A NECESSARY AND SUFFICIENT CONDITION FOR GLOBAL EXISTENCE FOR A QUASILINEAR REACTION-DIFFUSION SYSTEM

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We show that the reaction-diffusion system $u_t = \Delta\varphi(u) + f(v)$, $v_t = \Delta\psi(v) + g(u)$, with homogeneous Neumann boundary conditions, has a positive global solution on $\Omega \times [0,\infty)$ if and only if $\int^{\infty} ds/f(F^{-1}(G(s))) = \infty$ (or, equivalently, $\int^{\infty} ds/g(G^{-1}(F(s))) = \infty$), where $F(s) = \int_0^s f(r)dr$ and $G(s) = \int_0^s g(r)dr$. The domain $\Omega \subseteq \mathbb{R}^N$ $(N \geq 1)$ is bounded with smooth boundary. The functions φ, ψ, f, and g are nondecreasing, nonnegative $C([0,\infty))$ functions satisfying $\varphi(s)\psi(s)f(s)g(s) > 0$ for $s > 0$ and $\varphi(0) = \psi(0) = 0$. Applied to the special case $f(s) = s^p$ and $g(s) = s^q$, $p > 0$, $q > 0$, our result proves that the system has a global solution if and only if $pq \leq 1$.

1. Introduction

We consider the reaction-diffusion system

$$\begin{gathered} u_t = \Delta\varphi(u) + f(v), \qquad v_t = \Delta\psi(v) + g(u) \quad \text{on } Q_\infty \equiv \Omega \times (0,\infty), \\ \partial_\nu\varphi(u) = \partial_\nu\psi(v) = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, \qquad v(x,0) = v_0(x) \geq 0 \quad \text{on } \overline{\Omega}, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, $\partial_\nu = \partial/\partial\nu$ is the derivative in the direction ν of the outward normal to $\partial\Omega$, and the functions φ, ψ, f, and g are nondecreasing, nonnegative $C([0,\infty))$ functions satisfying

$$\varphi(s)\psi(s)f(s)g(s) > 0 \quad \text{for } s > 0, \qquad \varphi(0) = \psi(0) = 0. \tag{1.2}$$

We show that the problem (1.1) has a global solution if and only if f and g satisfy

$$\int^{\infty} \frac{ds}{f(F^{-1}(G(s)))} = \infty \qquad \left(\text{or equivalently, } \int^{\infty} \frac{ds}{g(G^{-1}(F(s)))} = \infty\right), \tag{1.3}$$

where $F(s) \equiv \int_0^s f(\xi)d\xi$ and $G(s) \equiv \int_0^s g(\xi)d\xi$. This, in turn, is exactly the necessary and sufficient condition needed to guarantee the existence of a global solution to the initial

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value problem for the system (see Lemma 2.2.)

$$y'(t) = f(z(t)), \quad z'(t) = g(y(t)), \quad 0 < t < \infty,$$
$$y(0) = a \geq 0, \quad z(0) = b \geq 0, \quad a + b > 0. \tag{1.4}$$

In the special case $f(s) = s^p$ and $g(s) = s^q$ ($p > 0, q > 0$), condition (1.3) becomes $pq \leq 1$.

Reaction-diffusion systems have been studied for decades (see, e.g., [4, 5, 10, 11], and their references). The particular problem of determining conditions under which such systems have global solutions has been the object of studies for almost as long. (See [6, 7, 8, 9, 10, 11, 15, 16, 17, 18, 19] and their references.) For the system ($p > 0, q > 0$)

$$u_t = \Delta u + v^p,$$
$$v_t = \Delta v + u^q, \tag{1.5}$$

it is well known that the existence of global solutions in general depends on more than just the values of the exponents p and q. In particular, when homogeneous Dirichlet boundary conditions are imposed, it is well known [6, 8] that for $pq \leq 1$, the system has only global solutions, but if $pq > 1$, the system will have a global solution for "small" initial data but *not* for "large" initial data. A similar phenomenon occurs for the Cauchy problem [5, 7]. However, our results (see Theorems 2.1 and 3.2) show that this cannot occur with homogeneous Neumann boundary data, where blowup (i.e., no global solution) depends exclusively on the reaction terms and occurs (for (1.5)) precisely if $pq > 1$. We show that this is true also in the presence of nonlinear diffusion. Thus the existence of a global solution is also independent of the diffusion term, although the diffusion rate may well determine the *nature* of blowup as in the scalar case (see [12, 14]). On the other hand, with homogenous Dirichlet boundary data, Galaktionov et al. [10, 11] have shown that the quasilinear system

