LOCAL EXTREMA IN RANDOM TREES

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The number of local maxima (resp., local minima) in a tree $T \in \mathcal{T}_n$ rooted at $r \in [n]$ is denoted by $M_r(T)$ (resp., by $m_r(T)$). We find exact formulas as rational functions of n for the expectation and variance of $M_1(T)$ and $m_n(T)$ when $T \in \mathcal{T}_n$ is chosen randomly according to a uniform distribution. As a consequence, a.a.s. $M_1(T)$ and $m_n(T)$ belong to a relatively small interval when $T \in \mathcal{T}_n$.

1. Introduction

The extension of permutation statistics to labelled trees is the subject of a number of articles. Generating functions for the number of labelled trees of several types according to the number of ascents and descents are given in [4]. A functional equation satisfied by the generating function for the number of labelled trees according to the number of descents and leaves is given in [5]. Central and local limit theorems for the number of ascents or of descents in uniformly random labelled trees are given in [1]. A functional equation satisfied by the generating function for the number of labelled trees according to the number of inversions is given in [7]. Related results are contained in [6]. A formula for the expected number of inversions of a uniformly random labelled tree is given in [9]. Formulas for the expectation and variance of the number of inversions of a uniformly random labelled tree are given in [2].

Local extrema (in the literature as local maxima and local minima; peaks and troughs; collectively turning points; related to phases) in permutations have a long history; see [10] and references there in. The examination of local maxima (equivalently, local minima) in permutations is more recent. A recurrence relation and a generating function for the number of permutations according to the number of local maxima are given in [10]. A central limit theorem for the number of local maxima in a uniformly random permutation also is given in [10]. In this note, we extend local extrema in permutations to labelled trees and examine local maxima (equivalently, local minima) in uniformly random labelled trees.

For $n \geq 2$, let $\mathcal{T}_n$ denote the set of trees with vertex set $[n] := \{1,\dots,n\}$. When $T_1, T_2 \in \mathcal{T}_n$, $T_1 = T_2$ if and only if T_1 and T_2 have the same edge set. Let $T \in \mathcal{T}_n$, $r \in [n]$, and

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Table 1.1

	$T_1(n,k) = t_n(n,k)$	k					
		0	1	2	3	4	$\cdots$
n	2	1	0	0	0	0	$\cdots$
	3	2	1	0	0	0	$\cdots$
	4	6	9	1	0	0	$\cdots$
	5	24	73	27	1	0	$\cdots$

distinct $i,j,k \in [n]$. We say T rooted at r has a *local maximum at path* ijk if and only if $j > i,k$ and the path in T from r to k contains the path ijk. Similarly, T rooted at r has a *local minimum at path* ijk if and only if $j < i,k$ and the path in T from r to k contains the path ijk. Let $M_r(T) = M_{r,n}(T)$ (resp., $m_r(T) = m_{r,n}(T)$) denote the number of local maxima (resp., local minima) of $T \in \mathcal{T}_n$ rooted at r. Then $M_r(T), m_r(T) \in \{0,\dots,n-2\}$. Let $T_r(n,k)$ (resp., $t_r(n,k)$) denote the number of trees in $\mathcal{T}_n$ rooted at r with precisely k local maxima (resp., k local minima). Then $T_r(n,k) = t_r(n,k) = 0$ for $k \notin \{0,\dots,n-2\}$ and $\sum_{k=0}^{n-2} T_r(n,k) = \sum_{k=0}^{n-2} t_r(n,k) = n^{n-2}$. As with the other statistics extended to labelled trees, roots $r = 1,n$ are appropriate. The values of $T_1(n,k) = t_n(n,k)$ (see Lemma 2.1) are given in Table 1.1 for $2 \le n \le 5$ and $0 \le k \le n-2$.

We work in the probability space Ω_n consisting of all trees in $\mathcal{T}_n$, where each tree is chosen randomly according to a uniform distribution. Hence, $\Pr(T) = 1/n^{n-2}$ for $T \in \mathcal{T}_n$. A property Q of trees in $\{\mathcal{T}_n\}$ holds *asymptotically almost surely* (a.a.s.) on $\{\Omega_n\}$ if and only if $\lim_{n\to\infty} \Pr(T \in \mathcal{T}_n : T \text{ has property } Q) \to 1$ as $n \to \infty$.

The parameters M_r and m_r are then random variables on Ω_n whose exact expectations $E(M_r)$, $E(m_r)$ and variances $\sigma^2(M_r)$, $\sigma^2(m_r)$ we find as rational functions of n ($r = 1,n$). From Theorem 2.5,

$$E(M_1) = E(m_n) = \frac{2n^3 - 3n^2 - 5n + 6}{6n^2},$$
$$\sigma^2(M_1) = \sigma^2(m_n) = \frac{7n^5 - 20n^4 + 75n^3 - 40n^2 - 322n + 300}{60n^4}. \tag{1.1}$$

As a consequence, a.a.s. on $\{\Omega_n\}$,

$$E(M_1) - \omega(n)\sigma(M_1) < M_1, m_n < E(M_1) + \omega(n)\sigma(M_1), \tag{1.2}$$

where $\omega(n) \to \infty$ arbitrarily slowly as $n \to \infty$. (See Corollary 2.6 for this and further results.)

We mention that should $(M_1 - E(M_1))/\sigma(M_1) \xrightarrow{d} N(0,1)$, we could only conclude the above inequality for M_1, m_n a.a.s. on $\{\Omega_n\}$. Of course, asymptotic normality of M_1, m_n gives more information about the distribution of M_1, m_n than their a.a.s. properties.

Let $\mathbb{N}$ denote the nonnegative integers and let $\mathbb{R}$ denote the real numbers. The expectation of a random variable X is denoted by $E(X)$ and its variance by $\text{Var}(X)$. We refer the reader to Moon [8] for trees and Durrett [3] for probability.

2. Results

We first show that local maxima in trees rooted at r and local minima in trees rooted at $n+1-r$ are equidistributed.

Lemma 2.1. *For* $r \in [n]$,

$$T_r(n,k) = t_{n+1-r}(n,k) \quad (0 \le k \le n-2). \tag{2.1}$$

Proof. The bijection $i \mapsto n+1-i$ ($i \in [n]$) induces a bijection $T \in \mathcal{T}_n \mapsto T' \in \mathcal{T}_n$, where $T \simeq T'$. Then $r,\dots,i,j,k$ (with $j > i,k$) is a path in T if and only if $n+1-r,\dots,n+1-i,n+1-j,n+1-k$ (with $n+1-j < n+1-i$, $n+1-k$) is a path in T'. Hence, $M_r(T) = m_{n+1-r}(T')$. Consequently, $T_r(n,k) = t_{n+1-r}(n,k)$ for $0 \le k \le n-2$. □

In view of Lemma 2.1, we consider only M_1.

