ON THE SPECTRUM AND EIGENFUNCTIONS OF THE SCHRÖDINGER OPERATOR WITH AHARONOV-BOHM MAGNETIC FIELD

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We explicitly compute the spectrum and eigenfunctions of the magnetic Schrödinger operator $H(\vec{A},V) = (i\nabla + \vec{A})^2 + V$ in $L^2(\mathbb{R}^2)$, with Aharonov-Bohm vector potential, $\vec{A}(x_1,x_2) = \alpha(-x_2,x_1)/|x|^2$, and either quadratic or Coulomb scalar potential V. We also determine sharp constants in the CLR inequality, both dependent on the fractional part of α and both greater than unity. In the case of quadratic potential, it turns out that the LT inequality holds for all $\gamma \geq 1$ with the classical constant, as expected from the non-magnetic system (harmonic oscillator).

1. Introduction

The main aim of this paper is to determine explicit constants in the Lieb-Thirring (LT) and Cwikel-Lieb-Rozenblyum (CLR) inequalities for a class of exactly solvable quantum-mechanical models. We consider the magnetic Schrödinger operator

$$H(\vec{A},V) = (i\nabla + \vec{A})^2 + V \tag{1.1}$$

in $L^2(\mathbb{R}^2)$ with Aharonov-Bohm vector potential,

$$\vec{A}(x_1,x_2) = \frac{\alpha(-x_2,x_1)}{|x|^2}, \quad \alpha \in \mathbb{R} \setminus \mathbb{Z}, \tag{1.2}$$

and with two different choices of scalar potential. In both cases, the optimal CLR constant depends on $|\alpha - m_1|$, where m_1 is the best integer approximation of α.

We initially use a quadratic scalar potential, $V(x_1,x_2) = \beta|x|^2$, where $\beta \in \mathbb{R}_+ = (0,\infty)$. The operator is then unitarily equivalent to the two-dimensional harmonic oscillator if the magnitude α is an integer. Such an operator has already been considered, for instance, in [2, 6]. In the latter work, the authors construct a solution of the time-dependent Schrödinger equation. In the corresponding classical system, whose trajectories are given by Hamilton's equation, the particles move in periodic orbits around the singularity, unaffected by the Aharonov-Bohm field. Quantum-mechanically, however, the effect of

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the magnetic field can be observed in the solutions of the Schrödinger equation. It turns out that the spectrum and eigenfunctions of the operator (1.1) can be computed explicitly (Theorem 2.1). Here again, one sees a contribution of the Aharonov-Bohm effect insofar as the eigenfunctions differ from those of the harmonic oscillator when the magnitude α is noninteger.

We moreover prove that the LT inequality, that is,

$$\operatorname{Tr}\left(H(\vec{A},V)-\lambda\right)_-^{\gamma} \le \frac{R_\gamma}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(a(x,\xi)-\lambda\right)_-^{\gamma}\,\mathrm{d}x\,\mathrm{d}\xi, \tag{1.3}$$

holds true for the operator with the classical constant $R_\gamma = 1$ for all $\gamma \ge 1$ (Theorem 2.3). Such a result could not have been deduced from the results in [3] or [5], where the authors consider nonmagnetic Schrödinger operators. It is known that nonmagnetic systems cannot satisfy the CLR inequality ($\gamma = 0$) in two dimensions. With the Aharonov-Bohm field, however, this inequality is sharp with

$$R_0 = \begin{cases} \dfrac{2}{\left(1+\left|\alpha-m_1\right|\right)^2} & \text{if } 0<\left|\alpha-m_1\right|\le 3\sqrt{2}-4, \\[2ex] \dfrac{1}{\left(1-(1/2)\left|\alpha-m_1\right|\right)^2} & \text{if } 3\sqrt{2}-4\le\left|\alpha-m_1\right|\le\dfrac{1}{2}, \end{cases} \tag{1.4}$$

which is always greater than unity (Theorem 2.2).

Parallel results are obtained in the second part for the Coulomb potential, $V(x_1,x_2) = -\beta/|x|$. Unlike the quadratic potential it is not confining, and consequently the point spectrum is entirely negative (Theorem 3.1). The LT inequality is trivial if $\gamma \ge 1$, and we establish (Theorem 3.2) that the sharp CLR constant is

$$R_0 = \begin{cases} \dfrac{1}{\left(1/2+\left|\alpha-m_1\right|\right)^2} & \text{if } 0<\left|\alpha-m_1\right|\le 2\sqrt{2}-\dfrac{5}{2}, \\[2ex] \dfrac{2}{\left(3/2-\left|\alpha-m_1\right|\right)^2} & \text{if } 2\sqrt{2}-\dfrac{5}{2}\le\left|\alpha-m_1\right|\le\dfrac{1}{2}. \end{cases} \tag{1.5}$$

Again $R_0 > 1$ for all α.

2. Quadratic potential

2.1. Spectrum and eigenfunctions. In this section, we will see that the eigenvalue problem for $H(\vec{A},V)$ with quadratic potential can be reduced to Whittaker's differential equation. The spectrum of the operator turns out to have a close connection with that of the harmonic oscillator.

2.1.1. Separation of variables. We may use the decomposition

$$L^2(\mathbb{R}^2) = L^2(\mathbb{R}_+, r\,\mathrm{d}r)\otimes L^2(\mathbb{S}^1) = \bigoplus_{m\in\mathbb{Z}}\left(L^2(\mathbb{R}_+, r\,\mathrm{d}r)\otimes\left[\frac{e^{im\theta}}{\sqrt{2\pi}}\right]\right), \tag{2.1}$$

where $[\cdot]$ denotes the linear span, to express the Aharonov-Bohm operator as

$$H(\vec{A},V) = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(i\frac{\partial}{\partial\theta}+\alpha\right)^2 + \beta r^2 = \bigoplus_{m\in\mathbb{Z}} (H_m \otimes I_m), \tag{2.2}$$

where I_m is the identity on $[e^{im\theta}/\sqrt{2\pi}]$ and

$$H_m = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r^2}(\alpha - m)^2 + \beta r^2. \tag{2.3}$$

To remove the weight r, we introduce the unitary mapping

$$\begin{aligned} U : L^2(\mathbb{R}_+, r\,\mathrm{d}r) &\longrightarrow L^2(\mathbb{R}_+, \mathrm{d}r), \\ f(r) &\longmapsto \sqrt{r}f(r), \end{aligned} \tag{2.4}$$

which transforms H_m into

$$\widetilde{H}_m = UH_mU^{-1} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(\alpha-m)^2 - 1/4}{r^2} + \beta r^2. \tag{2.5}$$