$$u_t = \Delta u^{\nu+1} + v^p,$$
$$v_t = \Delta v^{\mu+1} + u^q \tag{1.6}$$

has only global solutions if $pq < (1+\mu)(1+\nu)$, but for $pq \geq (1+\mu)(1+\nu)$, the existence of global solutions depends on the initial data and the size of the domain. In the present case, this does not occur. Indeed applying Theorem 3.2 to the system (1.6) with homogeneous Neumann boundary conditions, we find that a global solution exists if and only if $pq \leq 1$.

We note also that some authors (e.g., [9, 16]) have been concerned with whether a diffusion-free system can have a global solution while the corresponding diffusive system does not. Obviously, this cannot occur with the present system.

2. Smooth constitutive functions

Before establishing the general case, we first consider the case where the constitutive functions and the initial and boundary data are smooth. Thus we prove the following theorem.

Theorem 2.1. *Let u_0 and v_0 be nonnegative $C^\infty(\overline{\Omega})$ functions, at least one of which is nontrivial. Assume that the functions φ, ψ, f, and g are nonnegative nondecreasing $C^\infty([0,\infty))$ functions satisfying (1.2) and $\varphi'\psi' > 0$. Then problem (1.1) has a nonnegative classical solution if and only if f and g satisfy condition (1.3).*

Before proving this, we establish a preliminary lemma.

Lemma 2.2. *Suppose that f and g are nonnegative nondecreasing functions on $[0,\infty)$ satisfying $f(s)g(s) > 0$ for $s > 0$. Then the system of ordinary differential equation (1.4) has a nonnegative classical solution if and only if (1.3) holds.*

Proof. Necessity. Without loss of generality, assume that $a > 0$, and problem (1.4) has solution (y,z). Then $y'g(y) = z'f(z)$, which gives $d/dt[G(y) - F(z)] = 0$. Thus there is a constant K so that $G(y) = F(z) + K$, and clearly from the initial values of y and z, we get $K = G(a) - F(b)$. Let $\widetilde{F}(z) = F(z) + K$ and note that $y(t) = G^{-1}(\widetilde{F}(z(t)))$, and hence $z'(t) = g(y(t)) = g(G^{-1}(\widetilde{F}(z(t)))) \geq g(G^{-1}(\widetilde{F}(b))) = g(a) > 0$, which implies that $\lim_{t\to\infty} z(t) = \infty$. Now, if $b > 0$, we get

$$\frac{d}{dt}\int_b^{z(t)} \frac{ds}{g(G^{-1}(\widetilde{F}(s)))} = 1 \qquad \text{which implies that } \int_b^{z(t)} \frac{ds}{g(G^{-1}(\widetilde{F}(s)))} = t. \tag{2.1}$$

Letting $t \to \infty$, we establish condition (1.3). If $b = 0$, then for every $\varepsilon > 0$, we get

$$\frac{d}{dt}\int_\varepsilon^{z(t)} \frac{ds}{g(G^{-1}(\widetilde{F}(s)))} = 1 \tag{2.2}$$

which, after integrating from $\delta > 0$ to t, gives

$$\int_\varepsilon^{z(t)} \frac{ds}{g(G^{-1}(\widetilde{F}(s)))} = t - \delta - \int_{z(\delta)}^{\varepsilon} \frac{ds}{g(G^{-1}(\widetilde{F}(s)))}. \tag{2.3}$$

Letting $\delta \to 0$ gives

$$\int_\varepsilon^{z(t)} \frac{ds}{g(G^{-1}(\widetilde{F}(s)))} = t - \int_0^{\varepsilon} \frac{ds}{g(G^{-1}(\widetilde{F}(s)))}, \tag{2.4}$$

which implies that the integral on the right converges, and hence letting $\varepsilon \to 0$, we establish that (2.1) holds for $b = 0$. The proof now continues as in the case $b > 0$.

