k^{k+1} V_{x-x/u}^x((f^{'})_x)du \\
&\leq x^{-1}\sum_{k=1}^{\lfloor\sqrt{n}\rfloor} V_{x-x/k}^x((f^{'})_x).
\end{aligned} \tag{3.12}$$

Thus we have

$$|A_n(f,x)| \leq \frac{1+x}{n-1}\sum_{k=1}^{\lfloor\sqrt{n}\rfloor} V_{x-x/k}^x((f^{'})_x) + \frac{x}{\sqrt{n}} V_{x-x/\sqrt{n}}^x((f^{'})_x). \tag{3.13}$$

Furthermore, we have

$$\begin{aligned}
|B_n(f,x)| &= \left| -\int_x^{2x}\int_x^t (f^{'})_x(u)du\, d_t(1-\lambda_n(x,t))\right| \\
&\leq \left|\int_x^{2x}(f^{'})_x(u)du\right| |1-\lambda_n(x,2x)| + \int_x^{2x} |(f^{'})_x(t)|\,|1-\lambda_n(x,t)|dt \\
&\leq \frac{1+x}{(n-1)x}|f(2x)-f(x)-xf^{'}(x+)| + \int_x^{x+x/\sqrt{n}} V_x^t((f^{'})_x)dt \\
&\quad + \frac{x(1+x)}{n-1}\int_{x+x/\sqrt{n}}^{2x}(t-x)^{-2}V_x^t((f^{'})_x)dt,
\end{aligned} \tag{3.14}$$

where we applied Lemma 2.2. By the substitution of $u = x/(t-x)$, we obtain

$$\begin{aligned}
\int_{x+x/\sqrt{n}}^{2x}(t-x)^{-2}V_x^t((f^{'})_x)dt &= x^{-1}\int_1^{\sqrt{n}} V_x^{x+x/u}((f^{'})_x)du \\
&\leq x^{-1}\sum_{k=1}^{\lfloor\sqrt{n}\rfloor}\int_k^{k+1} V_x^{x+x/u}((f^{'})_x)du \\
&\leq x^{-1}\sum_{k=1}^{\lfloor\sqrt{n}\rfloor} V_x^{x+x/k}((f^{'})_x).
\end{aligned} \tag{3.15}$$

Thus we have

$$\begin{aligned}|B_n(f,x)| \le{}& \frac{1+x}{(n-1)x}|f(2x)-f(x)-xf'(x+)| + \frac{1+x}{n-1}\sum_{k=1}^{\lfloor\sqrt{n}\rfloor} V_x^{x+x/k}((f')_x)\\ &+\frac{x}{\sqrt{n}}V_x^{x+x/\sqrt{n}}((f')_x).\end{aligned} \tag{3.16}$$

Finally, we have

$$\begin{aligned}|C_n(f,x)| &= \left|\int_{2x}^{\infty} K_n(x,t)(f(t)-f(x)-(t-x)f'(x+))\,dt\right|\\ &\le 2^{-r}M_{r,x}(f)\int_{2x}^{\infty}K_n(x,t)t^r\,dt + |f'(x+)|\int_{2x}^{\infty}K_n(x,t)|t-x|\,dt.\end{aligned} \tag{3.17}$$

Using the obvious inequalities $t \le 2(t-x)$ and $x \le t-x$ for $t \ge 2x$, we obtain

$$\begin{aligned}|C_n(f,x)| &\le M_{r,x}(f)\int_{2x}^{\infty}K_n(x,t)(t-x)^r\,dt + x^{-1}|f'(x+)|\int_{2x}^{\infty}K_n(x,t)(t-x)^2\,dt\\ &\le M_{r,x}(f)\cdot(L_n|\psi_x^r|)(x)+x^{-1}|f'(x+)|(L_n\psi_x^2)(x).\end{aligned} \tag{3.18}$$

By (2.3), we conclude that

$$|C_n(f,x)| = M_{r,x}(f)\cdot c_{r,x}n^{-r/2}+\frac{1+x}{n-1}|f'(x+)|. \tag{3.19}$$

Combining the estimates (3.13)–(3.19) with (3.9), we get the desired result. This completes the proof of the theorem. □

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