A GENERALIZED LOGISTIC DISTRIBUTION

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A generalized logistic distribution is proposed, based on the fact that the difference of two independent Gumbel-distributed random variables has the standard logistic distribution.

1. Introduction

If X_1 and X_2 are independent Gumbel-distributed random variables with the common cdf

$$F(x) = \exp\{-\exp(-x)\}, \tag{1.1}$$

then it is well known that the difference $Z = X_1 - X_2$ has the standard logistic distribution with the pdf

$$f_Z(z) = \frac{\exp(z)}{\{1+\exp(z)\}^2} \tag{1.2}$$

for $-\infty < z < \infty$. The properties of this distribution and its generalizations have been studied by several authors. Of particular eminence are the numerous papers on this topic by Professor N. Balakrishnan and his colleagues; see, for example, Balakrishnan [1, 2, 3], Balakrishnan and Aggarwala [4], Balakrishnan et al. [5, 7, 12], Balakrishnan and Chan [6], Balakrishnan and Joshi [8], Balakrishnan and Kocherlakota [9], Balakrishnan and Leung [10], Balakrishnan and Malik [11], Balakrishnan and Puthenpura [13], Balakrishnan and Sandhu [14], and Balakrishnan and Wong [15].

In this short note, we construct a new generalization of (1.2) by taking X_i, $i = 1,2$, to have the general Gumbel distribution with the cdf

$$F_i(x) = \exp\left\{-\exp\left(-\frac{x-\mu_i}{\sigma_i}\right)\right\} \tag{1.3}$$

for $-\infty < x < \infty$, $-\infty < \mu_i < \infty$, and $\sigma_i > 0$. This distribution (which is also known as the extreme-value distribution of type I) has received special attention in the probabilistic-statistical literature and in various applications in the second half of the twentieth century.

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A recent book by Kotz and Nadarajah [16], which describes this distribution, lists over fifty applications ranging from accelerated life testing through to earthquakes, floods, horse racing, rainfall, queues in supermarkets, sea currents, wind speeds, and track race records (to mention just a few).

2. The generalization

The pdf corresponding to (1.3) is

$$f_i(x) = \frac{1}{\sigma_i}\exp\left(-\frac{x-\mu_i}{\sigma_i}\right)\exp\left\{-\exp\left(-\frac{x-\mu_i}{\sigma_i}\right)\right\}, \tag{2.1}$$

and thus the pdf of $Z = X_1 - X_2$ can be written as

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_1(x) f_2(x-z)\,dx \\ &= \frac{1}{\sigma_1\sigma_2}\int_{-\infty}^{\infty}\exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\exp\left\{-\exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\right\} \\ &\qquad \times\exp\left(-\frac{x-z-\mu_2}{\sigma_2}\right)\exp\left\{-\exp\left(-\frac{x-z-\mu_2}{\sigma_2}\right)\right\}dx. \end{aligned} \tag{2.2}$$

Setting $y = \exp(-x/\sigma_1)$, (2.2) can be expressed as

$$f_Z(z) = \frac{1}{\sigma_2}\exp\left(\frac{\mu_1}{\sigma_1}+\frac{\mu_2+z}{\sigma_2}\right) I(\mu_1,\mu_2,\sigma_1,\sigma_2), \tag{2.3}$$

where I denotes the integral

$$I(\mu_1,\mu_2,\sigma_1,\sigma_2) = \int_0^{\infty} y^{\sigma_1/\sigma_2}\exp\left[-\left\{\exp\left(\frac{\mu_1}{\sigma_1}\right)y+\exp\left(\frac{\mu_2+z}{\sigma_2}\right)y^{\sigma_1/\sigma_2}\right\}\right]dy. \tag{2.4}$$

We refer to (2.3) as the *generalized logistic* distribution. The integral term in (2.4) is difficult to calculate. However, for some particular choices of $(\mu_1,\mu_2,\sigma_1,\sigma_2)$, one can obtain the following explicit expressions.