Sufficiency. Now suppose that (1.3) holds. Define $\widetilde{F}(s) = \int_b^s f(t)dt$ and $\widetilde{G}(s) = \int_a^s g(t)dt$. We need to prove that the problem (1.4) has a classical solution. Once again, we assume that $a > 0$. Define

$$H(s) = \int_a^s \frac{d\sigma}{f(\widetilde{F}^{-1}(\widetilde{G}(\sigma)))}. \tag{2.5}$$

Clearly, $H(a) = 0$, $H'(s) > 0$ for $s > 0$. Thus H is one-to-one and from (1.3), which holds for F replaced by $\widetilde{F}$ and G replaced by $\widetilde{G}$, we know that $H([a,\infty)) = [0,\infty)$. Now define $y : [0,\infty) \to [a,\infty)$ by $y(t) = H^{-1}(t)$ and define $z(t) = \widetilde{F}^{-1}(\widetilde{G}(y(t)))$. We now show that

y,z satisfy (1.4). Clearly, $y(0)=a$ and $H(y(t))=t$. Thus $H'(y(t))y'(t)=1$ so that $y'(t)=f(\widetilde{F}^{-1}(\widetilde{G}(y(t))))=f(z(t))$. Likewise, $z(0)=\widetilde{F}^{-1}(\widetilde{G}(y(0)))=\widetilde{F}^{-1}(\widetilde{G}(a))=\widetilde{F}^{-1}(0)=b$, and

$$z'=\frac{\widetilde{G}'(y)y'}{\widetilde{F}'(\widetilde{F}^{-1}(\widetilde{G}(y)))}=\frac{\widetilde{G}'(y)y'}{\widetilde{F}'(z)}=g(y). \tag{2.6}$$

This completes the proof. □

Proof of Theorem 2.1. Necessity. Suppose that problem (1.1) has a nonnegative classical solution (u,v). From [13, Theorem 5.1], there exist $T_0>0$ and $a>0$ such that $\min\{u(x,t),v(x,t)\}>a$ on $\Omega\times[T_0,\infty)$. We now consider the system

$$\begin{gathered}\alpha'(t)=\frac{f(\beta(t))}{2},\quad \beta'(t)=\frac{g(\alpha(t))}{2}\quad \text{for } t>T_0,\\ \alpha(T_0)=\beta(T_0)=\frac{a}{2}.\end{gathered} \tag{2.7}$$

We will show that this system has a solution, and then invoke Lemma 2.2 to yield that (1.3) holds, which will complete the proof of necessity. Clearly, the system (2.7) has a solution on some, perhaps small, interval. Let $t_0>T_0$ be the supremum of all values τ such that a solution exists on $[T_0,\tau)$. If $t_0=\infty$, then the system (2.7) has a solution and (1.3) holds as a result of Lemma 2.2. Thus suppose that $t_0<\infty$. We will first show that

$$\alpha(t)<u(x,t),\qquad \beta(t)<v(x,t)\quad \text{on } \overline{\Omega}\times[T_0,t_0). \tag{2.8}$$

Thus suppose that there exists $(\tilde{x},T)\in\overline{\Omega}\times[T_0,t_0)$, where at least one of the two inequalities (2.8) fails to hold. Let $\zeta\in C^2(\overline{\Omega})$ such that $\partial_\nu\zeta<0$ on $\partial\Omega$ and $\zeta\geq 1$ on $\overline{\Omega}$. Clearly, inequalities (2.8) hold for t near T_0 since they hold for $t=T_0$. Hence $T>T_0$. Define $W(x,t)=u(x,t)-\varepsilon\zeta(x)$ and $Z(x,t)=v(x,t)-\varepsilon\zeta(x)$, where $\varepsilon>0$ is chosen small so that the following conditions hold for all $s\in[0,s_0]$, $s_0\equiv\max_{\overline{\Omega}\times[T_0,T]}(u+v)$:

$$\begin{aligned}\zeta(x)&<\frac{m_0}{2\varepsilon},\\ -\varphi'(s)\Delta\zeta(x)-\varepsilon\varphi''(s)\left|\nabla\zeta(x)\right|^2&\leq\frac{m_0}{4\varepsilon},\\ -\psi'(s)\Delta\zeta(x)-\varepsilon\psi''(s)\left|\nabla\zeta(x)\right|^2&\leq\frac{m_0}{4\varepsilon},\end{aligned} \tag{2.9}$$

where $m_0=\min\{a,g(a/2),f(a/2)\}$. Then since $W(x,T_0)-\alpha(T_0)=u(x,T_0)-\varepsilon\zeta(x)-\alpha(T_0)\geq a-\varepsilon\zeta(x)-a/2>0$ and similarly for $Z(x,T_0)-\beta(T_0)$, we get $W(x,T_0)>\alpha(T_0)$ and $Z(x,T_0)>\beta(T_0)$ on $\overline{\Omega}$. Now let

$$t_1=\sup\{\tau\in[T_0,T]\mid W(x,t)\geq\alpha(t),\ Z(x,t)\geq\beta(t)\ \forall(x,t)\in\overline{\Omega}\times[T_0,\tau]\}. \tag{2.10}$$

Clearly $t_1 > T_0$ and at $t = t_1$, either $W - \alpha$ or $Z - \beta$ is zero for some $x_0 \in \overline{\Omega}$. Without loss of generality, we assume that it is $Z - \beta$; that is, $\min_{x\in\overline{\Omega}} Z(x,t_1) - \beta(t_1) = Z(x_0,t_1) - \beta(t_1) = 0$ and $W(x,t_1) - \alpha(t_1) \geq 0$ on $\overline{\Omega}$. Since $\partial_\nu Z = \partial_\nu v - \varepsilon\partial_\nu\zeta = -\varepsilon\partial_\nu\zeta > 0$ on $\partial\Omega \times (T_0,t_0)$, we must have $x_0 \in \Omega$, and hence $\nabla Z(x_0,t_1) = 0$ and $\Delta Z(x_0,t_1) \geq 0$. Thus, at (x_0,t_1), we have the following:

$$\begin{aligned}
0 \geq Z_t - \beta' = v_t - \beta' &= \Delta\psi(v) + g(u) - \frac{g(\alpha)}{2} \\
&= \psi'(v)\Delta v + \psi''(v)|\nabla v|^2 + g(u) - \frac{g(\alpha)}{2} \\
&= \psi'(v)\Delta(Z+\varepsilon\zeta) + \psi''(v)|\nabla Z + \varepsilon\nabla\zeta|^2 + g(u) - \frac{g(\alpha)}{2} \\
&\geq \varepsilon\psi'(v)\Delta\zeta + \varepsilon^2\psi''(v)|\nabla\zeta|^2 + g(u) - \frac{g(\alpha)}{2} \\
&\geq -\frac{m_0}{4} + g(u) - \frac{g(\alpha)}{2} \\
&\geq -\frac{m_0}{4} + \frac{g(u)}{2} + \frac{[g(u)-g(\alpha)]}{2} \\
&\geq -\frac{m_0}{4} + \frac{g(\alpha)}{2} \geq -\frac{m_0}{4} + \frac{g(a/2)}{2} > 0.
\end{aligned} \tag{2.11}$$

We thus arrive at a contradiction. Therefore inequalities (2.8) hold. Hence, the solution of (2.7) can be extended to an interval $[0,t_0^*)$, where $t_0^* > t_0$. This contradicts the fact that t_0 is the supremum of all such values. Therefore, our assumption that $t_0 < \infty$ cannot hold. Thus (2.7) has a global solution, and therefore Lemma 2.2 implies that (1.3) holds.

