EIGENVALUE PROBLEMS FOR A QUASILINEAR ELLIPTIC EQUATION ON $\mathbb{R}^N$

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We prove the existence of a simple, isolated, positive principal eigenvalue for the quasilinear elliptic equation $-\Delta_p u = \lambda g(x)|u|^{p-2}u$, $x \in \mathbb{R}^N$, $\lim_{|x|\to+\infty} u(x) = 0$, where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator and the weight function $g(x)$, being bounded, changes sign and is negative and away from zero at infinity.

1. Introduction

In this paper, we prove the existence of a positive principal eigenvalue of the following quasilinear elliptic problem:

$$-\Delta_p u(x) = \lambda g(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

$$\lim_{|x|\to+\infty} u(x) = 0, \tag{1.2}$$

where $\lambda \in \mathbb{R}$. Next, we state the general hypotheses which will be assumed throughout the paper.

(E) *Assume that N, p satisfy the following relation $N > p > 1$.*

(G) *g is a smooth function, at least $C^{1,\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$, such that $g \in L^\infty(\mathbb{R}^N)$ and $g(x) > 0$, on Ω^+, with measure of Ω^+, $|\Omega^+| > 0$. Also there exist R_0 sufficiently large and $k > 0$ such that $g(x) < -k$, for all $|x| > R_0$.*

Generally, problems where the operator $-\Delta_p$ is present arise both from pure mathematics (e.g., the theory of quasiregular and quasiconformal mappings), as well as from a variety of applications (e.g., non-Newtonian fluids, reaction-diffusion problems, flow through porous media, nonlinear elasticity, glaciology, astronomy, etc.).

On various types of bounded domains, there is an extensive literature on eigenvalue problems and the picture for “the principal eigenpair” seems to be fairly complete.

Papers on unbounded domains have appeared quite recently. These problems are of a more complex nature, as the equation may give rise to a noncompact operator. Such a problem is the one presented in [7].

The main aim of this paper is to study the quasilinear elliptic problem (1.1)-(1.2), by generalizing ideas introduced in [9], for the case $p = 2$. In Section 2, we study the space

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setting of problem (1.1)-(1.2), and give some equivalent norm results to be used later. A generalized version of Poincaré's inequality plays a crucial role. Some of the ideas developed in this section appear also in a different context in [9]. In Section 3, we define the basic operators for the construction of the first positive eigenvalue, the proof which is based on Ljusternik-Schnirelmann's theory. Also here, we derive some regularity results. Finally, in Section 4, we establish the simplicity and isolation of the principal eigenvalue.

Notation. We *denote* by B_R the open ball of $\mathbb{R}^N$ with center 0 and radius R and $B_R^* =: \mathbb{R}^N \setminus B_R$. For *simplicity reasons*, sometimes we use the symbols C_0^∞, L^p, $W^{1,p}$, respectively, for the spaces $C_0^\infty(\mathbb{R}^N)$, $L^p(\mathbb{R}^N)$, $W^{1,p}(\mathbb{R}^N)$ and $\|\cdot\|_p$ for the norm $\|\cdot\|_{L^p(\mathbb{R}^N)}$. Also, sometimes when the domain of integration is not stated, it is assumed to be all of $\mathbb{R}^N$. Equalities introducing definitions are denoted by "=:". Denote $g_\pm =: \max\{\pm g, 0\}$.

2. Space setting

In this section, we are going to characterize the space $\mathcal{V}_g$ (introduced below) in terms of classical Sobolev spaces. Let B be a ball centered at the origin of $\mathbb{R}^N$, such that $\int_B g(x)dx < 0$ and $g(x) \le -k$, for all $x \in B^*$. First, we prove the following type of Poicaré's inequality.

Theorem 2.1. *Suppose that* $\int_{\mathbb{R}^N} g(x)dx < 0$. *Then there exists* $\alpha > 0$ *such that* $\int_{\mathbb{R}^N} |\nabla u|^p dx > \alpha \int_{\mathbb{R}^N} g(x)|u|^p dx$, *for all* $u \in W^{1,p}(\mathbb{R}^N)$.

Proof. (i) If $\int_{\mathbb{R}^N} g(x)|u|^p dx \le 0$, then obviously the inequality holds.

