A NONLINEAR BOUNDARY PROBLEM INVOLVING THE p-BILAPLACIAN OPERATOR

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We show some new Sobolev's trace embedding that we apply to prove that the fourth-order nonlinear boundary conditions $\Delta_p^2 u + |u|^{p-2}u = 0$ in Ω and $-(\partial/\partial n)(|\Delta u|^{p-2}\Delta u) = \lambda\rho|u|^{p-2}u$ on $\partial\Omega$ possess at least one nondecreasing sequence of positive eigenvalues.

1. Introduction and notations

Let Ω be a bounded domain of class C^2 in $\mathbb{R}$, $N \geq 2, 1 < p < +\infty$, and $\rho \in L^r(\partial\Omega)$ a weight function which can change its sign, with $r = r(N,p)$ satisfying

$$r > \frac{N-1}{2p-1} \quad \text{for } \frac{N}{p} \geq 2, \qquad r = 1 \quad \text{for } \frac{N}{p} < 2. \tag{1.1}$$

We assume that $|(\partial\Omega)^+| \neq 0$, where $(\partial\Omega)^+ = \{x \in \partial\Omega,\ \rho(x) > 0\}$ and $\lambda \in \mathbb{R}$. We consider the following problem:

$$\begin{gathered} \Delta_p^2 u + |u|^{p-2}u = 0 \quad \text{in } \Omega, \\ -\frac{\partial}{\partial n}\left(|\Delta u|^{p-2}\Delta u\right) = \lambda\rho(x)|u|^{p-2}u \quad \text{on } \partial\Omega, \\ u \in W_0^{2,p}(\Omega). \end{gathered} \tag{1.2}$$

$\Delta_p^2 := \Delta(|\Delta u|^{p-2}\Delta u)$ is the operator of fourth order, so-called the p-biharmonic (or p-bilaplacian) operator. For $p = 2$, the linear operator $\Delta_2^2 = \Delta^2 = \Delta \cdot \Delta$ is the iterated Laplacian that to a multiplicative positive constant appears often in the equations of Navier-Stokes as being a viscosity coefficient, and its reciprocal operator noted $(\Delta^2)^{-1}$ is the celebrated Green's operator (see [8]).

Existence results for nonlinear boundary problem have only been considered in recent years. For the second-order p-Laplacian with nonlinear boundary conditions of different type, see [5], see also [3]. For a fourth-order elliptic equation with the ordinary boundary conditions, we cite [2] and with nonlinear boundary conditions, see [4].

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In this paper, we study in Theorem 2.5 the Sobolev's trace embedding $W^{m,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain of class C^m, $N \geq 2$, $q \in [1, p_m^\star[$ such that $p_m^\star = (N-1)p/(N-mp)$ if $mp < N$ and $p_m^\star = +\infty$ if $mp \geq N$. This embedding leads to a nonlinear eigenvalue problem (1.2), where the eigenvalue appears at the nonlinear boundary condition. Other main objective of this work, formulated by Theorem 3.3, is to show that problem (1.2) has at least one nondecreasing sequence of positive eigenvalues $(\lambda_k)_{k\geq 1}$, by using some technical lemmas and the Ljusternick-Schnirelmann theory on C^1-manifolds, see [9]. So we give a direct characterization of λ_k involving a minimax argument over sets of genus greater than k.

We set

$$\lambda_1 = \inf\left\{ \|u\|_{2,p}^p,\ u \in W^{2,p}(\Omega);\ \int_{\partial\Omega} \rho(x)|u|^p dx = 1 \right\}, \tag{1.3}$$

where $\|u\|_{2,p} = (\|u\|_p^p + \|\Delta u\|_p^p)^{1/p}$ is the norm of $W^{2,p}(\Omega)$.

This paper is organized as follows. In Section 2, we establish the Sobolev's trace embedding in the general case, that is, for any $m \in \mathbb{N}$. In Section 3, we use a variational technique to prove the existence of a sequence of the positive eigenvalues of problem (1.2).

2. The Sobolev's trace embedding

We begin with the following definition and lemmas that will be helpful to prove the Sobolev's trace embedding.