Following (2.1), we write

$$u(r,\theta) = \sum_{m=-\infty}^{\infty} u_m(r)e^{im\theta}, \tag{2.6}$$

and the corresponding quadratic form decomposes accordingly:

$$\widetilde{a}[u] = \sum_{m=-\infty}^{\infty} \widetilde{a}_m[u_m], \tag{2.7}$$

where

$$\widetilde{a}_m[u] = \int_0^\infty \left(\left| \frac{\mathrm{d}u}{\mathrm{d}r} \right|^2 + \frac{(\alpha-m)^2 - 1/4}{r^2}|u|^2 + \beta r^2|u|^2 \right) \mathrm{d}r. \tag{2.8}$$

The operator $H(\vec{A},V)$ will be considered as the Friedrichs extension of the differential expression (2.2) on $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$. By an application of the classical Hardy inequality

$$\int_0^\infty \frac{|f|^2}{4r^2}\mathrm{d}r \le \int_0^\infty |f'|^2 \mathrm{d}r \quad \forall f \in H_0^1(\mathbb{R}_+), \tag{2.9}$$

(and a standard density argument), one can prove that its domain consists of all H_0^1 functions such that the quadratic form (2.7) is finite.

2.1.2. Eigenfunctions. The spectrum of this operator is discrete and can be calculated explicitly. Our goal is to find all eigenfunctions of $H(\vec{A},V)$, that is, all $\phi_m e^{im\theta}$ which are eigenfunctions of $H_m \otimes I_m$. Taking into account the mapping (2.4), we have

$$H_m\phi_m = E\phi_m \Longleftrightarrow \widetilde{H}_m\widetilde{\phi}_m = E\widetilde{\phi}_m, \tag{2.10}$$

where $\widetilde{\phi}_m = U\phi_m$. Substituting further

$$\widetilde{\phi}_m(r) = \frac{\widetilde{\widetilde{\phi}}_m(r^2)}{\sqrt{r}} \tag{2.11}$$

in (2.10), we obtain the equation

$$\begin{gathered} 4r^2\left[\widetilde{\widetilde{\phi}}_m''(r^2) + \left(-\frac{\beta}{4} + \frac{E/4}{r^2} + \frac{1/4 - \left((\alpha - m)/2\right)^2}{r^4}\right)\widetilde{\widetilde{\phi}}_m(r^2)\right] = 0 \\ \Longleftrightarrow 4r^2\left[\widetilde{\widetilde{\phi}}_m''\left(\sqrt{\beta}r^2\right) + \left(-\frac{1}{4} + \frac{E/4\sqrt{\beta}}{\sqrt{\beta}r^2} + \frac{1/4 - \left((\alpha - m)/2\right)^2}{\left(\sqrt{\beta}r^2\right)^2}\right)\widetilde{\widetilde{\phi}}_m\left(\sqrt{\beta}r^2\right)\right] = 0. \end{gathered} \tag{2.12}$$

Setting $z = \sqrt{\beta}r^2$, we see that this is exactly Whittaker's equation,

$$\widetilde{\widetilde{\phi}}_m''(z) + \left(-\frac{1}{4} + \frac{\lambda}{z} + \frac{1/4 - \mu^2}{z^2}\right)\widetilde{\widetilde{\phi}}_m(z) = 0 \tag{2.13}$$

with parameters $\lambda = E/4\sqrt{\beta}$, $\mu = (1/2)|\alpha - m|$. As shown by Whittaker and Watson [7], when $2\mu \notin \mathbb{Z} \setminus \{0\}$ this differential equation has two linearly independent solutions, namely,

$$M_{\lambda,\pm\mu}(z) = z^{\pm\mu+1/2}e^{-z/2}\Phi\left(\pm\mu - \lambda + \frac{1}{2}, 2\mu + 1; z\right), \tag{2.14}$$

where Φ is a hypergeometric series given by

$$\Phi(\gamma,\delta;z) = 1 + \frac{\gamma}{\delta}\frac{z}{1!} + \frac{\gamma(\gamma+1)}{\delta(\delta+1)}\frac{z^2}{2!} + \frac{\gamma(\gamma+1)(\gamma+2)}{\delta(\delta+1)(\delta+2)}\frac{z^3}{3!} + \cdots. \tag{2.15}$$

We deduce that

$$\widetilde{\phi}_m^{\pm}(r) = \frac{M_{E/4\sqrt{\beta},\pm(1/2)|\alpha-m|}\left(\sqrt{\beta}r^2\right)}{\sqrt{r}} \tag{2.16}$$

form a fundamental set of solutions of (2.10). These solutions are, however, not necessarily eigenfunctions of the Friedrichs extension. We will now examine this via the quadratic form.

It is easy to see that

$$M_{\lambda,\mu}(z) = z^{\pm\mu+1/2}\left(1 + \mathbb{O}(z)\right)^2 = \mathbb{O}\left(z^{\pm\mu+1/2}\right) \tag{2.17}$$

for *small* z. Hence,

$$\frac{\widetilde{\phi}_m^{\pm}(r)}{r} = \mathbb{O}\left(\left(r^2\right)^{\pm(1/2)|\alpha-m|+1/2}r^{-3/2}\right) = \mathbb{O}\left(r^{\pm|\alpha-m|-1/2}\right) \tag{2.18}$$

and

$$\frac{\mathrm{d}\tilde{\phi}_m^{\pm}}{\mathrm{d}r} = \mathcal{O}\big(r^{-1/2 \pm |\alpha - m|}\big). \tag{2.19}$$

As $r^a \in L^2([0,1],\mathrm{d}r)$ if and only if $a > -1/2$, the quadratic form (2.7) is unbounded for all $\tilde{\phi}_m^-$, which therefore cannot be eigenfunctions. Indeed, linear combinations $a_+\tilde{\phi}_m^+ + a_-\tilde{\phi}_m^-$ can also be excluded, since $\tilde{\phi}_m^+$ is always integrable at the origin and no cancellation can occur.