Let $(x)_0 = x^0 = 1$ ($x \in \mathbb{R}$) and $(x)_k = (x)\cdots(x-k+1)$ ($k \ge 1$, $x \in \mathbb{R}$). For $n \in \mathbb{N}$ and $a,x \in \mathbb{R}$, let

$$E_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}, \qquad P_n(x) = \sum_{k=0}^{n} (n)_k x^k, \qquad Q_{n,a}(x) = \sum_{k=0}^{n} (n)_k (k+a) x^k,$$
$$R_n(x) = \sum_{k=0}^{n} (n)_k (k+1)(k+7) x^k. \tag{2.2}$$

We require the following technical result which allows us to calculate the exact expectation and variance of M_1 as rational functions of n.

Lemma 2.2. *For* $n, m-1 \in \mathbb{N}$,

$$P_n\left(\frac{1}{m}\right) = \frac{n!}{m^n} E_n(m), \tag{2.3}$$

for $n-1, m-1 \in \mathbb{N}$, *and* $a \in \mathbb{R}$,

$$Q_{n,a}\left(\frac{1}{m}\right) = \frac{n!}{m^n}(n+a)E_n(m) - \frac{n!}{m^{n-1}} E_{n-1}(m), \tag{2.4}$$

and for $n-2, m-1 \in \mathbb{N}$,

$$R_n\left(\frac{1}{m}\right) = \frac{n!}{m^n}(n^2+8n+7)E_n(m) - \frac{n!}{m^{n-1}}(2n+7)E_{n-1}(m) + \frac{n!}{m^{n-2}} E_{n-2}(m). \tag{2.5}$$

Proof. (All derivatives are with respect to real x). First,

$$\frac{x^n P_n(x^{-1})}{n!} = \sum_{k=0}^{n} \frac{x^{n-k}}{(n-k)!} = E_n(x) \tag{2.6}$$

so that

$$P_n(x) = n! x^n E_n(x^{-1}), \tag{2.7}$$

and hence

$$P_n\left(\frac{1}{m}\right) = \frac{n!}{m^n}E_n(m). \tag{2.8}$$

Next, (2.7) gives

$$\begin{aligned} Q_{n,a}(x) &= xP_n'(x) + aP_n(x) \\ &= n!x^n(n+a)E_n(x^{-1}) - n!x^{n-1}E_{n-1}(x^{-1}), \end{aligned} \tag{2.9}$$

and hence

$$Q_{n,a}\left(\frac{1}{m}\right) = \frac{n!}{m^n}(n+a)E_n(m) - \frac{n!}{m^{n-1}}E_{n-1}(m). \tag{2.10}$$

Finally, (2.7) gives

$$\begin{aligned} R_n(x) &= x^2P_n''(x) + 9xP_n'(x) + 7P_n(x) \\ &= n!x^n(n^2+8n+7)E_n(x^{-1}) - n!x^{n-1}(2n+7)E_{n-1}(x^{-1}) \\ &\quad + n!x^{n-2}E_{n-2}(x^{-1}), \end{aligned} \tag{2.11}$$

and hence

$$R_n\left(\frac{1}{m}\right) = \frac{n!}{m^n}(n^2+8n+7)E_n(m) - \frac{n!}{m^{n-1}}(2n+7)E_{n-1}(m) + \frac{n!}{m^{n-2}}E_{n-2}(m). \tag{2.12}$$

□

Corollary 2.3. *For $j, n-1 \in \mathbb{N}$ with $0 \le j \le n$,*

$$Q_{n-j,j}\left(\frac{1}{n}\right) = n. \tag{2.13}$$

Proof. For $0 \le j \le n-1$, our result follows from Lemma 2.2. For $j = n$, our result follows from the definition of $Q_{n-j,j}(x)$. □

We require the following result of Moon [8].

Theorem 2.4 (Moon [8]). *Let F be a forest with vertex set $[n]$ having ω components of orders $p_1,\dots,p_\omega$. Then the number of distinct trees in $\mathcal{T}_n$ containing F is $pn^{\omega-2}$, where $p = p_1\cdots p_\omega$.*

We now give our main result. Here $M_1 = M_{1,n}$.

Theorem 2.5. *For Ω_n $(n \ge 2)$,*

$$\begin{aligned} E(M_1) &= \frac{2n^3-3n^2-5n+6}{6n^2}, \\ E(M_1^2) &= \frac{20n^6-39n^5-115n^4+495n^3-175n^2-1266n+1080}{180n^4}, \end{aligned} \tag{2.14}$$

hence,

$$\sigma^2(M_1) = \operatorname{Var}(M_1) = \frac{7n^5-20n^4+75n^3-40n^2-322n+300}{60n^4}. \tag{2.15}$$

Proof. The theorem can be seen to be true for $2 \le n \le 5$ using Table 1.1. Assume $n \ge 6$. Let $I_{n,1} = \{(i,j,k) : 1 \le k < i < j \le n\}$, $I_{n,2} = \{(i,j,k) : 1 \le i < k < j \le n\}$, and $I_n = I_{n,1} \cup I_{n,2}$. For $(i,j,k) \in I_n$ and $T \in \mathcal{T}_n$, let

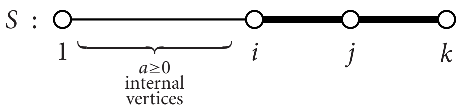
$$X_{(i,j,k)}(T) = \begin{cases} 1, & ijk \text{ is a local maximum in } T \text{ rooted at } 1; \\ 0, & \text{otherwise;} \end{cases} \tag{2.16}$$

hence,

$$M_1 = \sum_{(i,j,k)\in I_n} X_{(i,j,k)}. \tag{2.17}$$

We remind the reader that ij, jk are always edges in a tree by using thick lines in our diagrams.