Definition 2.1. A domain Ω is of class C^k if $\partial\Omega$ can be covered by bounded open sets Θ_j such that there is a mapping $f_j : \overline{\Theta_j} \to \overline{B}$, where B is the unit ball centered at the origin and

$$\begin{aligned} f_j(\Theta_j \cap \Omega) &= B \cap \mathbb{R}_+^N, \\ f_j(\Theta_j \cap \partial\Omega) &= B \cap \partial\mathbb{R}_+^N, \\ f_j \in C^k(\overline{\Omega}), &\qquad f_j^{-1} \in C^k(\overline{B}). \end{aligned} \tag{2.1}$$

Lemma 2.2. *Let $u \in W^{1,1}(\mathbb{R}^N)$, $N > 1$. For all $y \in \mathbb{R}$, $v(\overline{x}) := u(\overline{x}, y) \in L^1(\mathbb{R}^{N-1})$ and*

$$\|v\|_{L^1(\mathbb{R}^{N-1})} \leq \|u\|_{L^1(\mathbb{R}^N)} + \left\| \frac{\partial u}{\partial x_N} \right\|_{L^1(\mathbb{R}^N)}. \tag{2.2}$$

Proof. $W^{1,1}(\mathbb{R}^N) = \overline{C_c^\infty(\mathbb{R}^N)}$. So, it suffices to prove the lemma for $u \in C_c^\infty(\mathbb{R}^N)$. Thus,

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} u(\overline{x}, y) d\overline{x} &\leq \int_{\mathbb{R}^{N-1}} \int_y^{+\infty} \left| \frac{\partial u}{\partial x_N}(\overline{x}, t) \right| dt\, d\overline{x} \\ &\leq \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \left| \frac{\partial u}{\partial x_N}(\overline{x}, t) \right| dt\, d\overline{x}, \end{aligned} \tag{2.3}$$

that is,

$$\|v\|_{L^1(\mathbb{R}^{N-1})} \leq \left\| \frac{\partial u}{\partial x_N} \right\|_{L^1(\mathbb{R}^N)} \leq \left\| \frac{\partial u}{\partial x_N} \right\|_{L^1(\mathbb{R}^N)} + \|u\|_{L^1(\mathbb{R}^N)}. \tag{2.4}$$

□

Lemma 2.3. *Let $u \in W^{1,p}(\mathbb{R}^N)$, $p < N$. For all $y \in \mathbb{R}$, $v(\overline{x}) := u(\overline{x}, y) \in L^t(\mathbb{R}^{N-1})$, where*

$$t = \frac{(N-1)p}{N-p} = 1 + \frac{(p-1)N}{N-p}, \tag{2.5}$$

and there exists a positive constant depending only on p and N such that

$$\|v\|_{L^t(\mathbb{R}^{N-1})} \leq c\|u\|_{1,p,\mathbb{R}^N}. \tag{2.6}$$

Proof. $W^{1,p}(\mathbb{R}^N) = \overline{C_c^\infty(\mathbb{R}^N)}$. So it suffices to prove the lemma for $u \in C_c^\infty(\mathbb{R}^N)$. If we set $w = |u|^t$, then $w \in W^{1,1}(\mathbb{R}^N)$ and

$$\begin{gathered} \|w\|_{L^1(\mathbb{R}^N)} \leq c\|\nabla u\|_{L^p(\mathbb{R}^N)}^{(t-1)} \|u\|_{L^p(\mathbb{R}^N)}, \\ \left\| \frac{\partial w}{\partial x_j} \right\|_{L^1(\mathbb{R}^N)} \leq c\|\nabla u\|_{L^p(\mathbb{R}^N)}^t. \end{gathered} \tag{2.7}$$

Indeed, let $q = p/(p-1)$, $(t-1)q = Np/(N-p)$ and by using the Sobolev inequalities, see [6],

$$\left\| |u|^{t-1} \right\|_{L^q(\mathbb{R}^N)} \leq c\|\nabla u\|_{L^p(\mathbb{R}^N)}^{Np/(N-p)}. \tag{2.8}$$