Sufficiency. Now suppose that (1.3) holds. From [2, page 17], we know that there exists a maximal time $T_0 \in (0,\infty]$ such that problem (1.1) has a (unique) solution, (u,v), and furthermore, if u and v remain bounded and positive on $\Omega \times (0,T')$ for all $T' < T_0$, then $T_0 = \infty$. Clearly u and v are positive. Thus we need only to show that they are bounded on $\overline{\Omega} \times (0,T')$. To do this, we let (y,z) satisfy the system (whose solution exists by Lemma 2.2)

$$\begin{gathered}
y'(t) = 2f(z(t)), \quad z'(t) = 2g(y(t)), \quad 0 < t < \infty, \\
y(0) = z(0) = M+1,
\end{gathered} \tag{2.12}$$

where $M = \|u_0\|_{\infty,\Omega} + \|v_0\|_{\infty,\Omega}$. We first show that

$$0 \leq u(x,t) < y(t), \qquad 0 \leq v(x,t) < z(t) \quad \text{on } Q_{T'}. \tag{2.13}$$

Thus suppose that there exists an $(x_0,T) \in Q_{T'}$ such that (2.13) does not hold. Clearly, inequalities (2.13) hold for t small since $u(x,0) \leq M < y(0)$ and $v(x,0) \leq M < z(0)$. Hence $T > 0$. Define the function ζ as in the proof of necessity and let $p(x,t) = u(x,t) + \varepsilon\zeta(x)$ and $q(x,t) = v(x,t) + \varepsilon\zeta(x)$ on $\Omega \times [0,T]$, where $\varepsilon > 0$ is chosen small so that each of the following hold on $\overline{\Omega}$ for every $s \in [0, s_0]$, where $s_0 = \max_{\overline{\Omega}\times[0,T]}(u+v)$:

$$\zeta(x) < \frac{1}{\varepsilon}, \tag{2.14}$$

$$-\varphi'(s)\Delta\zeta(x) + \varepsilon\varphi''(s)\,|\,\nabla\zeta(x)\,|^2 \le \frac{M_0}{\varepsilon}, \tag{2.15}$$

$$-\psi'(s)\Delta\zeta(x) + \varepsilon\psi''(s)\,|\,\nabla\zeta(x)\,|^2 \le \frac{M_0}{\varepsilon}, \tag{2.16}$$

where $M_0 = \min\{g(M+1),\ f(M+1)\}$. From (2.14), it is clear that

$$p(x,0) - y(0) < 0, \quad q(x,0) - z(0) < 0, \quad \forall x \in \overline{\Omega}. \tag{2.17}$$

By our assumption concerning T, there exists $\tau_0 \in (0,T]$ such that either $\max_{x\in\overline{\Omega}} p(x,\tau_0) = y(\tau_0)$ or $\max_{x\in\overline{\Omega}} q(x,\tau_0) = z(\tau_0)$. Let

$$t_0 = \sup\{\tau \in [0,T] \mid p(x,t) \le y(t),\ q(x,t) \le z(t)\ \forall (x,t) \in \overline{\Omega}\times[0,\tau]\}. \tag{2.18}$$

Then at $t = t_0$, either $p - y$ or $q - z$ is zero for some $x_0 \in \overline{\Omega}$. Without loss of generality, we assume that $\max_{x\in\overline{\Omega}} p(x,t_0) - y(t_0) = p(x_0,t_0) - y(t_0) = 0$, and hence $\max_{x\in\overline{\Omega}} q(x,t_0) - z(t_0) \le 0$. Notice that $\partial_\nu p = \varepsilon\partial_\nu\zeta < 0$ on $\partial\Omega\times(0,t_0)$ so that $x_0 \in \Omega$, and hence $\nabla p(x_0,t_0) = 0$ and $\Delta p(x_0,t_0) \le 0$. We now have, at (x_0,t_0), the following:

$$\begin{aligned}
0 \le p_t - y' &= u_t - y' = \Delta\varphi(u) + f(v) - 2f(z) \\
&= \varphi'(u)\Delta(p-\varepsilon\zeta) + \varphi''(u)\,|\,\nabla(p-\varepsilon\zeta)\,|^2 + f(v) - 2f(z) \\
&\le -\varepsilon\varphi'(u)\Delta\zeta + \varepsilon^2\varphi''(u)|\nabla\zeta|^2 + \big(f(v) - f(z)\big) - f(z) \\
&\le M_0 - f(z) \le f(M+1) - f(z) < 0,
\end{aligned} \tag{2.19}$$