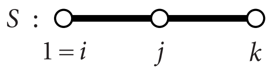
Expectation of M_1. We consider the following two cases according to the path S of T from 1 through ijk. Only $E(X_{(i,j,k)}) \ne 0$ need to be considered.
Case 1 ($i \ne 1$). Here



There are $(a+4)n^{n-a-5}$ trees in $\mathcal{T}_n$ containing a specific tree S by Theorem 2.4; there are $(n-4)_a$ specific trees containing a vertices between 1, i; and there are $2\binom{n-1}{3}$ choices for (i,j,k). Hence,

$$\sum_{\substack{(i,j,k)\in I_n \\ 1\ne i}} E(X_{(i,j,k)}) = \frac{(n-1)_3}{3n^3}\sum_{a=0}^{n-4}(n-4)_a\frac{a+4}{n^a} = \frac{(n-1)_3}{3n^2} \tag{2.18}$$

by Lemma 2.2.
Case 2 ($i = 1$). Here



There are $3n^{n-4}$ trees in $\mathcal{T}_n$ containing a specific tree S by Theorem 2.4; and there are $\binom{n-1}{2}$ choices for (j,k). Hence,

$$\sum_{(1,j,k)\in I_n} E(X_{(i,j,k)}) = \frac{3(n-1)_2}{2n^2}. \tag{2.19}$$

From (2.18), (2.19),

$$E(M_1) = \frac{(n-1)_3}{3n^2} + \frac{3(n-1)_2}{2n^2} = \frac{2n^3 - 3n^2 - 5n + 6}{6n^2}. \tag{2.20}$$

Variance of M_1. Here

$$M_1^2 = \left(\sum_{(i,j,k)\in I_n} X_{(i,j,k)} \right)^2 = M_1 + \sum_{((i_1,j_1,k_1),(i_2,j_2,k_2))\in I_n^*} X_{(i_1,j_1,k_1)} X_{(i_2,j_2,k_2)}, \tag{2.21}$$

where $I_n^* = \{((i_1,j_1,k_1),(i_2,j_2,k_2)) \in I_n \times I_n : (i_1,j_1,k_1) \neq (i_2,j_2,k_2)\}$.

First, we describe how we calculate $E(M_1^2) - E(M_1)$.

We first consider $3 \cdot 2 = 6$ cases according to $\#\{i_1,j_1,k_1,i_2,j_2,k_2\} = 6$, 5, or 4, and, whether $1 \notin \{i_1,j_1,k_1,i_2,j_2,k_2\}$ or $1 \in \{i_1,j_1,k_1,i_2,j_2,k_2\}$. In each of these six cases, we further partition as described below.

For $((i_1,j_1,k_1),(i_2,j_2,k_2)) \in I_n^*$, we consider the possible subtrees $S = S_{((i_1,j_1,k_1),(i_2,j_2,k_2))}$ of $[n]$ determined by the path from 1 to the second coordinate $i_2 j_2 k_2$ relative to the path from 1 to the first coordinate $i_1 j_1 k_1$. The possible subtrees $S' = S'_{((i_2,j_2,k_2),(i_1,j_1,k_1))}$ are included above by definition. Only $E(X_{(i_1,j_1,k_1)} X_{(i_2,j_2,k_2)}) \neq 0$ need to be considered. This gives nine types of subtrees of $[n]$ total among these six cases.

For the symmetric types 1, 3, 5, 7, and 9, S "looks like" S'. We count the number t_S of trees $T \in \mathcal{T}_n$ containing $S = S_{((i_1,j_1,k_1),(i_2,j_2,k_2))}$ and the number i_S of such $((i_1,j_1,k_1),(i_2,j_2,k_2))$. The product $i_S t_S$ counts each tree $T \in \mathcal{T}_n$ containing S twice; once for S and once for S'. For each such tree T, $X_{(i_1,j_1,k_1)}(T) X_{(i_2,j_2,k_2)}(T) = 1 = X_{(i_2,j_2,k_2)}(T) X_{(i_1,j_1,k_1)}(T)$.

For the asymmetric types 2, 4, 6, and 8, S "looks different" than S'. We introduce subtypes $S^x_{((i_1,j_1,k_1),(i_2,j_2,k_2))}$ and $S^y_{((i_1,j_1,k_1),(i_2,j_2,k_2))}$ so that $T \in \mathcal{T}_n$ contains $S^x = S^x_{((i_1,j_1,k_1),(i_2,j_2,k_2))}$ if and only if T contains $S^y = S^y_{((i_2,j_2,k_2),(i_1,j_1,k_1))}$; note the different orders. We count the number t_{S^z} of trees $T \in \mathcal{T}_n$ containing S^z and the number i_{S^z} of such $((i_1,j_1,k_1),(i_2,j_2,k_2))$ for $z = x, y$. The sum $i_{S^x} t_{S^x} + i_{S^y} t_{S^y}$ counts each tree $T \in \mathcal{T}_n$ containing S^x and S^y; once for S^x and once for S^y. For each such tree T, $X_{(i_1,j_1,k_1)}(T) X_{(i_2,j_2,k_2)}(T) = 1 = X_{(i_2,j_2,k_2)}(T) X_{(i_1,j_1,k_1)}(T)$.

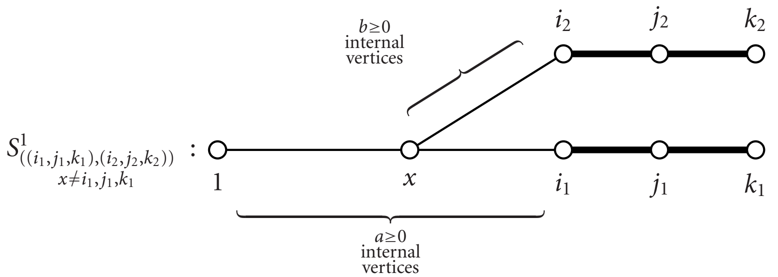
For each type, the above count(s) are divided by n^{n-2} then simplified using Lemma 2.2 and Corollary 2.3. Summing over the types of a particular case i ($1 \le i \le 6$) gives

$$\sum_{\substack{((i_1,j_1,k_1),(i_2,j_2,k_2))\in I_n^* \\ \text{case } i}} E\big(X_{(i_1,j_1,k_1)} X_{(i_2,j_2,k_2)}\big). \tag{2.22}$$

The sum over all six cases is then $E(M_1^2) - E(M_1)$.

In what follows, $(n-1)_6 = E_{n-7}(n) = 0$ for $n = 6$ and $E_{n-9}(n) = 0$ for $n = 6,7,8$ as usual. All cases appear for $n \ge 9$.