By Hölder and (2.8),

$$\|w\|_1 = \int_{\mathbb{R}^N} |u|^{(t-1)}|u|\,dx \leq \|u\|_p \left\| |u|^{(t-1)} \right\|_q \leq c\|u\|_p \|\nabla u\|_{L^p(\mathbb{R}^N)}^{(t-1)}. \tag{2.9}$$

On the other hand, $\partial w/\partial x_j = \pm t|u^{(t-1)}|(\partial u/\partial x_j)$. By Hölder and (2.8),

$$\left\| \frac{\partial w}{\partial x_j} \right\|_{L^1(\mathbb{R}^N)} \leq t\|u\|^{(t-1)}\|_{L^q(\mathbb{R}^N)} \left\| \frac{\partial u}{\partial x_j} \right\|_{L^p(\mathbb{R}^N)} \leq c\|\nabla u\|_{L^p(\mathbb{R}^N)}^t, \tag{2.10}$$

where c is a positive constant.

Now, applying (2.9), (2.10), and Lemma 2.3, we find

$$\|u\|_{L^t(\mathbb{R}^{N-1})} \leq c\big(\|u\|_{L^p(\mathbb{R}^N)} + \|\nabla u\|_{L^p(\mathbb{R}^N)}\big) \leq c\|u\|_{1,p,\mathbb{R}^N}. \tag{2.11}$$

□

Lemma 2.4. *Let $u \in W^{m,p}(\mathbb{R}^N)$, $N > 1$, $m \in \mathbb{R}$, and $mp < N$. For all $y \in \mathbb{R}$, $v(\overline{x}) := u(\overline{x}, y) \in L^{p_m^\star}(\mathbb{R}^{N-1})$, with $p_m^\star = (N-1)p/(N-mp)$ and there exists a positive constant c depending only on p and N such that*

$$\|v\|_{L^{p_m^\star}(\mathbb{R}^{N-1})} \leq c\|u\|_{m,p,\mathbb{R}^N}. \tag{2.12}$$

Proof. By applying Sobolev inequality [6] to $\partial u/\partial x_j$, $1 \leq j \leq N$, we obtain that $u \in W^{1,Np/(N-(m-1)p)}(\mathbb{R}^N)$. By Lemma 2.3, we deduce that $v \in L^{p_m^\star}(\mathbb{R}^{N-1})$ with $p_m^\star = (N-1)p/(N-mp)$. □

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain of class C^m. For all $u \in W^{m,p}(\Omega)$, $mp < N$. The restriction of u to $\partial\Omega$ denoted also by u belongs to $L^q(\partial\Omega)$, for all $q \in [1, p_m^\star]$,*

$$p_m^\star = \frac{(N-1)p}{N-mp} \tag{2.13}$$

and there exists a positive constant c depending only on p, m, and Ω such that

$$\|u\|_{p_m^\star(\partial\Omega)} \leq c\|u\|_{m,p,\Omega}. \tag{2.14}$$

Proof. There exists a continuous linear operator P that operates from $W^{m,p}(\Omega)$ to $W^{m,p}(\mathbb{R}^N)$, (cf. [1, 6]), such that to every u element of $W^{m,p}(\Omega)$ is associated an element $P(u) \in W^{2,p}(\mathbb{R}^N)$. By density, it is sufficient to study the properties of the trace on $\partial\Omega$ of the function $C_c^\infty(\mathbb{R}^N)$.

Let θ_j and f_j be as in the definition (2.2). $\partial\Omega$ is compact, therefore we can suppose that there exists a finite θ_j, $1 \leq j \leq k$, which covers $\partial\Omega$. Let $(P_j,\ 1 \leq j \leq k)$ be a partition of unity of $\partial\Omega$ subordinate to this covering, see, for example, [1]. If $u \in C_c^\infty(\mathbb{R}^N)$, then $P_j u o f_j^{-1} \in C_0^m(B)$. We extend $P_j u o f_j^{-1}$ to $C_0^m(\mathbb{R}^N)$. By Lemma 2.4, the trace w_j of $P_j u o f_j^{-1}$ on the hyperplane $\{(x_1, x_2, \dots, x_{N-1}, 0),\ x_i \in \mathbb{R}\}$ satisfies the inequality