For *large* z, the Whittaker functions have the following asymptotics [7]:

$$M_{\lambda,\mu}(z) = \left(\frac{e^{i\pi\lambda}\Gamma(2\mu+1)}{\Gamma(\mu-\lambda+1/2)}(-z)^{-\lambda}e^{z/2} + \frac{e^{i\pi(\mu-\lambda+1/2)}\Gamma(2\mu+1)}{\Gamma(\mu+\lambda+1/2)}z^{\lambda}e^{-z/2}\right)\big(1+\mathcal{O}(z^{-1})\big). \tag{2.20}$$

We deduce that

$$\begin{aligned}\tilde{\phi}_m^+(r) = &\left(\frac{e^{i\pi(E/4\sqrt{\beta})}\Gamma\big(|\alpha-m|+1\big)}{\Gamma\big((1/2)|\alpha-m| - E/4\sqrt{\beta}+1/2\big)}\big(-\sqrt{\beta}r^2\big)^{-E/4\sqrt{\beta}}r^{-1/2}e^{\sqrt{\beta}r^2/2}\right.\\ &\left.+\frac{e^{i\pi((1/2)|\alpha-m|-E/4\sqrt{\beta}+1/2)}\Gamma\big(|\alpha-m|+1\big)}{\Gamma\big((1/2)|\alpha-m|+E/4\sqrt{\beta}+1/2\big)}\big(\sqrt{\beta}r^2\big)^{E/4\sqrt{\beta}}r^{-1/2}e^{-\sqrt{\beta}r^2/2}\right)\\ &\times\big(1+\mathcal{O}(r^{-2})\big).\end{aligned} \tag{2.21}$$

The first term in this expression is not integrable. To make it vanish, we choose E in order that the denominator's gamma function be singular, that is,

$$\frac{1}{2}|\alpha-m| - \frac{E}{4\sqrt{\beta}} + \frac{1}{2} = -n \Longleftrightarrow E = 2\sqrt{\beta}\big(1+|\alpha-m|\big)+4\sqrt{\beta}n \tag{2.22}$$

for some n in $\mathbb{N}_0 = \{0,1,2,\ldots\}$. With this choice of E, we obtain the finite number

$$\frac{e^{-i\pi n}\Gamma\big(|\alpha-m|+1\big)}{\Gamma\big(1+|\alpha-m|+n\big)} = \left(\frac{-1}{1+|\alpha-m|}\right)^n \tag{2.23}$$

as a coefficient of the integrable term. It remains to verify that the derivative is also integrable for large r. Differentiating the second term in (2.21) gives us two terms of the form $r^a e^{-\sqrt{\beta}r^2/2}$. Clearly both terms are square integrable away from zero.

The preceding discussion can be summarised in the following theorem.

THEOREM 2.1. *The $L^2(\mathbb{R}^2)$ eigenfunctions of the operator (1.1) with*

$$\vec{A}(x_1,x_2) = \frac{\alpha(-x_2,x_1)}{|x|^2}, \qquad V(x_1,x_2) = \beta|x|^2, \tag{2.24}$$

where $\alpha \in \mathbb{R}\setminus\mathbb{Z}$ and $\beta \in \mathbb{R}_+$, are

$$\frac{e^{im\theta}}{r}M_{E(m,n)/4\sqrt{\beta},(1/2)|\alpha-m|}\big(\sqrt{\beta}r^2\big), \tag{2.25}$$

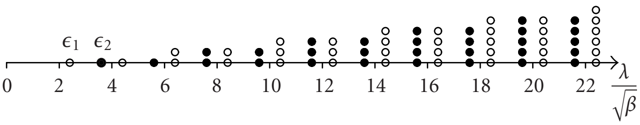

Figure 2.1. The first eigenvalues, normalised by $\sqrt{\beta}$.

where $m \in \mathbb{Z}$ and $M_{\lambda,\mu}$ is defined in (2.14). The eigenvalues are

$$E(m,n) = 2\sqrt{\beta}(1+|\alpha - m| + 2n), \quad n \in \mathbb{N}_0. \tag{2.26}$$

The multiplicity of a given eigenvalue equals the number of times it appears as m runs over $\mathbb{Z}$ and n over $\mathbb{N}_0$.

2.1.3. Eigenvalues. For future convenience, we will write the eigenvalues as two increasing sequences:

$$E_{j,p} = \epsilon_j + 2\sqrt{\beta}p, \quad j = 1,2,\ p \in \mathbb{N}_0. \tag{2.27}$$

Here ϵ_j denotes the lowest eigenvalues,

$$\begin{aligned} \epsilon_1 &= \min_{m\in\mathbb{Z}} 2\sqrt{\beta}(1+|\alpha - m|) = 2\sqrt{\beta}(1+|\alpha - m_1|), \\ \epsilon_2 &= \min_{m_1 \neq m \in \mathbb{Z}} 2\sqrt{\beta}(1+|\alpha - m|) = 6\sqrt{\beta} - \epsilon_1, \end{aligned} \tag{2.28}$$

which coincide if α is a half-integer. In fact,

$$1 + |\alpha - m| + 2n = \epsilon_j + m' + 2n = \epsilon_j + p, \tag{2.29}$$

and since $p = m' + 2n$ has $\lfloor p/2 \rfloor + 1$ solutions in $\mathbb{N}_0 \times \mathbb{N}_0$, the multiplicity of the eigenvalue $E_{j,p}$ will be $N(p) = \lfloor p/2 \rfloor + 1$.

In Figure 2.1, we have plotted the first eigenvalues. The spectrum has a close connection with that of the two-dimensional harmonic oscillator,

$$E_{\text{h.o.}}(p) = 2p, \quad N_{\text{h.o.}}(p) = p, \quad p = 1,2,\dots. \tag{2.30}$$

The eigenvalues have moved apart from their original positions by a distance which is proportional to the fractional part of α.

2.2. Eigenvalue inequalities. We now consider the two-dimensional Lieb-Thirring inequality

$$\operatorname{Tr}(H(\vec{A}, V) - \lambda)_-^\gamma \le \frac{R_\gamma}{(2\pi)^2} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (a(x,\xi) - \lambda)_-^\gamma \, \mathrm{d}x\, \mathrm{d}\xi, \tag{2.31}$$

which is known to hold for all $\gamma > 0$ for the harmonic oscillator in the absence of a magnetic field. In this case, the constant $R_\gamma = 1$ if $\gamma \geq 1$ [3], but as a general fact $R_\gamma > 1$ if $\gamma < 1$ [4]. In the special case $\gamma = 0$, the inequality is usually named for Cwikel, Lieb, and Rozenblyum. It fails for nonmagnetic systems unless the number of dimensions is at least 3.