(i) If $\sigma_1 = \sigma_2 = \sigma$, then by standard integration one can obtain

$$I = \frac{1}{\left\{\exp\left(\mu_1/\sigma\right)+\exp\left(\left(\mu_2+z\right)/\sigma\right)\right\}^2}. \tag{2.5}$$

If, in addition, $\mu_1 = \mu_2 = \mu$, then the above reduces to

$$I = \frac{\exp(-2\mu/\sigma)}{\{1+\exp(z/\sigma)\}^2}. \tag{2.6}$$

(ii) If $\sigma_1 = 2\sigma_2$, then one can show by [17, equation (2.3.15.7)] that

$$I = \frac{\alpha\sqrt{\pi\alpha}}{8}\exp\left(\frac{\alpha\beta^2}{4}\right)\left\{2\,\mathrm{erfc}\left(\frac{\sqrt{\alpha}\beta}{2}\right) + \alpha\beta^2\,\mathrm{erfc}\left(\frac{\sqrt{\alpha}\beta}{2}\right)\right\} - \frac{\alpha^2\beta}{4}, \tag{2.7}$$

where $\alpha = \exp\{(\mu_2 + z)/\sigma_2\}$, $\beta = \exp\{\mu_1/(2\sigma_2)\}$, and $\mathrm{erfc}(\cdot)$, denotes the complementary error function defined by

$$\mathrm{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)\,dt. \tag{2.8}$$

(iii) If $0 < \sigma_1/\sigma_2 = p/q < 1$ (where $p \geq 1$ and $q \geq 1$ are co-prime integers), then one can show by [17, equation (2.3.1.13)] that

$$\begin{aligned} I = {} & \sum_{j=0}^{q-1} \frac{(-\alpha)^j}{j!}\Gamma\left(1 + \frac{p(1+j)}{q}\right)\beta^{-(1+p(1+j)/q)} \\ & \times {}_{p+1}F_q\left(1, \Delta\left(p, 1 + \frac{p(1+j)}{q}\right); \Delta(q, 1+j); \frac{(-1)^q p^p \alpha^q}{q^q \beta^p}\right), \end{aligned} \tag{2.9}$$

where $\alpha = \exp\{(\mu_2 + z)/\sigma_2\}$, $\beta = \exp(\mu_1/\sigma_1)$, $\Delta(k, a)$ denotes the sequence

$$\Delta(k, a) = \frac{a}{k}, \frac{a+1}{k}, \ldots, \frac{a+k-1}{k}, \tag{2.10}$$

${}_mF_n$ denotes the generalized hypergeometric function defined by

$${}_mF_n(\alpha_1, \ldots, \alpha_m; \beta_1, \ldots, \beta_n; x) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_m)_k}{(\beta_1)_k \cdots (\beta_n)_k}\frac{x^k}{k!}, \tag{2.11}$$

and $(c)_k = c(c+1)\cdots(c+k-1)$ denotes the ascending factorial.

(iv) If $\sigma_1/\sigma_2 = p/q > 1$ (where $p \geq 1$ and $q \geq 1$ are coprime integers), then one can show again by [17, equation (2.3.1.13)]that

$$\begin{aligned} I = {} & \sum_{j=0}^{p-1} \frac{q(-\beta)^j}{pj!}\Gamma\left(1 + \frac{q(1+j)}{p}\right)\alpha^{-(1+q(1+j)/p)} \\ & \times {}_{q+1}F_p\left(1, \Delta\left(q, 1 + \frac{q(1+j)}{p}\right); \Delta(p, 1+j); \frac{(-1)^p q^q \beta^p}{p^p \alpha^q}\right), \end{aligned} \tag{2.12}$$

where $\alpha = \exp\{(\mu_2 + z)/\sigma_2\}$ and $\beta = \exp(\mu_1/\sigma_1)$.