$$||w_j||_{L^{p_m^\star}(\mathbb{R}^{N-1})} \leq c||P_j u o f_j^{-1}||_{m,p,B} \leq c_j\|u\|_{m,p,\mathbb{R}^N}, \tag{2.15}$$

where c_j is a positive constant. We estimate the trace $v_j := w_j o f_j$ of the function $P_j u$ on $\Gamma_j := \theta_j \cap \partial\Omega$. Then

$$||v_j||_{\Gamma_j}^{p_m^\star} \leq c_j \int_{\mathbb{R}^{N-1}\cap B} |w_j|^{p_m^\star} dx, \tag{2.16}$$

where c_j is a positive constant. We combine (2.15) and (2.16) as follows:

$$||v_j||_{L^{p_m^\star}(\Gamma_j)} \leq M_j\|u\|_{m,p,\mathbb{R}^N}. \tag{2.17}$$

On the other hand, $u = \sum_{j=1}^{j=k} P_j u = \sum_{j=1}^{j=k} v_j$, where $v_j = P_j u$, and $\operatorname{supp} v_j \subset \Gamma_j$, $\partial\Omega \subset \bigcup_{j=1}^{j=k} \Gamma_j$. So

$$\|u\|_{L^{p_m^\star}(\partial\Omega)} \leq \sum_{j=1}^{j=k} \|v_j\|_{L^{p_m^\star}(\partial\Omega)} = \sum_{j=1}^{j=k} \|v_j\|_{L^{p_m^\star}(\Gamma_j)}. \tag{2.18}$$

From (2.18),

$$\|u\|_{L^{p_m^\star}(\partial\Omega)} \leq \left(\sum_{j=1}^{j=k} M_j\right) \|u\|_{m,p,\mathbb{R}^N}. \tag{2.19}$$

On the other hand, $\partial\Omega$ is bounded, so $u \in L^q(\partial\Omega)$, for all $q \in [1, p_m^\star]$. □

By using Theorem 2.5, the next corollary follows exactly as in the classical compact Sobolev embedding established in [1, 6].

Corollary 2.6. *Under the same hypotheses at the last theorem, $W^{m,p}(\Omega)$ is compactly embedding in $L^q(\partial\Omega)$ for all $q \in [1, p_m^\star[$.*

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain of class C^m. For all $u \in W^{m,p}(\Omega)$, $mp \geq N$. The restriction of u to $\partial\Omega$ denoted also by u belongs to $L^q(\partial\Omega)$, for all $q \in [1, +\infty[$.*

Proof. Let an arbitrary $q \in [1, \infty[$. We can find $\overline{q}$ such that

(a) $m\overline{q} < N$;

(b) $\overline{q} = Nq/(N - 1 + mq)$.

From (b), $q = \overline{q}_m^\star = (N-1)\overline{q}/(N - m\overline{q})$. Since $m\overline{q} < N$, then $\overline{q} < p$ (because $mp \geq N$). So

(1) $W^{m,p}(\Omega)$ is continuously embedding in $W^{m,\overline{q}}(\Omega)$.

Since $q = \overline{q}_m^\star$ and $m\overline{q} < N$, thus from Theorem 2.5,

(2) $W^{m,\overline{q}}(\Omega)$ is continuously embedding in $L^t(\partial\Omega)$ for all $t \in [1, q]$,

and from Corollary 2.6,

(3) $W^{m,\overline{q}}(\Omega)$ is compactly embedding in $L^t(\partial\Omega)$ for all $t \in [1, q[$.