By unitary equivalence, (2.31) holds for the Aharonov-Bohm operator if the magnetic potential has integer magnitude α. We will address the question whether this is true also in the case of noninteger magnitude. We are led to study the cases $\gamma = 0$ and $\gamma = 1$ by the above prediction and the well-known result by Aizenman and Lieb [1]: *if R_γ is finite for some $\gamma \geq 0$, then $R_{\gamma'} \leq R_\gamma$ for all $\gamma' \geq \gamma$.*

2.2.1. Right-hand side. Let us first calculate the right-hand side of (2.31). The Schrödinger operator $H(\vec{A}, V)$ is a pseudodifferential operator with symbol

$$a(x,\xi) = \left(-\xi_1 - \frac{\alpha x_2}{|x|^2}, -\xi_2 + \frac{\alpha x_1}{|x|^2}\right)^2 + \beta |x|^2. \tag{2.32}$$

By means of the substitution,

$$y_1 = \sqrt{\beta} x_1, \qquad y_2 = \sqrt{\beta} x_2, \qquad \eta_1 = -\xi_1 - \frac{\alpha x_2}{|x|^2}, \qquad \eta_2 = -\xi_2 + \frac{\alpha x_1}{|x|^2}, \tag{2.33}$$

the symbol simplifies to $|\eta|^2 + |y|^2$. The integral is therefore zero for $\lambda \leq 0$. For positive λ, we have

$$\begin{aligned}
\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (a(x,\xi) - \lambda)_-^\gamma \, \mathrm{d}x\, \mathrm{d}\xi &= \frac{1}{\beta}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (|y|^2 + |\eta|^2 - \lambda)_-^\gamma \, \mathrm{d}y\, \mathrm{d}\eta \\
&= \frac{1}{\beta}\iint_{|y|^2+|\eta|^2 \leq \lambda} (\lambda - |y|^2 - |\eta|^2)^\gamma \, \mathrm{d}y\, \mathrm{d}\eta \\
&= \frac{(2\pi)^2}{\beta}\iint_{\substack{r,\rho \geq 0 \\ r^2+\rho^2 \leq \lambda}} (\lambda - r^2 - \rho^2)^\gamma r\rho \, \mathrm{d}r\, \mathrm{d}\rho \\
&= \frac{(2\pi)^2}{\beta}\int_0^{\pi/2} \cos\psi \sin\psi \, \mathrm{d}\psi \underbrace{\int_0^{\sqrt{\lambda}} (\lambda - R^2)^\gamma R^3 \, \mathrm{d}R}_{=(1/2)\lambda^{\gamma+2}B(\gamma+1,2)} \\
&= (2\pi)^2 \frac{\lambda^{\gamma+2}}{4\beta(\gamma+1)(\gamma+2)}.
\end{aligned} \tag{2.34}$$

The result is independent of the magnetic field.

2.2.2. Left-hand side, case $\gamma = 0$. The left-hand side can be written as

$$\operatorname{Tr}\left(H(\vec{A}, V) - \lambda\right)_-^\gamma = \sum_{j=1}^{2}\sum_{p=0}^{\infty} N(p)(\lambda - E_{j,p})_+^\gamma, \tag{2.35}$$

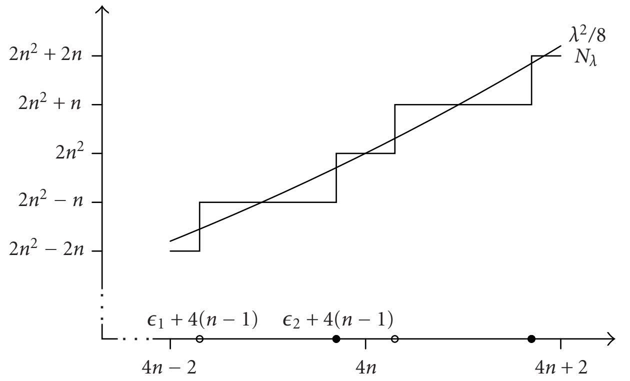

Figure 2.2. Plots of $\lambda^2/8$ and N_λ on the interval $[4n-2, 4n+2]$.

which, if $\gamma = 0$, is simply the number N_λ of eigenvalues (counted with their multiplicities) less than or equal to λ. For any γ, we can restrict the computations to the case $\beta = 1$ because $\sum_{j,p} N(p)(\lambda - E_{j,p})_+^\gamma \le R_\gamma \lambda^{\gamma+2}/4(\gamma+1)(\gamma+2)$ implies that

$$\begin{aligned}\sum_{j,p} N(p)\left(\lambda - \sqrt{\beta}E_{j,p}\right)_+^\gamma &= \beta^{\gamma/2}\sum_{j,p} N(p)\left(\frac{\lambda}{\sqrt{\beta}} - E_{j,p}\right)_+^\gamma \\ &\le \beta^{\gamma/2}\frac{R_\gamma\left(\lambda/\sqrt{\beta}\right)^{\gamma+2}}{4(\gamma+1)(\gamma+2)} = \frac{R_\gamma \lambda^{\gamma+2}}{4\beta(\gamma+1)(\gamma+2)}.\end{aligned} \tag{2.36}$$

Since $2 < \epsilon_j \le 3$ irrespectively of α, there is exactly one point in the spectrum between two consecutive integers. The sum (2.35) is particularly easy to compute if λ is an even integer. Recall that the spectrum begins at 2 and that the interval $[4p-2, 4p+2]$ contains four eigenvalue points, each with multiplicity p. Thus, if $\lambda = 4n+2$,

$$N_\lambda = \sum_{p=1}^{n} 4p = 2n(n+1) = \left(\frac{\lambda}{2} - 1\right)\left(\frac{1}{2}\left(\frac{\lambda}{2} - 1\right) + 1\right) = \frac{\lambda^2}{8} - \frac{1}{2}. \tag{2.37}$$