(ii) Let $\int_{\mathbb{R}^N} g(x)|u|^p dx > 0$ then we can rewrite the above inequality as follows:

$$\int_{\mathbb{R}^N} |\nabla u|^p dx > \alpha\left(\int_B g(x)|u|^p dx + \int_{B^*} g(x)|u|^p dx\right). \tag{2.1}$$

To complete the proof of the theorem, since $g(x) \le -k < 0$ for all $x \in B^*$, it is enough to prove that there exists $\alpha > 0$ such that

$$\int_B |\nabla u|^p dx > \alpha \int_B g|u|^p dx, \tag{2.2}$$

where B is such that $\int_B g(x)dx < 0$ and $g(x) \le -k < 0$, for all $x \in B^*$. Suppose that the result is not true. This means that there exists a sequence $\{u_n\}$ in $W^{1,p}(B)$ such that $\int_B |\nabla u_n|^p dx \le (1/n)\int_B g(x)|u_n|^p dx$, for all $n \in \mathbb{N}$. Define $v_n =: u_n/\|u_n\|^p_{L^p(B)}$. This implies that $\int_B |v_n|^p dx = 1$ and $\int_B g(x)|v_n|^p dx > 0$. Therefore, we have that

$$\int_B |\nabla v_n|^p dx \le \frac{1}{n}\int_B g|v_n|^p dx \le \frac{K_B}{n}\int_B |v_n|^p dx \le \frac{K_B}{n}, \tag{2.3}$$

where $K_B =: \max\{|g(x)| : x \in B\}$. Hence $\{v_n\}$ is a bounded sequence in $W^{1,p}(B)$. Thus there is a subsequence—denoted again by $\{v_n\}$—which will converge strongly to some v in $L^p(B)$. We also know that

$$||v_n - v_m||^p_{W^{1,p}(B)} = ||v_n - v_m||^p_{L^p(B)} + ||\nabla v_n - \nabla v_m||^p_{L^p(B)}. \tag{2.4}$$

Furthermore, for all $p \in [1,+\infty)$, we have that

$$\|\nabla v_n - \nabla v_m\|^p_{L^p(B)} \leq \left(\|\nabla v_n\|^p_{L^p(B)} + \|\nabla v_m\|^p_{L^p(B)}\right)^p \leq K_B\left(\frac{1}{n^{1/p}} + \frac{1}{m^{1/p}}\right)^p. \quad (2.5)$$

Therefore $\{v_n\}$ is a Cauchy sequence in $W^{1,p}(B)$. So $\{v_n\}$ converges strongly to some $v \in W^{1,p}(B)$. From (2.3), we also have that

$$\int_B |\nabla v|^p dx = \lim_{n\to\infty} \int_B |\nabla v_n|^p dx = 0, \quad (2.6)$$

which means that $\nabla v = 0$ and $v \equiv c$. However,

$$|c|^p \int_B g(x)dx = \lim_{n\to\infty} \int_B g(x)|v_n|^p dx \geq 0. \quad (2.7)$$

Since $\int_B g(x) < 0$, we have that $c = 0$, that is, $v \equiv 0$. But on the other hand, we have that

$$\int_B v^p dx = \lim_{n\to\infty} \int_B v_n^p dx = 1, \quad (2.8)$$

which is a contradiction and the proof is completed. □

By the above result, we may introduce the following norm:

$$\|u\|_g =: \left(\int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{\alpha}{2}\int_{\mathbb{R}^N} g(x)|u|^p dx\right)^{1/p}. \quad (2.9)$$

We define the space $\mathcal{V}_g$ to be the completion of C_0^∞ with respect to the norm $\|\cdot\|_g$. Let $\mathcal{V}_g^*$ be the dual space of $\mathcal{V}_g$ with the pairing $\langle\cdot,\cdot\rangle_{\mathcal{V}}$. Note that $\mathcal{V}_g$ is a uniformly convex Banach space. Although the space $\mathcal{V}_g$ would seem to depend on g, we will prove that the space is independent of g. To achieve this result, we need the following two lemmas.

Corollary 2.2. *Under the assumptions of Theorem 2.1, for all $u \in C_0^\infty(\mathbb{R}^N)$,*

(i) $\int_{\mathbb{R}^N} |\nabla u|^p \leq 2\|u\|_g^p$,

(ii)

$$\left|\int_{\mathbb{R}^N} g|u|^p dx\right| \leq \frac{2}{\alpha}\|u\|_g^p. \quad (2.10)$$

Lemma 2.3. *Assume that the hypotheses of Theorem 2.1 are valid. Let $\{u_n\} \subset C_0^\infty(\mathbb{R}^N)$ be a bounded sequence in $\mathcal{V}_g$. Then $\{\int_B g|u_n|^p dx\}$ is bounded in $\mathcal{V}_g$.*