By combining (1), (2), and (3), we conclude that

(i) $W^{m,p}(\Omega)$ is continuously embedding in $L^t(\partial\Omega)$ for all $t \in [1, q]$,

(ii) $W^{m,p}(\Omega)$ is compactly embedding in $L^t(\partial\Omega)$ for all $t \in [1, q[$.

q being arbitrary, then we have the desired result. □

Theorem 2.8. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain of class C^m, $mp > N$. $W^{m,p}(\Omega)$ is compactly embedding $L^\infty(\partial\Omega) \cap C(\partial\Omega)$.*

Proof. By using the Sobolev embedding, $W^{m,p}(\Omega)$ is compactly embedding in $L^\infty(\Omega) \cap C(\overline{\Omega})$. So the functions of $W^{m,p}(\Omega)$ are continuous on $\overline{\Omega}$ and bounded, therefore their traces are well defined, continuous, and bounded. So we have

($*$) $W^{m,p}(\Omega)$ is compactly embedding in $L^\infty(\Omega) \cap C(\overline{\Omega})$,

($**$) $L^\infty(\Omega) \cap C(\overline{\Omega})$ is continuously embedding in $L^\infty(\partial\Omega) \cap C(\partial\Omega)$.

By ($*$) and ($**$), we have the desired result. □

3. Main results

Through this paper, all solutions are weak, that is, $u \in W^{2,p}(\Omega)$ is a solution of (1.2), if for all $v \in W^{2,p}(\Omega)$, we have

(S_1) $\langle \Delta_p^2 u, v\rangle + \int_\Omega |u|^{p-2}uv = 0$;

(S_2) $-\int_{\partial\Omega}(\partial/\partial n)(|\Delta u|^{p-2}\Delta u)v = \lambda\int_{\partial\Omega}\rho(x)|u|^{p-2}uv$.

If we replace S_2 in (S_1), then we deduce that

$$\int_\Omega |\Delta u|^{p-2}\Delta u\Delta v + \int_\Omega |u|^{p-2}uv = \lambda\int_{\partial\Omega}\rho|u|^{p-2}uv\,d\sigma. \tag{3.1}$$

If $u \in W^{2,p}(\Omega) - \{0\}$, then u is called the eigenfunction of (1.2) associated to the eigenvalue λ.

We will use the Ljusternick-Schnirelmann theory on C^1-manifolds [9].

Consider the following two functionals defined on $W^{2,p}(\Omega)$:

$$A(u) = \frac{1}{p}\|u\|_{W^{2,p}(\Omega)}, \qquad B(u) = \frac{1}{p}\int_{\partial\Omega}\rho(x)|u|^p d\sigma, \tag{3.2}$$

where $\|u\|_{W^{2,p}(\Omega)} = (\|u\|_p + \|\Delta u\|_p)^{1/p}$. We set

$$\mathcal{M} = \{u \in W^{2,p}(\Omega);\ pB(u) = 1\}. \tag{3.3}$$

Lemma 3.1. (i) *A and B are even and of class C^1 on $W^{2,p}(\Omega)$.*

(ii) *$\mathcal{M}$ is a closed C^1-manifold.*

Proof. (i) It is clear that A and B are even and of class C^1 on $W^{2,p}(\Omega)$, $A'(u) = \Delta_p^2 u + |u|^{p-2}u$, and $B'(u) = \rho|u|^{p-2}u$.

(ii) $\mathcal{M} = B^{-1}\{1/p\}$, so B is closed. Its derivative operator B' satisfies $B'(u) \neq 0$, for all $u \in \mathcal{M}$ (i.e., $B'(u)$ is onto for all $u \in \mathcal{M}$), so B is a submersion, then $\mathcal{M}$ is a C^1-manifolds. □

The following lemma is the key to show the existence.

Lemma 3.2. (i) *$B' : W^{2,p}(\Omega) \to (W^{2,p}(\Omega))'$ is completely continuous.*

(ii) *The functional A satisfies the Palais-Smale condition on $\mathcal{M}$, that is, for $\{u_n\} \subset \mathcal{M}$, if $A(u_n)$ is bounded and*

$$\epsilon_n := A'(u_n) - g_n B'(u_n) \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty, \tag{3.4}$$

where $g_n = \langle A'(u_n), u_n\rangle / \langle B'(u_n), u_n\rangle$. Then $\{u_n\}_{n\geq 1}$ has a convergent subsequence in $W^{2,p}(\Omega)$.