Similarly, if $\lambda = 4n$,

$$N_\lambda = \sum_{p=1}^{n} 4p - 2n = 2n^2 = \frac{\lambda^2}{8}. \tag{2.38}$$

Figure 2.2 contains all the information needed to determine a lower bound on the constant

$$R_0 = \sup_\lambda \frac{N_\lambda}{\lambda^2/8}. \tag{2.39}$$

N_λ being non-decreasing, this supremum is necessarily attained at some point of the spectrum, where N_λ has a jump increase. Formulae (2.28) tell us that, for example,

$$\frac{N_{\epsilon_1+4(n-1)}}{\left(\epsilon_1+4(n-1)\right)^2/8}=\frac{8\left(2n^2-n\right)}{2^2\left(1+2(n-1)+|\alpha-m_1|\right)^2}=\frac{2n(2n-1)}{\left(2n-1+|\alpha-m_1|\right)^2}. \tag{2.40}$$

Hence, the interval $[4n-2,4n+2]$ provides the bound

$$\begin{aligned}R_0\geq\max\Bigg\{&\frac{2n(2n-1)}{\left(2n-1+|\alpha-m_1|\right)^2},\frac{4n^2}{\left(2n-|\alpha-m_1|\right)^2},\\&\frac{2n(2n+1)}{\left(2n+|\alpha-m_1|\right)^2},\frac{4n(n+1)}{\left(2n+1-|\alpha-m_1|\right)^2}\Bigg\}.\end{aligned} \tag{2.41}$$

We may view these expressions as functions of $n=1,2,3,\ldots$. They are decreasing if, respectively,

$$n>\frac{|\alpha-m_1|-1}{2\left(2|\alpha-m_1|-1\right)};\quad n>0;\quad n>\frac{|\alpha-m_1|}{2-4|\alpha-m_1|};\quad n>\frac{1-|\alpha-m_1|}{2|\alpha-m_1|}. \tag{2.42}$$

A somewhat lengthy but altogether elementary examination of all possible cases shows that it is enough to consider $n=1$, that is, to solve the maximisation problem on the interval $[2,6]$. The conclusion is the following theorem.

Theorem 2.2. *When $\gamma=0$, inequality (2.31) is sharp with*

$$R_0=\begin{cases}\dfrac{2}{\left(1+|\alpha-m_1|\right)^2} & \text{if } 0<|\alpha-m_1|\leq 3\sqrt{2}-4,\\[2ex] \dfrac{1}{\left(1-(1/2)|\alpha-m_1|\right)^2} & \text{if } 3\sqrt{2}-4\leq|\alpha-m_1|\leq\dfrac{1}{2}.\end{cases} \tag{2.43}$$

Apparently R_0 is always greater than or equal to $2(1+\sqrt{2})^2/9\approx 1.295$ (which indeed confirms the result in [4]) and $R_0\uparrow 2$ as α approaches an integer. This fact can also be established by direct calculations with the nonmagnetic eigenvalues (2.30).

2.2.3. Left-hand side, case $\gamma=1$. We attempt to show that $R_1=1$, as in the case of the harmonic oscillator. Taking $\beta=1$ as previously, we will prove that the quantity (2.35) does not exceed $\lambda^3/24$. Since each point in the spectrum stays between the same consecutive integers when α varies, and because $\epsilon_1+\epsilon_2=6$, the sum is independent of α when λ is an even integer. We will compute the sum for such λ and then use convexity to determine the value of the constant.

Consider first $\lambda=4n+2$ and write $[2,4n+2]=\bigcup_{p=1}^{n}[4p-2,4p+2]$. The four points in the spectrum located on $[4p-2,4p+2]$ all have multiplicity p. They contribute to the sum in the following way:

$$\begin{aligned}&4(p-1)+\epsilon_1,\\&4(p-1)+6-\epsilon_1,\\&4(p-1)+2+\epsilon_1,\\&4(p-1)+8-\epsilon_1,\end{aligned} \tag{2.44}$$

give, respectively,

$$\begin{aligned}p(4n+2-(4(p-1)+\epsilon_1)) &= p(4(n-p)+6-\epsilon_1),\\ p(4n+2-(4(p-1)+6-\epsilon_1)) &= p(4(n-p)+\epsilon_1),\\ p(4n+2-(4(p-1)+\epsilon_1+2)) &= p(4(n-p)+4-\epsilon_1),\\ p(4n+2-(4(p-1)+8-\epsilon_1)) &= p(4(n-p)-2+\epsilon_1).\end{aligned} \tag{2.45}$$

The sum of these terms is $8((2n+1)p-2p^2)$. Summing over all intervals, we get

$$\begin{aligned}\sum_{j=1}^{2}\sum_{p=0}^{\infty}N(p)(4n+2-E_{j,p})_+ &= 8\sum_{p=1}^{n}((2n+1)p-2p^2)\\ &= 8\left((2n+1)\frac{n(n+1)}{2}-2\left(\frac{(n+1)^3}{3}-\frac{(n+1)^2}{2}+\frac{(n+1)}{6}\right)\right)\\ &= 8\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right).\end{aligned} \tag{2.46}$$

Next, to treat $\lambda = 4n$, we split $[2,4n] = \bigcup_{p=1}^{n-1}[4p-2,4p+2]\cup[4n-2,4n]$. On each of the subintervals $[4p-2,4p+2]$, where the multiplicity is p, we note that

$$\begin{aligned}&4(p-1)+\epsilon_1,\\&4(p-1)+6-\epsilon_1,\\&4(p-1)+2+\epsilon_1,\\&4(p-1)+8-\epsilon_1,\end{aligned} \tag{2.47}$$

give, respectively,

$$\begin{aligned}&p(4(n-p)+4-\epsilon_1),\\&p(4(n-p)-2+\epsilon_1),\\&p(4(n-p)+2-\epsilon_1),\\&p(4(n-p)-4+\epsilon_1).\end{aligned} \tag{2.48}$$

These terms sum to $16p(n-p)$, and in all we get

$$\sum_{p=1}^{n-1}16p(n-p) = \frac{8n}{3}(n^2-1). \tag{2.49}$$

Finally on $[4n-2,4n]$, the eigenvalues $4n-4+\epsilon_1$ and $4n+2-\epsilon_1$, each with multiplicity n, contribute

$$n(4n-(4n-4+\epsilon_1))+n(4n-(4n+2-\epsilon_1)) = 2n. \tag{2.50}$$

Thus,

$$\sum_{j=1}^{2}\sum_{p=0}^{\infty} N(p)(4n-E_{j,p})_+ = \frac{8n}{3}(n^2-1)+2n = 8\left(\frac{n^3}{3}-\frac{n}{12}\right). \tag{2.51}$$