which provides a contradiction. Thus no such t_0 exists. Hence $p < y$ and $q < z$ on $\overline{\Omega}\times[0,T]$ which, in turn, yields $u < y$ and $v < z$ on $\overline{\Omega}\times[0,T]$. Thus there is no T where (2.13) fails to hold, and hence it holds on $Q_{T'}$ and therefore holds for all $T' < T_0$ giving $T_0 = \infty$. This completes the proof. □

3. Nonsmooth constitutive functions

We now consider the case where the data and constitutive functions are not smooth. In this case, it is well known that the system (1.1) does not, in general, have a classical solution even in the case of a single equation (see, e.g., [1]). Therefore, we will consider a weak formulation of a solution motivated by Bénilan et al. [3] and similar to that of [14].

Definition 3.1. The sequence of problems

$$\frac{\partial u_n}{\partial t} = \Delta\varphi_n(u_n) + f_n(v_n), \qquad \frac{\partial v_n}{\partial t} = \Delta\psi_n(v_n) + g_n(u_n) \quad \text{in } Q_\infty,$$
$$\partial_\nu\varphi_n(u_n) = \partial_\nu\psi_n(v_n) = 0 \quad \text{on } \partial\Omega\times[0,\infty), \tag{3.1}$$
$$u_n(x,0) = u_{0,n}(x), \qquad v_n(x,0) = v_{0,n}(x) \quad \text{on } \overline{\Omega}$$

is called a *sequence of approximating problems* for (1.1) if

$$\varphi_n,\psi_n,f_n,g_n \in C^\infty([0,\infty)), \qquad u_{0,n},v_{0,n} \in C^\infty(\overline{\Omega}),$$
$$\varphi_n(0) = \psi_n(0) = 0, \qquad f_n(0) \ge 0, \qquad g_n(0) \ge 0,$$
$$\varphi_n' > 0, \qquad \psi_n' > 0, \qquad f_n' > 0, \qquad g_n' > 0, \tag{3.2}$$
$$\lim_{n\to\infty}\left(||\varphi_n - \varphi||_{\infty,S} + ||\psi_n - \psi||_{\infty,S} + ||f_n - f||_{\infty,S} + ||g_n - g||_{\infty,S}\right) = 0,$$
$$\lim_{n\to\infty}\left(||u_{0,n} - u_0||_{\infty,\Omega} + ||v_{0,n} - v_0||_{\infty,\Omega}\right) = 0,$$

for every compact subset S of $[0,\infty)$. Furthermore, a sequence $\{(u_n,v_n)\}$ of classical solutions to the approximating problems (3.1) is called a *sequence of approximating solutions* to problem (1.1). Finally, a nonnegative function pair (u,v) defined on Q_∞ is a *generalized solution* of problem (1.1) if there exists a sequence $\{(u_n,v_n)\}$ of approximating solutions which, for every $T > 0$, converges to (u,v) weakly in $L^1(Q_T)$ and

$$\sup_n\left(||u_n||_{\infty,Q_T} + ||v_n||_{\infty,Q_T}\right) < \infty. \tag{3.3}$$

We prove the following theorem.

Theorem 3.2. *Let u_0 and v_0 be positive $C(\overline{\Omega})$ functions and assume that the functions φ, ψ, f, and g are nondecreasing, nonnegative $C([0,\infty))$ functions satisfying (1.2). Then problem (1.1) has a generalized solution if and only if f and g satisfy condition (1.3).*

Since much of the proof that follows is like that of Theorem 2.1 above, we merely point out important differences.