Proof. (i) *Step* 1 (definition of B').

First case. If $N/p > 2$, $r > (N-1)/(2p-1)$. Let $u, v \in W^{2,p}(\Omega)$. By Hölder's inequality, we have

$$\left|\int_{\partial\Omega}\rho(x)\,|\,u(x)\,|^{p-2}u(x)v(x)d\sigma\right| \leq \|\rho\|_r\|u\|_s^{p-1}\|v\|_{p_2^*}, \tag{3.5}$$

where $p_2^\star = (N-1)p/(N-2p)$, and s is given by

$$\frac{p-1}{s} + \frac{1}{p_2^\star} + \frac{1}{r} = 1. \tag{3.6}$$

Therefore,

$$\frac{p-1}{s} = 1 - \frac{1}{r} - \frac{1}{p_2^\star} > 1 - \frac{2p-1}{N-1} - \frac{N-2p}{(N-1)p} = \frac{p-1}{p_2^\star}. \tag{3.7}$$

Then it suffices that

$$\max(1, p-1) < s < p_2 \tag{3.8}$$

and B' is well defined.

Second case. If $N/p = 2$, $r > (N-1)/(2p-1)$. In this case, from Theorem 2.7,

$$W^{2,p}(\Omega) \hookrightarrow L^q(\partial\Omega) \tag{3.9}$$

for any $q \in [1, +\infty[$. There is $q \geq 1$ such that

$$\frac{1}{q} + \frac{1}{r} + \frac{p-1}{p} = \frac{1}{q} + \frac{1}{r} + \frac{1}{p'} = 1. \tag{3.10}$$

We obtain that

$$\frac{1}{q} = 1 - \left(\frac{1}{r} + \frac{1}{p'}\right) \leq 1. \tag{3.11}$$

By Hölder's inequality, we arrive at

$$\left| \int_{\partial\Omega} \rho(x) \,|\, u(x) \,|^{p-2} u(x) v(x) dx \right| \leq \|\rho\|_r \|u\|_p^{p-1} \|v\|_q \tag{3.12}$$

for any $u, v \in W^{2,p}(\Omega)$. Then in this case, B' is well defined.

Third case. If $N/p < 2$, $r = 1$. In this case, from Theorem 2.8,

$$W^{2,p}(\Omega) \hookrightarrow C(\partial\Omega) \cap L^\infty(\partial\Omega). \tag{3.13}$$

Therefore for any $u, v \in W^{2,p}(\Omega)$, we have

$$\left| \int_{\partial\Omega} \rho(x) \left| u(x) \right|^{p-2} u(x) v(x) dx \right| < \infty, \tag{3.14}$$

with $\rho \in L^1(\Omega)$, and B' is well defined also in this case.

Step 2. B' is completely continuous. Let $(u_n) \subset W^{2,p}(\Omega)$ be a sequence such that $u_n \to u$ weakly in $W^{2,p}(\Omega)$. We must show that $B'(u_n) \to B'(u)$ strongly in $(W^{2,p}(\Omega))'$, that is,

$$\sup_{\substack{v \in W^{2,p}(\Omega) \\ \|v\|_{2,p} \le 1}} \left| \int_{\partial\Omega} \rho \left[\left| u_n \right|^{p-2} u_n - |u|^{p-2} u \right] v \, dx \right| \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty. \tag{3.15}$$

For this end, we distinguish three cases as in Step 1 above for $N/p > 2$, and $r > (N-1)/(2p-1)$. Let s be as in (3.8). Then,

$$\begin{aligned} &\sup_{\substack{v \in W^{2,p}(\Omega) \\ \|v\|_{2,p} \le 1}} \left| \int_{\partial\Omega} \rho \left[\left| u_n \right|^{p-2} u_n - |u|^{p-2} u \right] v dx \right| \\ &\quad \le \sup_{\substack{v \in W^{2,p}(\Omega) \\ \|v\|_{2,p} \le 1}} \left[\|\rho\|_r \left\| \left| u_n \right|^{p-2} u_n - |u|^{p-2} u \right\|_{s/(p-1)} \|v\|_{p_2^\star} \right] \\ &\quad \le c \|\rho\|_r \left\| |u_n|^{p-2} u_n - |u|^{p-2} u \right\|_{s/(p-1)}, \end{aligned} \tag{3.16}$$

where c is the constant of Sobolev's embedding [1].