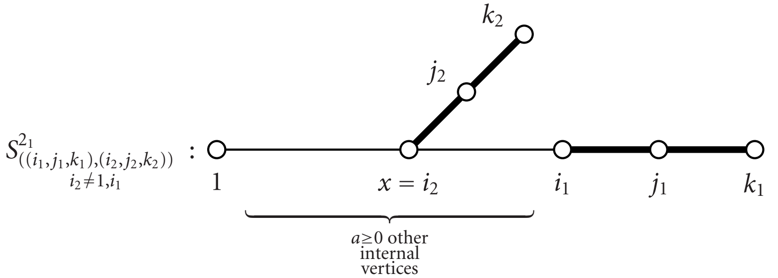
Case 3 ($\#\{i_1, j_1, k_1, i_2, j_2, k_2\} = 6$, $1 \notin \{i_1, j_1, k_1, i_2, j_2, k_2\}$).
Type 1. Here



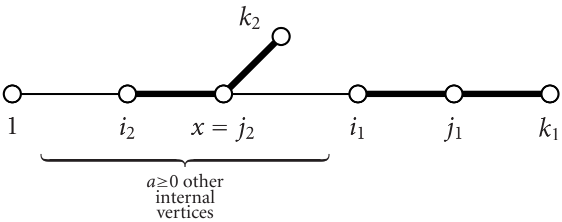
There are $(a+b+7)n^{n-a-b-8}$ trees in $\mathcal{T}_n$ containing a specific tree S^1 by Theorem 2.4; there are $(n-7)_{a+b}$ specific trees containing a vertices between 1, i_1 and b vertices between x, i_2; there are $a+1$ choices for x; and there are $2\binom{n-1}{3} \cdot 2\binom{n-4}{3}$ such pairs $((i_1, j_1, k_1), (i_2, j_2, k_2))$. Observe that T contains $S^1_{((i_1,j_1,k_1),(i_2,j_2,k_2))}$ for a, b, x if and only if T contains $S^1_{((i_2,j_2,k_2),(i_1,j_1,k_1))}$ for a', b', x. Hence, (each such pair appears once)

$$\begin{aligned}
\sum_{\substack{((i_1,j_1,k_1),(i_2,j_2,k_2))\in I_n^* \\ T \text{ Type } 1}} E\big(X_{(i_1,j_1,k_1)}X_{(i_2,j_2,k_2)}\big) &= \frac{(n-1)_6}{9n^6} \sum_{\substack{(a,b)\in\mathbb{N}^2 \\ 0\leq a+b\leq n-7}} (n-7)_{a+b}\frac{(a+1)(a+b+7)}{n^{a+b}} \\
&= \frac{(n-1)_6}{9n^6}\sum_{a=0}^{n-7}(n-7)_a\frac{a+1}{n^a}\sum_{b=0}^{n-a-7}(n-a-7)_b\frac{a+b+7}{n^b} \\
&= \frac{(n-1)_6}{9n^5}\sum_{a=0}^{n-7}(n-7)_a\frac{a+1}{n^a} \\
&= \frac{(n-1)_6}{9n^5}\left\{n - \frac{6(n-7)!}{n^{n-7}}E_{n-7}(n)\right\} \\
&= \frac{(n-1)_6}{9n^4} - \frac{2(n-1)!}{3n^{n-2}}E_{n-7}(n)
\end{aligned} \tag{2.23}$$

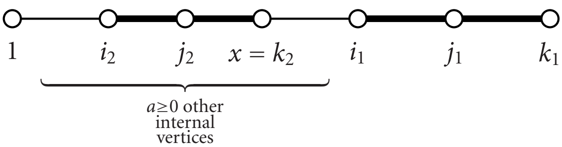
by Lemma 2.2, and Corollary 2.3.
Type 2. First subtypes 2_1, 2_2, 2_3 are



$S^{2_2}_{\substack{((i_1,j_1,k_1),(i_2,j_2,k_2))\\ i_2,j_2\neq 1,i_1}}$:

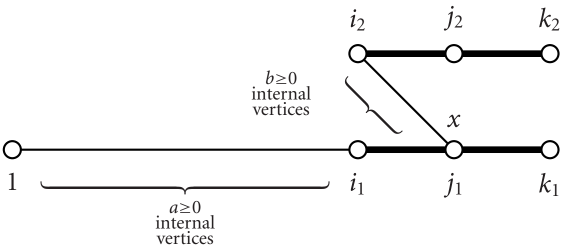


$S^{2_3}_{\substack{((i_1,j_1,k_1),(i_2,j_2,k_2))\\ i_2,j_2,k_2\neq 1,i_1}}$:



In each of the subcases, we have replaced one of the $a+1$ edges uv between 1, i_1, with the path $ux = i_2v$, $ui_2x = j_2v$ or $ui_2j_2x = k_2v$, where the rest of the path $i_2j_2k_2$ is as indicated. In each of these three subcases, there are $(a+7)n^{n-a-8}$ trees in $\mathcal{T}_n$ containing a specific tree S^{2_1}, S^{2_2}, S^{2_3} by Theorem 2.4; there are $(n-7)_a$ specific trees containing a other vertices between 1, i_1; there are $a+1$ choices for x, equivalently, uv; and there are $2\binom{n-1}{3}\cdot 2\binom{n-4}{3}$ such pairs $((i_1,j_1,k_1),(i_2,j_2,k_2))$. Next, subtype 2_4 is

$S^{2_4}_{\substack{((i_1,j_1,k_1),(i_2,j_2,k_2))\\ x=i_1,j_1 \text{ or } k_1}}$:



There are $(a+b+7)n^{n-a-b-8}$ trees in $\mathcal{T}_n$ containing a specific tree S^{2_4} by Theorem 2.4; there are $(n-7)_{a+b}$ specific trees containing a vertices between 1, i_1 and b vertices between x, i_2; there are 3 choices for $x = i_1, j_1$, or k_1; and there are $2\binom{n-1}{3}\cdot 2\binom{n-4}{3}$ such pairs $((i_1,j_1,k_1),(i_2,j_2,k_2))$. Observe that T contains $S^{2_{1,2\text{ or }3}}_{((i_1,j_1,k_1),(i_2,j_2,k_2))}$ for a, x if and only if T contains $S^{2_4}_{((i_2,j_2,k_2),(i_1,j_1,k_1))}$ for a', b', x. Hence, (each such pair appears once)

$$\sum_{\substack{((i_1,j_1,k_1),(i_2,j_2,k_2))\in I_n^*\\ T\text{ Type 2}}} E\big(X_{(i_1,j_1,k_1)}X_{(i_2,j_2,k_2)}\big)$$

$$= \frac{(n-1)_6}{9n^6}\left\{3\sum_{a=0}^{n-7}(n-7)_a\frac{(a+1)(a+7)}{n^a} + 3\sum_{\substack{(a,b)\in\mathbb{N}^2\\ 0\leq a+b\leq n-7}}(n-7)_{a+b}\frac{a+b+7}{n^{a+b}}\right\}$$