Proof. Necessity. Suppose that problem (1.1) has a generalized solution (u,v). Let (u_n, v_n) be a sequence of approximating solutions, thus satisfying (3.1). Since u_0 and v_0 are strictly positive on $\overline{\Omega}$ and the sequence $\{(u_{0,n},v_{0,n})\}$ converges uniformly on Ω, there exists a subsequence, which, for convenience, we will assume is the sequence itself, for which there exists a positive constant a such that $\min\{u_n(x,t),v_n(x,t)\} > a$ on $\Omega\times[0,\infty)$. (We note that in the smooth case (Theorem 2.1), the initial data did not need to be strictly positive. However, for nonsmooth data and constitutive functions, it is unknown whether a generalized solution with nonnegative, nontrivial initial data ever becomes strictly positive at a later time.) The proof may now proceed as with Theorem 2.1 (with $T_0 = 0$) to prove

$$\alpha_n(t) < u_n(x,t), \qquad \beta_n(t) < v_n(x,t) \quad \text{on } \overline{\Omega}\times[0,\infty), \tag{3.4}$$

where (α_n,β_n) is the solution to system (2.7) with f and g replaced with f_n and g_n, respectively, and hence for all $T>0$,

$$\alpha_n(t)+\beta_n(t) \le \sup_k \left(||u_k||_{\infty,Q_T}+||v_k||_{\infty,Q_T}\right) < \infty \quad n\in\mathbb{N},\ 0\le t\le T. \tag{3.5}$$

We can now use this inequality to show that (2.7) has a solution on $[0,\infty)$. To do this, we note that there is an interval, perhaps small, on which a solution (α,β) to (2.7) exists. In fact, from the proof of Lemma 2.2, it is clear that a solution exists on the interval $[0,t_0)$, where $t_0 = H(\infty)$ with

$$H(s) = \int_{a/2}^{s} \frac{d\sigma}{f(\widetilde{F}^{-1}(\widetilde{G}(\sigma)))} \tag{3.6}$$

and $\widetilde{F}$, $\widetilde{G}$ defined as in the proof of Lemma 2.2 with both a and b replaced by $a/2$. However, since the sequences $\{f_n\}$, $\{g_n\}$ converge uniformly on compact subsets of $[0,\infty)$, so do the sequences $\{F_n\}$, $\{G_n\}$, where $F_n(s) = \int_{a/2}^{s} f_n(\sigma)d\sigma$ and $G_n(s) = \int_{a/2}^{s} g_n(\sigma)d\sigma$. It is then straightforward to show that F_n^{-1}, G_n^{-1} converge uniformly on compact subsets of $[0,\infty)$, and therefore H_n^{-1} converges uniformly on compact subsets of $[0,t_0)$. Therefore, $(\alpha_n,\beta_n)\to(\alpha,\beta)$ as $n\to\infty$ uniformly on compact subsets of $[0,t_0)$, and hence (α,β) must satisfy

$$\alpha(t)+\beta(t) \le \sup_k \left(||u_k||_{\infty,Q_T}+||v_k||_{\infty,Q_T}\right) < \infty, \quad 0\le t\le T<t_0. \tag{3.7}$$

Therefore, the functions α and β must be defined on $[0,\infty)$. Indeed, the only way that α and β can fail to exist at t_0 is for $\lim_{t\to t_0^-}\alpha(t)=\infty$ (similarly for β), which is impossible because of (3.7). Therefore α and β exist on $[0,t_0]$ and can be extended to a larger interval $[0,t_0+\varepsilon)$, which contradicts the fact that t_0 was the extent of the existence. Therefore, we must have $t_0=\infty$ so that (1.3) holds.

Sufficiency. Now suppose that (1.3) holds. We show that problem (1.1) has a nonnegative generalized solution. We choose sequences $\{f_n\}$, $\{g_n\}$, $\{\varphi_n\}$, $\{\psi_n\}$, $\{u_{0,n}\}$, and $\{v_{0,n}\}$ as specified in the definition of a generalized solution. Such sequences are not difficult to construct using mollifiers and the properties of the functions f, g, φ, ψ, u_0, and v_0. Furthermore, the sequences $\{f_n\}$, $\{g_n\}$ may be (and are) chosen so that for each n, they satisfy (1.3) with f and g replaced by f_n and g_n, respectively. Let (u_n,v_n) be the smooth solution of (3.1), and let (y_n,z_n) be the solution of