If we substitute n as a function of λ in (2.46) or (2.51), we obtain

$$\sum_{j=1}^{2}\sum_{p=0}^{\infty} N(p)(\lambda-E_{j,p})_+ = \frac{\lambda^3}{24}-\frac{\lambda}{6} \quad \text{if } \lambda = 2,4,6,\ldots \tag{2.52}$$

in both cases. (Actually (2.46) and (2.51) are only valid if $n \geq 1$ but a simple calculation shows that $\lambda = 2$ need not be excluded.) In the intervals between even integers, we can prove the same thing by convexity. The Lieb-Thirring sum is a piecewise affine function of λ, and since the first-order coefficient equals the number of eigenvalues below λ, it is also convex. Assume that $\bar{\lambda}$ is an even integer and let $\lambda = \bar{\lambda}+2t$, $0<t<1$. By Jensen's inequality,

$$\begin{aligned}\sum_{j=1}^{2}\sum_{p=0}^{\infty} N(p)(\lambda-E_{j,p})_+ &\leq \left(\frac{\bar{\lambda}^3}{24}-\frac{\bar{\lambda}}{6}\right)(1-t)+\left(\frac{(\bar{\lambda}+2)^3}{24}-\frac{\bar{\lambda}+2}{6}\right)t \\ &= \frac{(\bar{\lambda}+2t)^3}{24}+h(t), \quad \text{where } h(t) = -\frac{t^3}{3}+\bar{\lambda}\left(-\frac{t^2}{2}+\frac{t}{2}-\frac{1}{6}\right).\end{aligned} \tag{2.53}$$

Noting that $h'(t) = -t^2+\bar{\lambda}(1/2-t)$, we see that h has a local maximum in $(0,1)$, namely

$$h\left(\frac{1}{1+\sqrt{1+2/\bar{\lambda}}}\right) = -\frac{\bar{\lambda}+2}{6\left(1+\sqrt{1+2/\bar{\lambda}}\right)^2} < 0 \quad \forall \bar{\lambda} \geq \epsilon_1 > 2. \tag{2.54}$$

Hence,

$$\sum_{j=1}^{2}\sum_{p=0}^{\infty} N(p)(\lambda-E_{j,p})_+ \leq \frac{\lambda^3}{24}\left(1-\frac{4(\bar{\lambda}+2)}{\lambda^3\left(1+\sqrt{1+2/\bar{\lambda}}\right)^2}\right). \tag{2.55}$$

The last factor will tend to one as $\lambda \to \infty$, which proves the following theorem.

Theorem 2.3. *When $\gamma = 1$, inequality (2.31) is sharp with $R_1 = 1$.*

3. Coulomb potential

3.1. Spectrum and eigenfunction. Treating now the case of Coulomb scalar potential, we will see that the eigenvalue problem can again be reduced to Whittaker's equation. The spectrum is, however, very dissimilar to what was found in the case of the quadratic potential.

3.1.1. Preparations. Using again the decomposition (2.1), we obtain the differential expression

$$H(\vec{A},V)=-\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\left(i\frac{\partial}{\partial\theta}+\alpha\right)-\frac{\beta}{r}=\bigoplus_{m\in\mathbb{Z}}(H_m\otimes I_m), \tag{3.1}$$

where

$$H_m=-\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}+\frac{1}{r^2}(\alpha-m)^2-\frac{\beta}{r}. \tag{3.2}$$

As earlier, the first-order term can be removed by unitary equivalence under the mapping (2.4). One then obtains

$$\widetilde{H}_m=UH_mU^{-1}=-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{(\alpha-m)^2-1/4}{r^2}-\frac{\beta}{r}. \tag{3.3}$$

This allows us to define the quadratic form of the operator in the Coulomb case:

$$\widetilde{a}[u]=\sum_{m=-\infty}^{\infty}\widetilde{a}_m[u_m], \tag{3.4}$$

where

$$\widetilde{a}_m[u]=\int_0^\infty\left(\left|\frac{\mathrm{d}u}{\mathrm{d}r}\right|^2+\frac{(\alpha-m)^2-1/4}{r^2}|u|^2-\beta\frac{|u|^2}{r}\right)\mathrm{d}r. \tag{3.5}$$

Using the one-dimensional classical Hardy inequality, one can easily show that the quadratic form (3.4) is lower semibounded and closed on the domain $H_0^1(\mathbb{R}^2)$. This observation will simplify the examination of which formal solutions are actually eigenfunctions of the Friedrichs extension of (3.1). We then merely have to verify that the solutions belong to $H_0^1(\mathbb{R}^2)$.

3.1.2. Eigenfunctions. We now turn to the equation

$$\widetilde{H}_m\widetilde{\phi}_m=E\widetilde{\phi}_m,\quad \widetilde{\phi}_m=U\phi_m. \tag{3.6}$$

It is equivalent to

$$\widetilde{\phi}_m''(r)-4E\left(-\frac{1}{4}-\frac{\beta/4E}{r}-\frac{1/4-(\alpha-m)^2}{4Er^2}\right)\widetilde{\phi}_m(r)=0. \tag{3.7}$$

Since the Coulomb potential is not confining, $E\geq 0$ corresponds to scattering states and so we can restrict our study to $E<0$. We then have

$$\widetilde{\phi}_m''(r)+4|E|\left(-\frac{1}{4}+\frac{\beta}{4|E|r}+\frac{1/4-(\alpha-m)^2}{4|E|r^2}\right)\widetilde{\phi}_m(r)=0, \tag{3.8}$$

which can be rewritten as Whittaker's equation,

$$\widetilde{\phi}_m''(z) + \left(-\frac{1}{4} + \frac{\lambda}{z} + \frac{1/4 - (\alpha - m)^2}{z^2} \right) \widetilde{\phi}_m(z) = 0, \tag{3.9}$$

with $z = 2\sqrt{|E|}r$, $\lambda = \beta/2\sqrt{|E|}$, and $\mu = |\alpha - m|$. Its solutions are (cf. Section 2.1.2) $M_{\lambda,\mu}(z)$ and $M_{\lambda,-\mu}(z)$, the latter of which is not defined if $2\mu \in \mathbb{Z} \setminus \{0\}$. With the new definition of μ, the exceptional case occurs whenever α is a half-integer, but as only either of the solutions obtained for each m is integrable, this will not cause any difficulties.