Proof. Suppose that $\{\int_B g|u_n|^p dx\}$ becomes unbounded, as $n \to \infty$. Since $g < 0$, for all $x \in B^*$, we have that

$$\int_{\mathbb{R}^N} g|u_n|^p dx < \int_B g|u_n|^p dx, \quad (2.11)$$

that is, $\int_B g|u_n|^p dx$ is bounded below. This implies that $\int_B g|u_n|^p dx \to +\infty$. Let $u_n = c_n v_n$, where $c_n \in \mathbb{R}$ such that $\int_B g|v_n|^p dx = 1$ and $c_n \to \infty$, as $n \to \infty$. Then,

$$\lim_{n\to\infty} \int_B |\nabla v_n|^p dx = \lim_{n\to\infty} \frac{1}{c_n^p}\int_B |\nabla u_n|^p dx. \quad (2.12)$$

But $\int_B |\nabla u_n|^p dx$ is bounded by relation (2.10), therefore $\lim_{n\to\infty} \int_B |\nabla v_n|^p dx = 0$, that is, $\{v_n\}$ is a bounded sequence in $W^{1,p}(B)$. Thus there exists a subsequence denoted again by $\{v_n\}$ such that $\{v_n\}$ converges in $(L^p(B))^N$. Since $\{\nabla v_n\}$ converges in $(L^p(B))^N$, $\{v_n\}$ is a Cauchy sequence in $W^{1,p}(B)$, and hence there exists a $v \in W^{1,p}(B)$ such that $v_n \to v$, that is, $\nabla v_n \to \nabla v = 0$ or $v = c$. However,

$$1 = \lim_{n\to\infty} \int_B g\,|\,v_n\,|^p dx = |c|^p \int_B g\,dx < 0, \tag{2.13}$$

which is a contradiction, and thereby the proof is complete. □

To prove the next results, we need to introduce the following notation: $D_1 =: \{x \in B : g(x) > 0\}$, $D_2 =: \{x \in B : g(x) \le 0\}$, and

$$\bar{g}(x) =: \begin{cases} g_+(x), & x \in D_1, \\ -g_-(x), & x \in D_2. \end{cases} \tag{2.14}$$

Lemma 2.4. *Assume that the hypotheses of Theorem 2.1 are valid. Then there exist constants $K_0 > 0$ and $K_1 > 0$ such that*

(i)

$$\int g_+(x)|u|^p dx \le K_0 \|u\|_g^p, \tag{2.15}$$

(ii)

$$-\int g_-(x)|u|^p dx \le K_1 \|u\|_g^p, \tag{2.16}$$

for all $u \in C_0^\infty(\mathbb{R}^N)$.

Proof. (i) Suppose that the inequality (2.15) is not true. Then there exists a sequence $\{u_n\} \subset C_0^\infty(\mathbb{R}^N)$ such that $\int g_+|u_n|^p dx = 1$ and $\|u_n\|_g \to 0$, as $n \to \infty$. By Corollary 2.2, we have that $\int_B |\nabla u_n|^p dx \to 0$, as $n \to \infty$. Hence there exists a subsequence—again denoted by $\{u_n\}$—converging to some constant function c in $W^{1,p}(B)$. But then,

$$\lim_{n\to\infty} \int_B g_+(x)|u|_n^p dx = |c|^p \int_B g_+(x) dx = 1. \tag{2.17}$$

Therefore, $c \ne 0$. Since $g < 0$ on $\mathbb{R}^N/B$, we obtain

$$\limsup_{n\to\infty} \int g(x)\,|\,u_n\,|^p dx < \lim_{n\to\infty} \int_B g(x)\,|\,u_n\,|^p dx = |c|^p \int_B g(x) dx < 0. \tag{2.18}$$

On the other hand, from relation (2.12), we have that $|\int g|u_n|^p dx| \le (2/\alpha)\|u_n\|_g^p \to 0$, as $n \to +\infty$, which is a contradiction.

(ii) Using relation (2.16) and $\int g_-|u_n|^p dx = \int g|u_n|^p dx - \int g_+|u_n|^p dx$, we complete the proof of the lemma. □

Next, we give the following uniform Sobolev characterization of the space $\mathcal{V}_g$.

Proposition 2.5. *Suppose that g satisfies ($\mathcal{G}$). Then* $\mathcal{V}_g = W^{1,p}(\mathbb{R}^N)$.

Proof. Because of density, we only compare the $\mathcal{V}_g$- and $W^{1,p}$-norms on the space $C_0^\infty(\mathbb{R}^N)$.

(i) For all $u \in C_0^\infty(\mathbb{R}^N)$, we have

$$\|u\|_g^p \leq \int |\nabla u|^p dx + \frac{\alpha}{2}\|g\|_\infty \int |u|^p dx \leq C(\alpha, \|g\|_\infty)\|u\|_{W^{1,p}}^p, \tag{2.19}$$

where $C(\alpha, \|g\|_\infty) = \max\{1, \alpha\|g\|_\infty/2\}$. Hence, we have that $W^{1,p} \subset \mathcal{V}_g$.