On other hand, the Nemytskii's operator $u \mapsto |u|^{p-2}u$ is continuous from $L^s(\partial\Omega)$ into $L^{s/(p-1)}(\partial\Omega)$, and $u_n \to u$ weakly in $W^{2,p}(\Omega)$. So, we deduce that $u_n \to u$ strongly in $L^s(\partial\Omega)$ because $s < p_2^\star$. Hence,

$$\left\| \left| u_n \right|^{p-2} u_n - |u|^{p-2} u \right\|_{s/(p-1)} \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty. \tag{3.17}$$

This completes the proof of the claim in this case.

If $N/p = 2$,

$$\left| \int_{\partial\Omega} \rho \left[\left| u_n \right|^{p-2} u_n - |u|^{p-2} u \right] v dx \right| \le \|\rho\|_r \left\| \left| u_n \right|^{p-2} u_n - |u|^{p-2} u \right\|_p^{p-1} \|v\|_q, \tag{3.18}$$

where q is given by (3.11). By Sobolev's trace embedding, there exists $c > 0$ such that

$$\|v\|_q \le c \|v\|_{2,p}, \quad \forall v \in W^{2,p}(\Omega). \tag{3.19}$$

Thus,

$$\sup_{\substack{v \in W^{2,p}(\Omega) \\ \|v\|_{2,p} \le 1}} \left| \int_{\partial\Omega} \rho \left[\left| u_n \right|^{p-2} u_n - |u|^{p-2} u \right] v \, dx \right| \le c \|\rho\|_r \left\| \left| u_n \right|^{p-2} u_n - |u|^{p-2} u \right\|_p^{p-1}. \tag{3.20}$$

From the continuity of $u \mapsto |u|^{p-1}u$ from $L^p(\partial\Omega)$ into $L^{p'}(\partial\Omega)$, and from the compact embedding of $W^{2,p}(\Omega)$ in $L^p(\partial\Omega)$, we have the desired result.

If $N/p < 2$, $r = 1$. $W^{2,p}(\Omega) \hookrightarrow C(\partial\Omega)$, we obtain

$$\sup_{\substack{v\in W^{2,p}(\Omega)\\ \|v\|_{2,p}\leq 1}} \left| \int_{\partial\Omega} \rho\Big[|u_n|^{p-2}u_n - |u|^{p-2}u \Big] v\,dx \right| \leq c\|\rho\|_1 \sup_{\partial\Omega} \Big| |u_n|^{p-2}u_n - |u|^{p-2}u \Big|, \tag{3.21}$$

where c is the constant given by embedding of $W^{2,p}(\Omega)$ in $C(\partial\Omega)\cap L^\infty(\partial\Omega)$.

It is clear that

$$\sup_{\partial\Omega} \Big| |u_n|^{p-2}u_n - |u|^{p-2}u \Big| \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty. \tag{3.22}$$

Hence B' is completely continuous also in this case.

(ii) $\{u_n\}$ is bounded in $W^{2,p}(\Omega)$. Hence without loss of generality, we can assume that u_n converges weakly in $W^{2,p}(\Omega)$ to some function $u \in W^{2,p}(\Omega)$ and $\|u_n\|_{2,p} \to c$. For the rest, we distinguish two cases. If $c = 0$, then u_n converges strongly to 0 in $W^{2,p}(\Omega)$.

If $c \neq 0$, the claim is to prove that u_n is of Cauchy in $W^{2,p}(\Omega)$.