We know that a fundamental system of solutions is

$$\widetilde{\phi}_m(r) = \frac{1}{\sqrt{r}} M_{\lambda,\pm\mu}\left(2\sqrt{|E|}r\right). \tag{3.10}$$

We use the same approach as in Section 2.1.2 to check that these functions lie in the domain of the Friedrichs extension, that is, in the closure of $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ with respect to (3.4) or, equivalently, the H_0^1 norm. For small r,

$$\widetilde{\phi}_m^\pm = \mathbb{O}\left(r^{\pm|\alpha-m|+1/2}\right), \qquad \frac{d\widetilde{\phi}_m^\pm}{dr} = \mathbb{O}\left(r^{-1/2\pm|\alpha-m|}\right), \tag{3.11}$$

and hence $\widetilde{\phi}_m^-$ can be excluded for all m. On the other hand, when r is large,

$$\begin{aligned} \widetilde{\phi}_m^+(r) = \Bigg(& \frac{e^{i\pi\lambda}\Gamma(2|\alpha-m|+1)}{\Gamma(|\alpha-m|-\lambda+1/2)} \left(-2\sqrt{|E|}r\right)^{-\lambda} e^{\sqrt{|E|}r} \\ & + \frac{e^{i\pi(|\alpha-m|-\lambda+1/2)}\Gamma(2|\alpha-m|+1)}{\Gamma(|\alpha-m|+\lambda+1/2)} \left(2\sqrt{|E|}r\right)^{\lambda} e^{-\sqrt{|E|}r} \Bigg) \left(1 + \mathbb{O}\left(r^{-1}\right)\right). \end{aligned} \tag{3.12}$$

Repeating our argument from Section 2.1.2, finiteness of the quadratic form requires that

$$|\alpha - m| - \frac{\beta}{2\sqrt{|E|}} + \frac{1}{2} = -n \Longleftrightarrow E = -\left(\frac{\beta/2}{n + |\alpha - m| + 1/2}\right)^2, \quad n \in \mathbb{N}_0. \tag{3.13}$$

Clearly, the operator H_m has a sequence of negative, discrete eigenvalues starting at $-(\beta/2(|\alpha - m| + 1/2))^2$ and accumulating towards zero.

Winding up, we arrive at the following theorem.

Theorem 3.1. *The $L^2(\mathbb{R}^2)$ eigenfunctions of the operator (1.1) with*

$$\vec{A}(x_1,x_2) = \frac{\alpha(-x_2,x_1)}{|x|^2}, \qquad V(x_1,x_2) = -\frac{\beta}{|x|}, \tag{3.14}$$

where $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ and $\beta \in \mathbb{R}_+$, are

$$\frac{e^{im\theta}}{\sqrt{r}} M_{\beta/2\sqrt{|E(m,n)|},|\alpha-m|}\left(2\sqrt{|E(m,n)|}r\right), \tag{3.15}$$

where $m \in \mathbb{Z}$ and $M_{\lambda,\mu}$ is defined in (2.14). The eigenvalues are

$$E(m,n) = -\left(\frac{\beta/2}{n+|\alpha-m|+1/2}\right)^2, \quad n \in \mathbb{N}_0. \tag{3.16}$$

The multiplicity of a given eigenvalue equals the number of times it appears as m runs over $\mathbb{Z}$ and n over $\mathbb{N}_0$.

3.2. Eigenvalue inequalities. In this section, we return to Lieb-Thirring's inequality (2.31) and examine when it holds for the Aharonov-Bohm operator with Coulomb potential. Since the discrete spectrum is entirely situated on the negative real axis, only negative values of λ are interesting.

3.2.1. Right-hand side. The symbol of the operator is now

$$a(x,\xi) = \left(-\xi_1 - \frac{\alpha x_2}{|x|^2}, -\xi_2 + \frac{\alpha x_1}{|x|^2}\right)^2 - \frac{\beta}{|x|}. \tag{3.17}$$

Proceeding the same way as in Section 2.2.1, we obtain for all $\lambda < 0$ and $0 \le \gamma < 1$

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (a(x,\xi)-\lambda)_-^\gamma \,\mathrm{d}x\,\mathrm{d}\xi \\
&\quad = \beta^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(|\eta|^2 - \frac{1}{|y|} - \lambda\right)_-^\gamma \mathrm{d}y\,\mathrm{d}\eta \\
&\quad = (2\pi\beta)^2 \int_0^{-1/\lambda}\int_0^{\sqrt{\lambda+1/r}} \left(\lambda - \rho^2 + \frac{1}{r}\right)^\gamma r\rho\,\mathrm{d}r\,\mathrm{d}\rho = \frac{(2\pi\beta)^2}{2(\gamma+1)} \int_0^{-1/\lambda} \left(\frac{1}{r} + \lambda\right)^{\gamma+1} r\,\mathrm{d}r \\
&\quad = \frac{(2\pi\beta)^2}{2(\gamma+1)} \int_0^\infty \frac{s^{\gamma+1}}{(s-\lambda)^3}\mathrm{d}s = \frac{(2\pi\beta)^2}{2(\gamma+1)} |\lambda|^{\gamma-1} \frac{\gamma\pi}{\sin\gamma\pi}\frac{\gamma+1}{2} = (2\pi)^2\left(\frac{\beta}{2}\right)^2 \frac{\gamma\pi}{\sin\gamma\pi}|\lambda|^{\gamma-1}.
\end{aligned} \tag{3.18}$$

(To compute the last integral, we used a contour situated on both sides of the branch cut.) The integral diverges for $\gamma \ge 1$, and then the Lieb-Thirring inequality is trivial.