(ii) Let $\{u_n\} \subset C_0^\infty(\mathbb{R}^N)$ be a Cauchy sequence in $\mathcal{V}_g$ converging in some $u \in \mathcal{V}_g$. Then, relations (2.10) and (2.16) imply that

$$\int_B |\nabla(u_n - u)|^p dx \longrightarrow 0, \quad \int_B g(x)|u_n - u|^p dx \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.20}$$

Suppose that $\int_B (u_n - u)^p dx \nrightarrow 0$. Then if $v_n =: (u_n - u)/\|u_n - u\|_B$, we have that $\lim_{n\to\infty} \int_B |\nabla v_n|^p dx = \lim_{n\to\infty} \int_B g(x)|v_n|^p dx = 0$. Since $\{\nabla v_n\}$ converges (strongly to zero) in $L^p(B)$, $\{v_n\}$ is a bounded sequence in $W^{1,p}(B)$. Hence, there is a subsequence, denoted again by $\{v_n\}$, such that $\{v_n\}$ strongly converges in $L^p(B)$. But $\lim_{n\to\infty} \int_B |\nabla v_n|^p dx = 0$, so $\{v_n\}$ is a Cauchy sequence in $W^{1,p}(B)$, that is, there exists some $v \in W^{1,p}(B)$ such that $v_n \to v$ in $W^{1,p}(B)$. On the other hand, since $\nabla v_n \to \nabla v$ in $(L^p(B))^N$, it implies that $\nabla v = 0$ or $v = c$, where $c \neq 0$ since $\int_B v^p dx = 1$. However,

$$0 = \lim_{n\to\infty} \int_B g(x)|v_n|^p dx = |c|^p \int_B g(x) dx \neq 0, \tag{2.21}$$

which is a contradiction. Hence we have

$$\int_B (u_n - u)^p dx \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.22}$$

By adding the two inequalities (2.15) and (2.16), we get

$$\int_B \bar{g}(x)|u_n - u|^p dx + \int_{B^*} (-g_-)(x)|u_n - u|^p dx \leq (K_0 + K_1)||u_n - u||_g^p, \tag{2.23}$$

for all $u \in C_0^\infty$. But, as $n \to \infty$, we have $\int_B \bar{g}(x)|u_n - u|^p dx = M_n \int_B |u_n - u|^p dx \to 0$, where the quantity M_n, given by the intermediate value theorem for integrals, is finite positive, for all $n \in \mathbb{N}$ ($g \in L^\infty$). Also we have that

$$k\int_{B^*} |u_n - u|^p dx \leq \int_{B^*} (-g_-)(x)|u_n - u|^p dx \leq (K_0 + K_1)||u_n - u||_g^p, \tag{2.24}$$

which implies that, as $n \to \infty$,

$$\int_{B^*} (u_n - u)^p dx \longrightarrow 0. \tag{2.25}$$

Therefore by relations (2.22) and (2.25), we get $\int_{\mathbb{R}^N} (u_n - u)^p dx \to 0$, as $n \to \infty$. Summarizing, we have that $u_n \to u$, in $W^{1,p}$, that is, $\mathcal{V}_g \subset W^{1,p}$, for every g satisfying hypothesis ($\mathcal{G}$), and the proof is completed. □

3. Principal eigenvalue and regularity results

In this section, we are going to define the basic operators and some of their characteristics, which will help to prove the existence of a positive principal eigenvalue of problem (1.1)-(1.2). Finally, we close this section by proving some regularity results.

For any r_0 large enough ($r_0 \geq R_0$), there exists $\sigma_0 > 0$ such that $g(x) \leq -k/\sigma_0$, for all $|x| \geq r_0$. For later needs, we introduce the following smooth splitting of the weight function g:

$$g_2(x) =: \begin{cases} g(x) & \text{for } |x| \geq r_0, \\ -\dfrac{k}{\sigma_0} & \text{for } |x| < r_0, \end{cases} \qquad g_1(x) =: g(x) - g_2(x). \tag{3.1}$$

Let us define the operator $A_\lambda : D(A_\lambda) \subset W^{1,p} \to W^{1,q}$ as follows:

$$(A_\lambda(u), v) = \int (|\nabla u|^{p-2}\nabla u \nabla v - \lambda g_2 |u|^{p-2} uv)\,dx, \quad \forall u, v \in W^{1,p}. \tag{3.2}$$

We can then define the mapping

$$a_\lambda : W^{1,p} \times W^{1,p} \longrightarrow \mathbb{R}, \quad \text{by } a_\lambda(u,v) =: (A_\lambda(u), v). \tag{3.3}$$