$$\begin{gathered} y_n'(t)=2f_n(z_n(t)),\quad z_n'(t)=2g_n(y_n(t)),\quad 0<t<\infty,\\ y_n(0)=z_n(0)=M+1, \end{gathered} \tag{3.8}$$

where $M=\sup_n(\|u_{0,n}\|_{\infty,\Omega}+\|v_{0,n}\|_{\infty,\Omega})$. It is then clear that, as in the proof of (2.12),

$$0\le u_n(x,t)<y_n(t), \qquad 0\le v_n(x,t)<z_n(t) \quad \text{on } Q_\infty. \tag{3.9}$$

Also, since (y_n,z_n) converges locally uniformly to (y,z), we know that (y_n,z_n) is locally bounded so that (3.3) holds. To complete the proof, we will prove that the sequence $\{(u_n, v_n)\}$ has a subsequence $\{(U_n, V_n)\}$ defined on Q_∞ and obviously satisfying (3.3), which converges weakly in $L^1(Q_T)$ to a function pair (u, v) for all $T > 0$. To do this, we note that (3.3) implies that $\{(u_n, v_n)\}$ is pointwise bounded on Q_T which, in turn, implies that for each $k \in \mathbb{N}$, the $L^2(Q_k)$ norm (and every $L^p(Q_k)$ norm for $p \geq 1$) of the sequence $\{(u_n, v_n)\}$ is bounded. In particular, the $L^2(Q_1)$ norm is bounded independent of n so the sequence $\{(u_n, v_n)\}$ has a weakly convergent subsequence in $L^2(Q_1)$. We denote this subsequence by $\{(u_{n,1}, v_{n,1})\}$, and we let (P_1, R_1) be its weak $L^2(Q_1)$ limit. Likewise, the sequence $\{(u_{n,1}, v_{n,1})\}$ is bounded in the $L^2(Q_2)$ norm, and hence has a subsequence $\{(u_{n,2}, v_{n,2})\}$ which is weakly convergent to a function pair (P_2, R_2) in $L^2(Q_2)$. Clearly $(P_1, R_1) = (P_2, R_2)$ on Q_1. We continue the process to produce for each $k \in \mathbb{N}$ the sequence $\{(u_{n,k}, v_{n,k})\}$, a subsequence of $\{(u_{n,k-1}, v_{n,k-1})\}$, which is weakly convergent to (P_k, R_k) in $L^2(Q_k)$ and $(P_k, R_k) = (P_{k-1}, R_{k-1})$ on Q_{k-1}. Clearly the sequence (P_k, R_k) converges weakly in $L^2(Q_T)$ for all $T > 0$ to the function pair (u, v) defined on Q_∞ by $(u, v) = (P_j, R_j)$ on Q_j, $j \in \mathbb{N}$. In addition, it is easy to prove that the sequence of diagonal entries of the double-indexed sequence $\{(u_{n,k}, v_{n,k})\}$, namely $\{(u_{n,n}, v_{n,n})\}$, converges weakly in $L^2(Q_T)$, and hence weakly in $L^1(Q_T)$ to (u, v) for all $T > 0$. Thus the desired sequence $\{(U_n, V_n)\}$ of approximating solutions which converges to (u, v) is $\{(u_{n,n}, v_{n,n})\}$, and therefore (u, v) is a generalized solution of (1.1). This completes the proof. □

An open problem. We note that there is an important difference regarding the initial data in the hypothesis of Theorem 2.1, the smooth case, and Theorem 3.2, the nonsmooth case. In the latter, the initial data is required to be strictly positive, whereas in the former it needs only to be nonnegative and nontrivial. This leaves open the problem: can Theorem 3.2 be extended to the case where u_0 and v_0 are merely nonnegative with at least one of them nontrivial? With smooth constitutive functions, the solution will, in time, become strictly positive with only nonnegative nontrivial initial data. It is unknown whether this will ever occur in the nonsmooth case. However, it may be possible that a different proof can be devised, as in the scalar case [14], where Theorem 3.2 can be extended.

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