3.2.2. Left-hand side, case $\gamma = 0$. As in the case of quadratic potential, we will write the eigenvalues (3.16) in an "ordered" way, by giving new meaning to the notation in Section 2.1.3. We redefine

$$\begin{aligned}
\epsilon_1 &= \min_{m\in\mathbb{Z}} |\alpha - m| + \frac{1}{2} = |\alpha - m_1| + \frac{1}{2}, \\
\epsilon_2 &= \min_{m_1 \ne m \in \mathbb{Z}} |\alpha - m| + \frac{1}{2} = 2 - \epsilon_1 \ge \epsilon_1.
\end{aligned} \tag{3.19}$$

The eigenvalues can then be written in the following way:

$$E_{j,p} = -\left(\frac{\beta/2}{\epsilon_j + p}\right)^2, \quad j = 1,2,\ p \in \mathbb{N}_0, \tag{3.20}$$

with multiplicity $N(p) = \lfloor p/2 \rfloor + 1$. The eigenvalues define the subintervals

$$I_{1,p} = [E_{1,p}, E_{2,p}), \qquad I_{2,p} = [E_{2,p}, E_{1,p+1}), \tag{3.21}$$

which clearly constitute a partition of the interval $[E_{1,0}, 0)$. If α is a half-integer, $E_{1,p}$ and $E_{2,p}$ coincide so that $I_{1,p} = \emptyset$. In the other limiting case, when α approaches an integer, $E_{2,p}$ will tend to $E_{1,p+1}$, thus making $I_{2,p}$ vanish.

The problem is to find a constant R_0 such that

$$N_\lambda \le R_0 \left(\frac{\beta}{2}\right)^2 |\lambda|^{-1} \quad \forall \lambda < 0, \tag{3.22}$$

or, equivalently, to determine

$$R_0 = \left(\frac{\beta}{2}\right)^2 \sup_{\lambda<0} N_\lambda |\lambda|. \tag{3.23}$$

By an argument similar to that in Section 2.2.2, R_0 is independent of β. To simplify the calculations, we therefore assume $\beta = 2$.

We first consider $p = 0$. On $I_{1,0}$ we have $N_\lambda = 1$, so that necessarily

$$R_0 \ge \sup_{I_{1,0}} N_\lambda |\lambda| = |E_{1,0}| = \frac{1}{\epsilon_1^2}. \tag{3.24}$$

From $I_{2,0}$, where $N_\lambda = 2$, we obtain the lower bound

$$R_0 \ge \sup_{I_{2,0}} N_\lambda |\lambda| = 2|E_{2,0}| = \frac{2}{(2-\epsilon_1)^2}. \tag{3.25}$$

Actually the supremum will always be attained either on $I_{1,0}$ or $I_{2,0}$. To see this, we will prove an upper bound on such values of R_0 that are obtained upon maximising (3.23) with λ restricted to intervals $I_{j,p}$, $p \ge 1$. We have

$$N_\lambda = \begin{cases} \displaystyle\sum_{q=0}^{p-1} 2\left(\left\lfloor \frac{q}{2} \right\rfloor + 1\right) + \left\lfloor \frac{p}{2} \right\rfloor + 1 & \text{if } \lambda \in I_{1,p}, \\ \displaystyle\sum_{q=0}^{p} 2\left(\left\lfloor \frac{q}{2} \right\rfloor + 1\right) & \text{if } \lambda \in I_{2,p}. \end{cases} \tag{3.26}$$

Since $\sum_{q=0}^{p} \lfloor q/2 \rfloor \le p^2/4$, we readily obtain

$$\begin{aligned} N_\lambda |\lambda| &\le \frac{p^2+3p+3}{2}\left(\frac{1}{\epsilon_1 + p}\right)^2 \quad \text{if } \lambda \in I_{1,p}, \\ N_\lambda |\lambda| &\le \frac{(p+2)^2}{2}\left(\frac{1}{\epsilon_2 + p}\right)^2 = \frac{1}{2\left(1 - \epsilon_1/(p+2)\right)^2} \quad \text{if } \lambda \in I_{2,p}. \end{aligned} \tag{3.27}$$

Clearly both bounds decrease as functions of p. It is also easy to verify that the value of R_0, as in (3.24) and (3.25), is always greater than that in (3.27) (with $p = 1$) for a given ϵ_1.

Hence,

$$R_0 = \max\left\{\frac{1}{\epsilon_1^2}, \frac{2}{(2-\epsilon_1)^2}\right\}. \tag{3.28}$$

Writing ϵ_1 explicitly, we can state the following theorem.

Theorem 3.2. *When $\gamma = 0$, inequality (2.31) is sharp with*

$$R_0 = \begin{cases} \dfrac{1}{(1/2 + |\alpha - m_1|)^2} & \text{if } 0 < |\alpha - m_1| \le 2\sqrt{2} - \dfrac{5}{2}, \\ \dfrac{2}{(3/2 - |\alpha - m_1|)^2} & \text{if } 2\sqrt{2} - \dfrac{5}{2} \le |\alpha - m_1| \le \dfrac{1}{2}. \end{cases} \tag{3.29}$$

We note that $R_0 \ge (\sqrt{2}+1)/2 \approx 1.207$ and $R_0 \uparrow 4$ when α tends to an integer. Another remark is that the leading term in the expansion of N_λ is $\lambda/2$, independently of α and β. This fact is suggested by the bounds (3.27) and we have been able to verify it by deriving closed expressions for finite sums over the multiplicities. Due to the positive higher-order terms, R_0 is however strongly influenced by the location of the lowest eigenvalues.

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References

[1] M. Aizenman and E. H. Lieb, *On semiclassical bounds for eigenvalues of Schrödinger operators*, Phys. Lett. A **66** (1978), no. 6, 427–429.

[2] A. A. Balinsky, W. D. Evans, and R. T. Lewis, *On the number of negative eigenvalues of Schrödinger operators with an Aharonov-Bohm magnetic field*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **457** (2001), no. 2014, 2481–2489.

[3] R. de la Bretèche, *Preuve de la conjecture de Lieb-Thirring dans le cas des potentiels quadratiques strictement convexes* [*Proof of the Lieb-Thirring conjecture in the case of strictly convex quadratic potentials*], Ann. Inst. H. Poincaré Phys. Théor. **70** (1999), no. 4, 369–380.

[4] B. Helffer and D. Robert, *Riesz means of bounded states and semi-classical limit connected with a Lieb-Thirring conjecture. II*, Ann. Inst. H. Poincaré Phys. Théor. **53** (1990), no. 2, 139–147.

[5] A. Laptev, *On the Lieb-Thirring conjecture for a class of potentials*, The Maz'ya Anniversary Collection, Vol. 2 (Rostock, 1998), Oper. Theory Adv. Appl., vol. 110, Birkhäuser, Basel, 1999, pp. 227–234.

[6] A. Laptev and I. M. Sigal, *Global Fourier integral operators and semiclassical asymptotics*, Rev. Math. Phys. **12** (2000), no. 5, 749–766.

[7] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge University Press, Cambridge, 1927.

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