It is easy to see that a_λ is bounded, for all $u, v \in D(A_\lambda)$ and $\lambda > \lambda_0$. Indeed, we have

$$\begin{aligned} |a_\lambda(u,v)| &= \left| \int (|\nabla u|^{p-2}\nabla u \nabla v - \lambda g_2 |u|^{p-2} uv)\,dx \right| \\ &\leq \int \left(|\nabla u|^{p-1} |\nabla v| + \frac{\lambda k}{\sigma_0} |u|^{p-1} |v| \right) dx \\ &\leq \left(\int |\nabla u|^p \right)^{(p-1)/p} \left(\int |\nabla v|^p \right)^{1/p} + \frac{\lambda k}{\sigma_0} \left(\int |u|^p \right)^{(p-1)/p} \left(\int |v|^p \right)^{1/p} \\ &\leq c \|u\|_{W^{1,p}}^{p-1} \|u\|_{W^{1,p}} < \infty. \end{aligned} \tag{3.4}$$

Also $a_\lambda(u,v)$ is coercive, that is,

$$a_\lambda(u,u) = \int (|\nabla u|^p - \lambda g_2 |u|^p)\,dx \geq \int \left(|\nabla u|^p + \frac{\lambda k}{\sigma_0} u^p \right) dx \geq \frac{\lambda k}{\sigma_0} \|u\|_{L^p}^p. \tag{3.5}$$

Next, we introduce the following form:

$$b(u,v) = \int g_1 |u|^{p-2} uv\,dx, \quad \forall u, v \in W^{1,p}(\mathbb{R}^N). \tag{3.6}$$

We see that $b(u,v)$ is bounded, that is, with the help of the Hölder inequality and the definition of g_1, for all $u, v \in W^{1,p}$, we have

$$\begin{aligned} |b(u,v)| &= \int g_1 |u|^{p-2} |uv| \leq \|g_1\|_{L^\infty} \left(\int |u|^{p-1} |v| \right) \\ &\leq c^* \left(\int |u|^p \right)^{(p-1)/p} \left(\int |v|^p \right)^{1/p} \\ &\leq c^* \|u\|_{W^{1,p}}^{p-1} \|v\|_{W^{1,p}}, \end{aligned} \tag{3.7}$$

where $c^* = \|g_1\|_{L^\infty}$. Therefore by the Riesz representation theory, we can define a nonlinear operator $B : D(B) \subset L^p \mapsto L^q$ such that $(B(u), v) = b(u, v)$, for all $u, v \in D(B)$ and $\lambda > 0$. It is easy to see that $D(B) \subset W^{1,p}$. Moreover, it is easy to see that the operators A_λ, B are well defined and A_λ is continuous.

Lemma 3.1. (i) *If $\{u_n\}$ is a sequence in $W^{1,p}$, with $u_n \rightharpoonup u$, then there is a subsequence, denoted again by $\{u_n\}$, such that $B(u_n) \to B(u)$.*

(ii) *If $B'(u) = 0$, then $B(u) = 0$.*

Proof. (i) Now suppose that $u_n \rightharpoonup u$ in $W^{1,p}$. We have

$$\begin{aligned} \|B(u_n) - B(u)\|_{W^{1,q}} &= \sup_{\|v\|_{W^{1,p}} \le 1} \left| (B(u_n) - B(u), v)_{W^{1,p}} \right| \\ &= \sup_{\|v\|_{W^{1,p}} \le 1} \left| \int g_1(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)v \right| \\ &\le \sup_{\|v\|_{W^{1,p}} \le 1} \left| \int_{|x| \le K} g_1(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)v \right| \\ &\quad + \sup_{\|v\|_{W^{1,p}} \le 1} \left| \int_{|x| > K} g_1(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)v \right|. \end{aligned} \tag{3.8}$$

Note that from the definition of $g_1(x)$ and for any $\epsilon > 0$, we can choose a $K > 0$ so that $|\int_{|x|>K} g_1(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)v| = 0$, while for this fixed K and by the strong convergence of $u_n \to u$ in L^q on any bounded region, the integral over $(|x| \le K)$ is smaller than ϵ, for n large enough. Hence, we have proved that $B(u_n) \to B(u)$ strongly in $W^{1,q}$, which means that B is a compact operator.