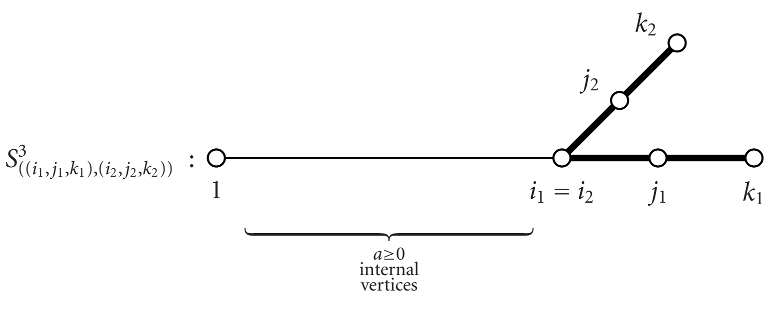
$$
\begin{aligned}
&= \frac{(n-1)_6}{3n^6}\left\{\sum_{a=0}^{n-7}(n-7)_a\frac{(a+1)(a+7)}{n^a} + \sum_{a=0}^{n-7}\frac{(n-7)_a}{n^a}\sum_{b=0}^{n-a-7}(n-a-7)_b\frac{a+b+7}{n^b}\right\}\\
&= \frac{(n-1)_6}{3n^6}\left\{\frac{(n-7)!}{n^{n-8}}(n-5)E_{n-7}(n) - \frac{(n-7)!}{n^{n-8}}(2n-7)E_{n-8}(n) + \frac{(n-7)!}{n^{n-9}}E_{n-9}(n)\right\}\\
&= \frac{(n-1)_6}{3n^6}\left\{\frac{2(n-7)!}{n^{n-8}}E_{n-9}(n) + 4n - 14\right\}\\
&= \frac{2(n-1)!}{3n^{n-2}}E_{n-9}(n) + (4n-14)\frac{(n-1)_6}{3n^6}
\end{aligned}
\tag{2.24}
$$

by Lemma 2.2 and Corollary 2.3. (The first 3 above is number of subcases and the second 3 is the number of choices for x.)

Summing (2.23), (2.24) gives the following equation:

$$
\begin{aligned}
&\sum_{\substack{((i_1,j_1,k_1),(i_2,j_2,k_2))\in I_n^* \\ \#\{i_1,j_1,k_1,i_2,j_2,k_2\}=6, 1\notin\{i_1,j_1,k_1,i_2,j_2,k_2\}}} E\left(X_{(i_1,j_1,k_1)}X_{(i_2,j_2,k_2)}\right)\\
&\quad= \frac{(n-1)_6}{9n^4} - \frac{2(n-1)!}{3n^{n-2}}E_{n-7}(n) + \frac{2(n-1)!}{3n^{n-2}}E_{n-9}(n) + (4n-14)\frac{(n-1)_6}{3n^6}\\
&\quad= \frac{(n-1)_6}{9n^4} - \frac{2}{3}\left\{\frac{(n-1)_6}{n^5} + \frac{(n-1)_7}{n^6} - (2n-7)\frac{(n-1)_6}{n^6}\right\}\\
&\quad= \frac{(n-1)_6}{9n^4}.
\end{aligned}
\tag{2.25}
$$

Case 4 ($\#\{i_1,j_1,k_1,i_2,j_2,k_2\} = 5$, $1 \notin \{i_1,j_1,k_1,i_2,j_2,k_2\}$).
Type 3. Here

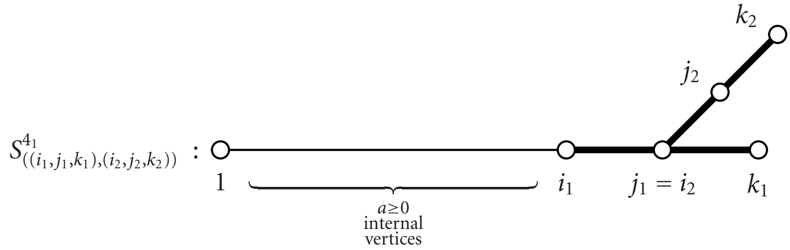


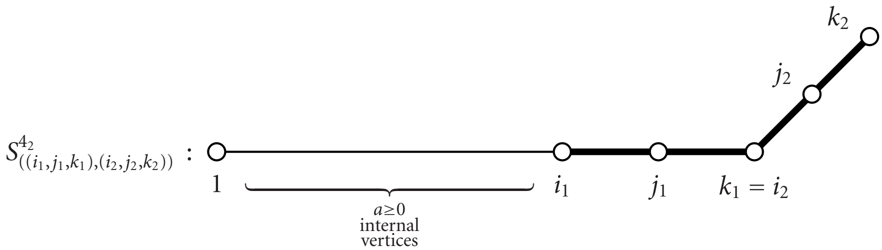
There are $(a+6)n^{n-a-7}$ trees in $\mathcal{T}_n$ containing a specific tree S^3 by Theorem 2.4; there are $(n-6)_a$ specific trees containing a vertices between $1, i_1$; and there are $16\binom{n-1}{5}$ such pairs $((i_1,j_1,k_1),(i_2,j_2,k_2))$ (for 5 elements in $\{2,\dots,n\}$, there are $2\cdot 6 = 12$ pairs with largest elements j_1, j_2, and there are $2 \cdot 2 = 4$ pairs with $j_1 > k_1 > j_2$ or $j_2 > k_2 > j_1$). Observe that T contains $S^3_{((i_1,j_1,k_1),(i_2,j_2,k_2))}$ if and only if T contains $S^3_{((i_2,j_2,k_2),(i_1,j_1,k_1))}$. Hence, (each such pair appears once)

$$\sum_{\substack{((i_1,j_1,k_1),(i_2,j_2,k_2))\in I_n^* \\ T \text{ Type } 3}} E\big(X_{(i_1,j_1,k_1)}X_{(i_2,j_2,k_2)}\big) = \frac{2(n-1)_5}{15n^5}\sum_{a=0}^{n-6}(n-6)_a\frac{a+6}{n^a} = \frac{2(n-1)_5}{15n^4} \tag{2.26}$$

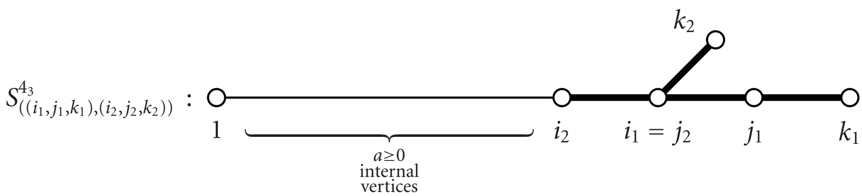
by Corollary 2.3.