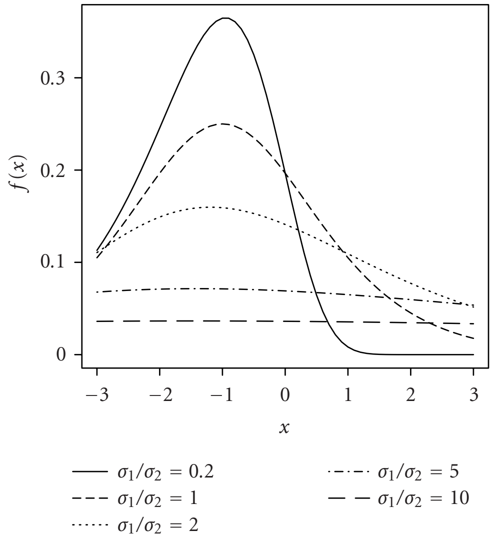

Figure 2.1. The generalized logistic pdf (2.3) for $\sigma_1/\sigma_2 = 0.2, 1, 2, 5, 10$, $\sigma_2 = 1$, $\mu_1 = 0$, and $\mu_2 = 1$.

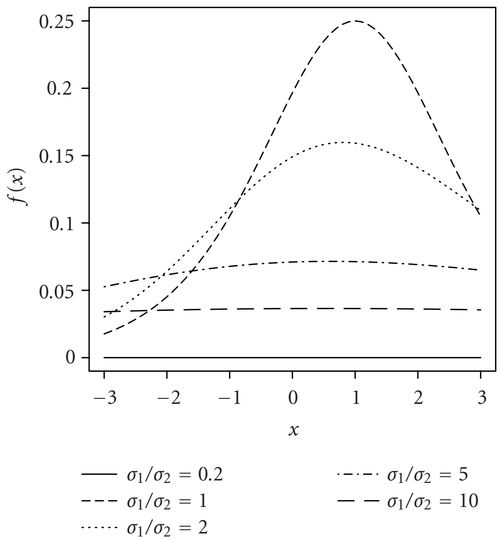

Figure 2.2. The generalized logistic pdf (2.3) for $\sigma_1/\sigma_2 = 0.2, 1, 2, 5, 10$, $\sigma_2 = 1$, $\mu_1 = 1$, and $\mu_2 = 0$.

Figures 2.1 and 2.2 illustrate possible shapes of the pdf (2.3) for selected values of $(\mu_1, \mu_2, \sigma_1, \sigma_2)$. The magnitude of σ_1/σ_2 clearly controls the shape of the pdf. In fact, if $\mu_1 = 0$, then

$$f_Z(z) \longrightarrow \frac{1}{\sigma_2} \exp\left(\frac{\mu_2 + z}{\sigma_2}\right) \exp\left\{-\exp\left(\frac{\mu_2 + z}{\sigma_2}\right)\right\} \tag{2.13}$$

as $\sigma_1/\sigma_2 \to 0$. Also, $f(z) \to 0$ for every $z \in (0,\infty)$ as $\sigma_1/\sigma_2 \to \infty$. On the other hand, if $\mu_1 \neq 0$, then $f(z) \to 0$ for every $z \in (0,\infty)$ irrespective of whether $\sigma_1/\sigma_2 \to 0$ or $\sigma_1/\sigma_2 \to \infty$.

3. Applications

The standard logistic distribution given by (1.2) has important uses in describing growth and as a substitute for the normal distribution. It has also attracted interesting applications in the modeling of the dependence of chronic obstructive respiratory disease prevalence on smoking and age, degrees of pneumoconiosis in coal miners, geological issues, hemolytic uremic syndrome data for children, physiochemical phenomenon, psychological issues, survival time of diagnosed leukemia patients, and weight gain data. The main feature of the generalized logistic distribution in (2.3) is that new parameters are introduced to control both location and scale. Thus, (2.3) allows for a greater degree of flexibility and we can expect this to be useful in many more practical situations.

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