(ii) For the proof, see [5, Theorem 1]. □

Theorem 3.2. *Let $1 < p < N$. Assume that g satisfies $(\mathcal{G})$. Then*

(i) *problem (1.1)-(1.2) has a sequence of solutions (λ_k, u_k) with $\int g(x)|u_k|^p = 1$, $0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_k \to \infty$, as $k \to \infty$,*

(ii) *the eigenfunction u_1 corresponding to the first eigenvalue can be taken positive in $\mathbb{R}^N$.*

Proof. (i) We will just sketch the proof. Denote that

$$\begin{gathered} G := \left\{ u \in W^{1,p} : \Psi(u) := \frac{1}{p} \int g|u|^p = \frac{1}{p} \right\}, \\ I(u) = \frac{1}{p} \int |\nabla u|^p. \end{gathered} \tag{3.9}$$

The functional I is even and bounded below on G. Since the critical points of $I(u)$ on G are solutions of problem (1.1)-(1.2) for certain value of λ, to continue the procedure, it is necessary to prove that $I(u)$ satisfies the Palais-Smale condition on G, that is, for any sequence $\{u_n\} \subset G$, if the sequence $\{I(u_n)\}$ is bounded and

$$I'(u_n) - a_n \Psi'(u_n) \longrightarrow 0, \quad a_n := \frac{\langle I'(u_n), u_n \rangle}{\langle \Psi'(u_n, u_n) \rangle}, \tag{3.10}$$

then $\{u_n\}$ has a convergent subsequence in $W^{1,p}$. This proof follows the same lines as in [1, Lemma 1]. Then we apply the Ljusternik-Schnirelmann theory.

(ii) This follows the standard maximum principle arguments (e.g., see [7]). □

The next theorem examines the regularity as well as the L^{p_k} character and asymptotic behavior of the $W^{1,p}$ solutions of problem (1.1)-(1.2).

Theorem 3.3. *Suppose that $u \in W^{1,p}$ is a solution of problem (1.1)-(1.2). Then $u \in L^{p_k}$, for all $p_k \in [pc, +\infty]$ and the solutions $u(x)$ decay uniformly to zero, as $|x| \to +\infty$.*

Proof. Let $c > 1$, $p_k = pc^k$, and $m_k = (c^k - 1)p$. Assume that $u \in L^{p_1}(\mathbb{R}^N)$; then we will prove by induction that $u \in L^{p_k}(\mathbb{R}^N)$, for all $k \geq 1$. Let $u \in L^{p_k}(\mathbb{R}^N)$, for some fixed k. Consider the following Sobolev-type inequality:

$$\|u\|_{L^q} \leq K_0 \|u\|_{W^{1,p}}, \quad \forall q \in [p, p^*]. \tag{3.11}$$

Rewriting the above inequality and multiplying problem (1.1) by $w = u^{1+m_k}$, we have

$$\begin{aligned} ||u^{c^k}||_{cp}^p &\leq K^p ||\nabla(u^{c^k})||_p^p \\ &\leq K_0 c^k \int \big| |\nabla u|^{p-2} \nabla u \cdot \nabla u^{1+m_k} \big| \, dx \\ &\leq K^p c^{k(p-1)} \int |\lambda g| |u|^{p-1} \big| u^{1+m_k} \big| \, dx \\ &\leq K_0 c^{k(p-1)} \|u\|_{p_k}^{p_k}, \end{aligned} \tag{3.12}$$

where $K_0 = K^p |\lambda| \|g\|_\infty$. Letting $k \to +\infty$, by the dominated convergence theorem, we obtain

$$\|u\|_{p_{k+1}}^{p_{k+1}/c} \leq K_0 c^{k(p-1)} \|u\|_{p_k}^{p_k}. \tag{3.13}$$

Therefore, $u \in L^{p_{k+1}}(\mathbb{R}^N)$. Hence we may deduce from the above inequality that $u \in L^\infty(\mathbb{R}^N)$. But, we already know that $u \in L^{p_1}(\mathbb{R}^N) \bigcap L^\infty(\mathbb{R}^N)$. Thereby, we have that $u \in L^{p_k}(\mathbb{R}^N)$, for all $p_k \in [p_1, +\infty]$. By Theorem 1 of Serrin [8], for any ball $B_r(x)$ of radius r centered at any $x \in \mathbb{R}^N$ and some constant $C(N, p_2)$, the solution $u \in W^{1,p}$ of the equation

$$-\Delta_p u = f \tag{3.14}$$

satisfies the estimate

$$\sup_{y \in B_1(x)} |u(y)| \leq C\{\|u\|_{L^p(B_2(x))} + \|f\|_{L^{p_2}(B_2(x))}\}. \tag{3.15}$$

For $q = p_k/(k-1) \geq p_2$, we obtain for the solution of problem (1.1)-(1.2)

$$|u(x)| \leq \sup_{y \in B_1(x)} |u(y)| \leq C_1 \Big\{ \|u\|_{L^{pc^1}(B_2(x))} + |\lambda| \|g\|_\infty || |u|^{p-1} ||_{L^q(B_2(x))}^{1/(p-1)} \Big\}, \tag{3.16}$$

for any $x \in \mathbb{R}^N$. Hence $|u|^{p-1}$ belongs to $L^q(\mathbb{R}^N)$ and the uniform decay of $u(x)$ to zero, as $|x| \to +\infty$, is proved. □

As a consequence of the above theorem, we have the following regularity characterization of the solutions of problem (1.1)-(1.2).