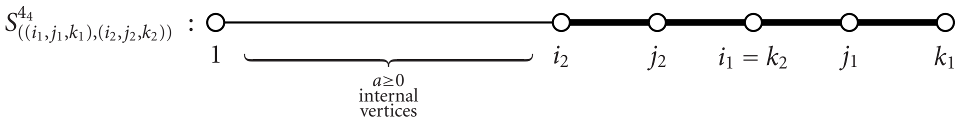
Type 4. First subtypes 4_1, 4_2 are





In either subcase, there are $(a+6)n^{n-a-7}$ trees in $\mathcal{T}_n$ containing a specific tree S^{4_1}, S^{4_2} by Theorem 2.4; there are $(n-6)_a$ specific trees containing a vertices between 1, i_1; there are $24\binom{n-1}{5}$ such pairs $((i_1,j_1,k_1),(i_2,j_2,k_2))$ total (for 5 elements in $\{2,\dots,n\}$; there are $6+2=8$ pairs with $j_2 > i_2 = j_1$ or $j_2 > k_2 > i_2 = j_1$ for 4_1; there are $2 \cdot 6 = 12$ pairs with largest elements j_1, j_2; and there are $2 \cdot 2 = 4$ pairs with $j_1 > i_1 > j_2$ or $j_2 > k_2 > j_1$ for 4_2). Next subtypes 4_3, 4_4 are





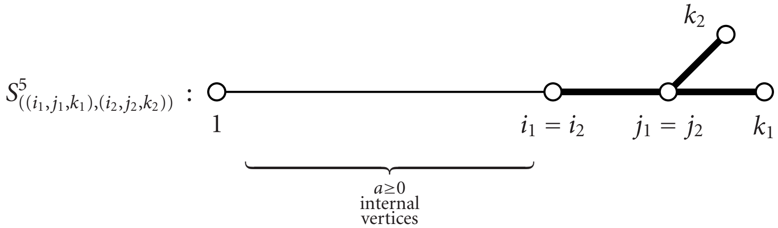
In either subcase, there are $(a+6)n^{n-a-7}$ trees in $\mathcal{T}_n$ containing a specific tree S^{4_3}, S^{4_4} by Theorem 2.4; there are $(n-6)_a$ specific trees containing a vertices between 1, i_2; and there are $24\binom{n-1}{5}$ such pairs $((i_1,j_1,k_1),(i_2,j_2,k_2))$ total (for 5 elements in $\{2,\dots,n\}$, there are $6+2=8$ pairs with $j_1 > i_1 = j_2$ or $j_1 > k_1 > i_1 = j_2$ for 4_3; there are $2\cdot 6 = 12$ pairs with largest elements j_1, j_2, and there are $2\cdot 2 = 4$ pairs with $j_2 > i_2 > j_1$ or $j_1 > k_1 > j_2$ for 4_4). Observe that T contains $S^{4_i}_{((i_1,j_1,k_1),(i_2,j_2,k_2))}$ if and only if T contains $S^{4_{i+2}}_{((i_2,j_2,k_2),(i_1,j_1,k_1))}$ for $i=1,2$. Hence, (each such pair appears once)

$$\sum_{\substack{((i_1,j_1,k_1),(i_2,j_2,k_2))\in I_n^* \\ T \text{ Type } 4}} E\big(X_{(i_1,j_1,k_1)}X_{(i_2,j_2,k_2)}\big) = \frac{2(n-1)_5}{5n^5}\sum_{a=0}^{n-6}(n-6)_a\frac{a+6}{n^a} = \frac{2(n-1)_5}{5n^4} \tag{2.27}$$

by Corollary 2.3. (The number of subcases has been accounted for.) Summing (2.26), (2.27) gives the following equation:

$$\sum_{\substack{((i_1,j_1,k_1),(i_2,j_2,k_2))\in I_n^* \\ \#\{i_1,j_1,k_1,i_2,j_2,k_2\}=5,\,1\notin\{i_1,j_1,k_1,i_2,j_2,k_2\}}} E\big(X_{(i_1,j_1,k_1)}X_{(i_2,j_2,k_2)}\big) = \frac{2(n-1)_5}{15n^4} + \frac{2(n-1)_5}{5n^4} = \frac{8(n-1)_5}{15n^4}. \tag{2.28}$$

Case 5 ($\#\{i_1,j_1,k_1,i_2,j_2,k_2\} = 4$, $1 \notin \{i_1,j_1,k_1,i_2,j_2,k_2\}$).
Type 5. Here



There are $(a+5)n^{n-a-6}$ trees in $\mathcal{T}_n$ containing a specific tree S^5 by Theorem 2.4; there are $(n-5)_a$ specific trees containing a vertices between 1, i_1; and there are $6\binom{n-1}{4}$ such pairs

$((i_1,j_1,k_1),(i_2,j_2,k_2))$. Observe that T contains $S^5_{((i_1,j_1,k_1),(i_2,j_2,k_2))}$ if and only if T contains $S^5_{((i_2,j_2,k_2),(i_1,j_1,k_1))}$. Hence, (each such pair appears once)

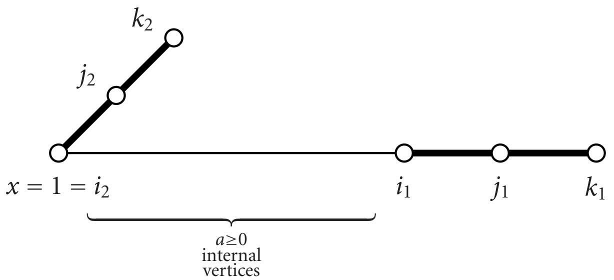
$$\sum_{\substack{((i_1,j_1,k_1),(i_2,j_2,k_2))\in I_n^* \\ \#\{i_1,j_1,k_1,i_2,j_2,k_2\}=4,1\notin\{i_1,j_1,k_1,i_2,j_2,k_2\}}} E\left(X_{(i_1,j_1,k_1)}X_{(i_2,j_2,k_2)}\right) = \frac{(n-1)_4}{4n^4}\sum_{a=0}^{n-5}(n-5)_a\frac{a+5}{n^a} = \frac{(n-1)_4}{4n^3} \tag{2.29}$$

by Corollary 2.3.