Corollary 3.4. *For any* $r > 0$, *the solutions of problem (1.1)-(1.2) belong to* $C^{1,\alpha}(B_r)$, *where* $\alpha = \alpha(r) \in (0,1)$.

4. Simplicity and isolation of the principal eigenvalue

In this section, first we are going to prove the simplicity of the principal eigenvalue of problem (1.1)-(1.2) by generalizing Picone's identity

$$|\nabla u|^2 + \frac{u^2}{v^2}|\nabla v|^2 - 2\frac{u}{v}\nabla u \nabla v = |\nabla u|^2 - \nabla\left(\frac{u^2}{v}\right)\nabla v \geq 0, \tag{4.1}$$

which holds for any differentiable functions $v > 0$ and $u \geq 0$, to the p-Laplacian operator $\Delta_p u$ with $p > 1$. The idea to use Picone's identity for the simplicity of the proof was firstly introduced in [2].

Theorem 4.1 (generalized Picone's identity). *Let* $v > 0$, $u \geq 0$ *be differentiable functions in* Ω, *where* Ω *is a bounded or an unbounded domain in* $\mathbb{R}^N$. *Denote that*

$$\begin{aligned} L(u,v) &= |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}\nabla u|\nabla v|^{p-2}\nabla v, \\ R(u,v) &= |\nabla u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v. \end{aligned} \tag{4.2}$$

Then $L(u,v) = R(u,v) \geq 0$. *Moreover,* $L(u,v) = 0$, *a.e. in* Ω, *if and only if* $\nabla(u/v) = 0$, *a.e. in* Ω, *that is,* $u = kv$, *for some constant* k *in each component of* Ω.

Proof. For the proof, we refer to Allegretto and Huang [2, Theorem 1.1]. □

Theorem 4.2. *Suppose that* $v \in C^1$ *satisfies* $-\Delta_p v \geq \lambda g v^{p-1}$ *and* $v > 0$ *in* $\mathbb{R}^N$, *for some* $\lambda > 0$. *Then, for* $u \geq 0$ *in* $W^{1,p}$,

$$\int |\nabla u|^p dx \geq \lambda \int g(x)|u|^p dx, \tag{4.3}$$

and $\lambda \leq \lambda_1^+$. *The equality in (4.3) holds if and only if* $\lambda = \lambda_1^+$, $u = kv$, *and* $v = cu_1$, *for some constants* k, c. *In particular, the principal eigenvalue* λ_1^+ *is simple.*

Proof. Let Ω_0 be a compact subset of $\mathbb{R}^N$. Let $\phi \in C_0^\infty(\mathbb{R}^N)$, with $\phi \geq 0$. Then, we have

$$\begin{aligned} 0 &\leq \int_{\Omega_0} L(\phi,v) \leq \int L(\phi,v) = \int R(\phi,v) \\ &= \int |\nabla\phi|^p + \int \left(\frac{\phi^p}{v^{p-1}}\right)\Delta_p v \leq \int |\nabla\phi|^p - \lambda\int g\phi^p. \end{aligned} \tag{4.4}$$

Now letting $\phi \to u$ in $W^{1,p}$, we obtain (4.3). Suppose that for some $0 \leq u_0 \in W^{1,p}$, we have $\int |\nabla u_0|^p = \lambda \int g|u_0|^p$. Then from (4.4), we conclude that $\int_{\Omega_0} L(u_0,v) = 0$, that is, $u_0 = kv$ on Ω_0 for some constant k. Since Ω_0 is arbitrary and u_0 is nontrivial, we have that

$u_0 = kv$ on $\mathbb{R}^N$, $k > 0$, and $v \in W^{1,p}$. Next, if we replace u by u_1 in (4.3), then following the above reasoning, by (4.4) we obtain that $v = cu_1$ and $\lambda_1^+ \geq \lambda$. Since $v \in W^{1,p}$, we can repeat the above arguments choosing v for u_0 and u_1 for v. Therefore, we come to the conclusion that $v = ku_1$ and simplicity of λ_1^+ is proved. □

Theorem 4.3. *The principal eigenvalue λ_1 of problem (1.1)-(1.2) is isolated in the following sense. There exists $\eta > 0$ such that the interval $(-\infty, \lambda_1 + \eta)$ does not contain any other eigenvalue than λ_1.*

Proof. Assume the contrary, that is, there exists a sequence of eigenpairs (λ_n, u_n) such that $\lambda_n \to \lambda_1$ and $u_n \in W^{1,p}$ with $\|u_n\|_{W^{1,p}} = 1$. Then from the simplicity of λ_1 and the variational characterization of the principal eigenvalue, we have that $\lambda_n > \lambda_1$. Also from the weak convergence, we have that $u_n \rightharpoonup u_1 > 0$ in $W^{1,p}$. We know that

$$\int |\nabla u_n|^{p-2} \nabla u_n \nabla \phi = \lambda_n \int g(x) |u_n|^{p-2} u_n \phi, \quad \text{for any } \phi \in C_0^\infty(\mathbb{R}^N). \tag{4.5}$$

Subtracting the two equations of the form (4.5) corresponding to n and m and taking $\phi = u_n - u_m$, we obtain

$$\begin{aligned} &\int (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_m|^{p-2} \nabla u_m) \nabla (u_n - u_m) dx \\ &\quad = \lambda_n \int g(x) (|u_n|^{p-2} u_n - |u_m|^{p-2} u_m)(u_n - u_m) dx \\ &\qquad + (\lambda_n - \lambda_m) \int g(x) (|u_n|^{p-2} u_n - |u_m|^{p-2} u_m)(u_n - u_m) dx \\ &\quad \leq \lambda_n \int g_1(x) (|u_n|^{p-2} u_n - |u_m|^{p-2} u_m)(u_n - u_m) dx \\ &\qquad + (\lambda_n - \lambda_m) \int g(x) |u_m|^{p-2} u_m (u_n - u_m) dx \longrightarrow 0, \quad \text{as } n, m \longrightarrow \infty. \end{aligned} \tag{4.6}$$

Indeed, it is clear that—due to the compact support of g_1 and the fact that $u_n \rightharpoonup u_1$—there exists a subsequence of $\{u_n\}$ such that

$$\int g_1(x) (|u_n|^{p-2} u_n - |u_m|^{p-2} u_m)(u_n - u_m) dx \longrightarrow 0, \quad \text{as } n, m \longrightarrow \infty. \tag{4.7}$$

Moreover, applying Hölder's inequality on the second integral of the last part of inequality (4.6), we see that it is bounded. Hence, we have that

$$(\lambda_n - \lambda_m) \int g(x) |u_m|^{p-2} u_m (u_n - u_m) dx \longrightarrow 0, \quad \text{as } n, m \longrightarrow \infty. \tag{4.8}$$

On the other hand, taking into consideration the inequality

$$|a - b|^p \leq c\{(|a|^{p-2} a - |b|^{p-2} b)(a - b)\}^{s/2} (|a|^p + |b|^p)^{1-s/2}, \quad a, b \in \mathbb{R}, \tag{4.9}$$

where $s = p$, if $p \in (1,2)$ and $s = 2$, if $p \geq 2$, we have that (4.6) becomes

$$\int |\nabla u_n - \nabla u_m|^p \leq c\left\{\int \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\right)\nabla(u_n - u_m)\right\}^{s/2} \times\left\{\int |\nabla u_n|^p dx + \int |\nabla u_m|^p dx\right\}^{1-s/2}. \tag{4.10}$$

Hence by (4.6), we see that the left-hand side of the inequality (4.10) tends to zero. Therefore, we have proved that $u_n \to u_1 \in W^{1,p}$. Let us define the following set $\Omega_{u_n}^- := \{x \in \mathbb{R}^N;\ u_n^- < 0\}$ with $|\Omega_{u_n}^-| > 0$. Moreover, we have that for any fixed number $K > 0$,

$$\operatorname{meas}(\Omega_{u_n}^- \cap B_K) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{4.11}$$

We also know that

$$(A_\lambda(u_n), u_n) \leq c_1||u_n||_{W^{1,p}}, \qquad (B(u_n), v) \leq c_2\|v\|_p^p. \tag{4.12}$$

On the other hand, since $W^{1,p}$ is continuously embedded in L^p, we have

$$c_1||u_n^-||_{W^{1,p}}^p \leq (A_\lambda(u_n), u_n^-) = \lambda_n(B(u_n), u_n^-) \leq c_2||u_{n^-}||_p^p \leq c_3||u_n^-||_{W^{1,p}}^p. \tag{4.13}$$

Finally, since $|\Omega_{u_n^-}| \neq 0$, (4.13) implies that $c_3 > \text{const} > 0$, for any $n \in \mathbb{N}$. But this contradicts (4.11) and the proof is complete. □

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