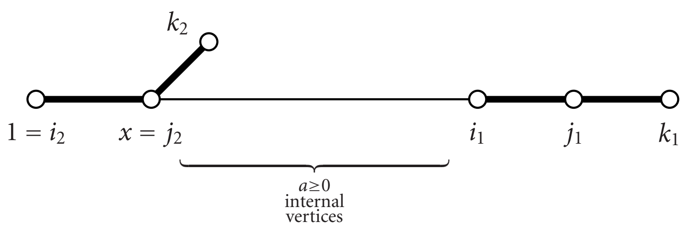
Case 6 ($\#\{i_1,j_1,k_1,i_2,j_2,k_2\}=6$, $1\in\{i_1,j_1,k_1,i_2,j_2,k_3\}$).

Type 6. First subtypes 6_1, 6_2, 6_3 are

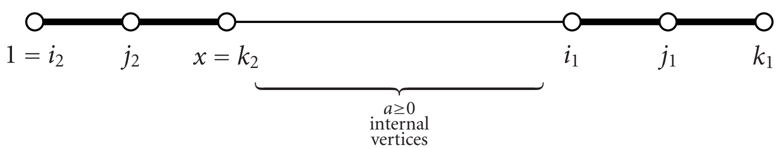
$S^{6_1}_{((i_1,j_1,k_1),(1,j_2,k_2))}$:



$S^{6_2}_{((i_1,j_1,k_1),(1,j_2,k_2))}$:

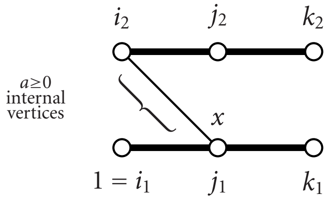


$S^{6_3}_{((i_1,j_1,k_1),(1,j_2,k_2))}$:



In each of these three subcases, there are $(a+6)n^{n-a-7}$ trees in $\mathcal{T}_n$ containing a specific tree S^{6_1}, S^{6_2}, S^{6_3} by Theorem 2.4; there are $(n-6)_a$ specific trees containing a vertices between x, i_1; and there are $\binom{n-1}{2}\cdot 2\binom{n-3}{3}$ such pairs $((i_1,j_1,k_1),(1,i_2,j_2))$. Next subtype 6_4 is

$S^{6_4}_{((1,j_1,k_1),(i_2,j_2,k_2))}$:
$x=1,j_1 \text{ or } k_1$



There are $(a+6)n^{n-a-7}$ trees in $\mathcal{T}_n$ containing a specific tree S^{6_4} by Theorem 2.4; there are $(n-6)_a$ specific trees containing a vertices between x, i_2; there are 3 choices for $x = 1, j_1$, or k_1; and there are $\binom{n-1}{2}\cdot 2\binom{n-3}{3}$ such pairs $((1,j_1,k_1),(i_2,j_2,k_2))$. Observe that T contains $S^{6_{1,2,3}}_{((i_1,j_1,k_1),(1,j_2,k_2))}$ for a, x if and only if T contains $S^{6_4}_{((1,j_2,k_2),(i_1,j_1,k_1))}$ for a, x. Hence, (each such pair appears once)

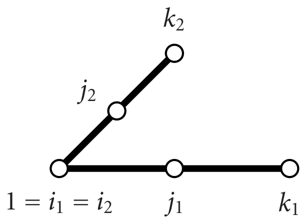
$$\begin{aligned}
&\sum_{\substack{((i_1,j_1,k_1),(i_2,j_2,k_2))\in I_n^* \\ \#\{i_1,j_1,k_1,i_2,j_2,k_2\}=6,1\in\{i_1,j_1,k_1,i_2,j_2,k_2\}}} E\big(X_{(i_1,j_1,k_1)}X_{(i_2,j_2,k_2)}\big) \\
&\quad = \frac{(n-1)_5}{6n^5}\left\{3\sum_{a=0}^{n-6}(n-6)_a\frac{a+6}{n^a}+3\sum_{a=0}^{n-6}(n-6)_a\frac{a+6}{n^a}\right\} \\
&\quad = \frac{(n-1)_5}{n^4}
\end{aligned} \tag{2.30}$$

by Corollary 2.3. (The first 3 and second 3 above are the number of subcases, i.e., choices for x.)

Case 7 ($\#\{i_1,j_1,k_1,i_2,j_2,k_2\}=5$, $1\in\{i_1,j_1,k_1,i_2,j_2,k_2\}$).

Type 7. Here

$S^{7}_{((1,j_1,k_1),(1,j_2,k_2))}$:

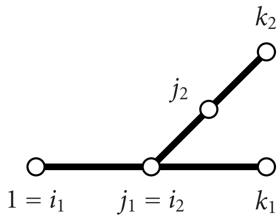


There are $5n^{n-6}$ trees in $\mathcal{T}_n$ containing a specific tree S^7 by Theorem 2.4; and there are $\binom{n-1}{2}\cdot\binom{n-3}{2}$ such pairs $((1,j_1,k_1),(1,j_2,k_2))$. Observe that T contains $S^7_{((1,j_1,k_1),(1,j_2,k_2))}$ if and only if T contains $S^7_{((1,j_2,k_2),(1,j_1,k_1))}$. Hence, (each such pair occurs once)

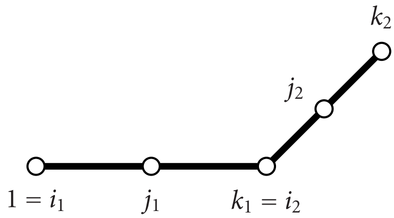
$$\sum_{\substack{((1,j_1,k_1),(1,j_2,k_2))\in I_n^* \\ T \text{ Type } 7}} E\big(X_{(1,j_1,k_1)}X_{(1,j_2,k_2)}\big) = \frac{5(n-1)_4}{4n^4}. \tag{2.31}$$

Type 8. First subtypes 8_1, 8_2 are

$S^{8_1}_{((1,j_1,k_1),(i_2,j_2,k_2))}$:

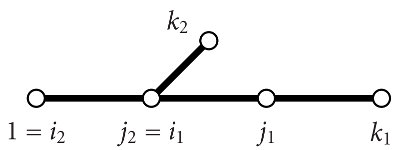


$S^{8_2}_{((1,j_1,k_1),(i_2,j_2,k_2))}$:

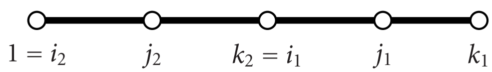


In either subcase, there are $5n^{n-6}$ trees in $\mathcal{T}_n$ containing a specific tree S^{8_1}, S^{8_2} by Theorem 2.4; and there are $8\binom{n-1}{4}$ such pairs $((1,j_1,k_1),(i_2,j_2,k_2))$ total (for 4 elements in $\{2,\dots,n\}$, there are $2+1=3$ pairs with $j_2 > j_1 = i_2$ or $j_2 > k_2 > i_2 = j_1$ for 8_1; and there are $2+2+1=5$ pairs with largest elements j_1, j_2 or $j_2 > k_2 > j_1$ for 8_2). Next subtypes 8_3, 8_4 are

$S^{8_3}_{((i_1,j_1,k_1),(1,j_2,k_2))}$:



$S^{8_4}_{((i_1,j_1,k_1),(1,j_2,k_2))}$:



In either subcase, there are $5n^{n-6}$ trees in $\mathcal{T}_n$ containing a specific tree S^{8_3}, S^{8_4} by Theorem 2.4; and there are $8\binom{n-1}{4}$ such pairs $((i_1,j_1,k_1),(1,j_2,k_2))$ total (for 4 elements in $\{2,\dots,n\}$, there are $2+1=3$ pairs with $j_1 > j_2 = i_1$ or $j_1 > k_1 > i_1 = j_2$ for 8_3; and there are $2+2+1=5$ pairs with largest elements j_1, j_2 or $j_1 > k_1 > j_2$ for 8_4). Observe that T contains $S^{8_i}_{((1,j_1,k_1),(i_2,j_2,k_2))}$ if and only if T contains $S^{8_{i+2}}_{((i_2,j_2,k_2),(1,j_1,k_1))}$ for $i=1,2$. Hence, (each such pair occurs once)

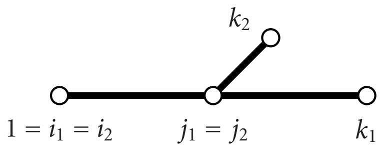
$$\sum_{\substack{((i_1,j_1,k_1),(i_2,j_2,k_2))\in I_n^* \\ T \text{ Type } 8}} E\left(X_{(i_1,j_1,k_1)}X_{(i_2,j_2,k_2)}\right) = \frac{5(n-1)_4}{3n^4} + \frac{5(n-1)_4}{3n^4} = \frac{10(n-1)_4}{3n^4}. \tag{2.32}$$

(The number of subcases has been accounted for.) Summing (2.31), (2.32) gives the following equation:

$$\sum_{\substack{((i_1,j_1,k_1),(i_2,j_2,k_2))\in I_n^* \\ \#\{i_1,j_1,k_1,i_2,j_2,k_2\}=5,1\in\{i_1,j_1,k_1,i_2,j_2,k_2\}}} E\big(X_{(i_1,j_1,k_1)}X_{(i_2,j_2,k_2)}\big) = \frac{5(n-1)_4}{4n^4} + \frac{10(n-1)_4}{3n^4} = \frac{55(n-1)_4}{12n^4}. \tag{2.33}$$

Case 8 ($\#\{i_1,j_1,k_1,i_2,j_2,k_2\} = 4$, $1 \in \{i_1,j_1,k_1,i_2,j_2,k_2\}$).
Type 9. Here

$S^9_{((1,j_1,k_1),(1,j_2,k_2))}$:



There are $4n^{n-5}$ trees in $\mathcal{T}_n$ containing a specific tree S^9 by Theorem 2.4; and there are $2\binom{n-1}{3}$ such pairs $((1,j_1,k_1),(1,j_2,k_2))$. Observe that T contains $S^9_{((1,j_1,k_1),(1,j_2,k_2))}$ if and only if T contains $S^9_{((1,j_2,k_2),(1,j_1,k_1))}$. Hence, (each such pair occurs once)

$$\sum_{\substack{((i_1,j_1,k_1),(i_2,j_2,k_2))\in I_n^* \\ \#\{i_1,j_1,k_1,i_2,j_2,k_2\}=4,1\in\{i_1,j_1,k_1,i_2,j_2,k_2\}}} E\big(X_{(i_1,j_1,k_1)}X_{(i_2,j_2,k_2)}\big) = \frac{4(n-1)_3}{3n^3}. \tag{2.34}$$

After all this preparation, we are now able to find the second moment and the variance of M_1. From (2.21), summing (2.20), (2.25), (2.28)–(2.30), (2.33), (2.34) gives

$$\begin{aligned} E(M_1^2) &= \frac{2n^3-3n^2-5n+6}{6n^2} + \frac{(n-1)_6}{9n^4} + \frac{8(n-1)_5}{15n^4} + \frac{(n-1)_4}{4n^3} \\ &\quad + \frac{(n-1)_5}{n^4} + \frac{55(n-1)_4}{12n^4} + \frac{4(n-1)_3}{3n^3} \\ &= \frac{20n^6-39n^5-115n^4+495n^3-175n^2-1266n+1080}{180n^4}. \end{aligned} \tag{2.35}$$

Hence, (2.20), (2.35) give

$$\sigma^2(M_1) = \operatorname{Var}(M_1) = \frac{7n^5-20n^4+75n^3-40n^2-322n+300}{60n^4}. \tag{2.36}$$

□

As a consequence of Theorem 2.5, a.a.s. on $\{\Omega_n\}$, $M_1(T)$ and $m_n(T)$ belong to a relatively small interval for $T \in \mathcal{T}_n$. Again, $M_1 = M_{1,n}$.

Corollary 2.6. *For* $\{\Omega_n\}$,

$$\Pr\big(\left|M_1 - E(M_1)\right| < \omega(n)\sigma(M_1)\big) \longrightarrow 1 \quad \text{as } n \longrightarrow \infty, \tag{2.37}$$

where $\omega(n) \to \infty$ *arbitarily slowly as* $n \to \infty$. *Hence, a.a.s. on* $\{\Omega_n\}$,

$$\frac{n}{3} - \omega(n)n^{0.5} < M_1 < \frac{n}{3} + \omega(n)n^{0.5}, \tag{2.38}$$

where $\omega(n) \to \infty$ *arbitarily slowly as* $n \to \infty$.

Proof. By Chebyshev's inequality,

$$\Pr\left(\left|M_1 - E(M_1)\right| \geq \omega(n)\sigma(M_1)\right) \leq \frac{1}{\omega^2(n)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{2.39}$$

provided that $\omega(n) \to \infty$ as $n \to \infty$. This implies our result. □

Acknowledgment

I wish to thank both referees for comments and suggestions. The detailed report by one referee in particular led to this much improved version of the note.

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