Set

$$\begin{aligned} G(u_n,u_m) &= \langle A'(u_n) - A'(u_m), u_n - u_m \rangle, \\ G_1(u_n,u_m) &= \langle \Delta_p^2 u_n - \Delta_p^2 u_m, u_n - u_m \rangle, \\ G_2(u_n,u_m) &= \Big\langle |u_n|^{p-2}u_n - |u_m|^{p-2}u_m, u_n - u_m \Big\rangle. \end{aligned} \tag{3.23}$$

We remark that

$$G(u_n,u_m) = G_1(u_n,u_m) + G_2(u_n,u_m). \tag{3.24}$$

On the other hand,

$$\begin{aligned} G(u_n,u_m) &= \langle A'(u_n) - A'(u_m), u_n - u_m \rangle \\ &= \langle \epsilon_n - \epsilon_m, u_n - u_m \rangle + \langle h_n - h_m, u_n - u_m \rangle, \end{aligned} \tag{3.25}$$

with ϵ_n defined as in (3.4), and $h_n = \|u\|_{2,p}^p B'(u_n)$.

$$G(u_n,u_m) \leq ||\epsilon_n - \epsilon_m||_* ||u_n - u_m||_{2,p} + ||h_n - h_m||_* ||u_n - u_m||_{2,p}, \tag{3.26}$$

where $\|\cdot\|_*$ is the dual norm associated to $\|\cdot\|_{2,p}$.

This implies that h_n converges, for a subsequence if necessary, in $W^{2,p}(\Omega)$. Indeed, from (i) of Lemma 3.2 $B' : W^{2,p}(\Omega) \to (W^{2,p}(\Omega))'$ is completely continuous. On the other hand, for a subsequence if necessary, $\|u_n\|_{2,p} \to c \geq 0$. It follows that $(h_n)_{n\geq 0}$ is convergent in $(W^{2,p}(\Omega))'$. Then,

$$G(u_n,u_m) \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty. \tag{3.27}$$

From [7], we have the inequality

$$|t_1 - t_2|^p \leq c\Big\{\Big(|t_1|^{p-2}t_1 - |t_2|^{p-2}t_2\Big)\cdot(t_1 - t_2)\Big\}^{\gamma/2}\Big(|t_1|^p + |t_2|^p\Big)^{1-\gamma/2}, \tag{3.28}$$

for any $t_1, t_2 \in \mathbb{R}$, with $\gamma = p$ if $1 < p < 2$ and $\gamma = 2$ if $p \geq 2$. By applying Hölder's inequality, we deduce that

$$||\Delta u_n - \Delta u_m||_p^p \leq c\{G_1(u_n, u_m)\}^{\gamma/2}\big(||\Delta u_n||_p^p + ||\Delta u_m||_p^p\big)^{1-\gamma/2}, \tag{3.29}$$

$$||u_n - u_m||_p^p \leq c\{G_2(u_n, u_m)\}^{\gamma/2}\Big(||u_n||_p^p + ||u_m||_p^p\Big)^{1-\gamma/2}, \tag{3.30}$$

where c is a positive constant independent of n and m, $\gamma = p$ if $1 < p < 2$, and $\gamma = 2$ if $p \geq 2$.

From [7], we have

$$\begin{aligned} \Big(|u_n|^{p-2}u_n - |u_m|^{p-2}u_m\Big)(u_n - u_m) &\geq c\frac{|u_n - u_m|^{(-\gamma+p+2)}}{(|u_n| + |u_m|)^{(2-\gamma)}}, \\ \Big(|\Delta u_n|^{p-2}\Delta u_n - |\Delta u_m|^{p-2}\Delta u_m\Big)(\Delta u_n - \Delta u_m) &\geq c\frac{|\Delta u_n - \Delta u_m|^{(-\gamma+p+2)}}{(|\Delta u_n| + |\Delta u_m|)^{(2-\gamma)}}, \end{aligned} \tag{3.31}$$

where $\gamma = p$ if $1 < p < 2$ and $\gamma = 2$ if $p \geq 2$. By integrating these two relations over Ω, we find

$$G_1(u_n, u_m) \geq 0, \qquad G_2(u_n, u_m) \geq 0. \tag{3.32}$$

On the other hand, $G_1 \leq G$ and $G_2 \leq G$. Then from (3.27) and (3.32),

$$G_1(u_n, u_m) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \qquad G_2(u_n, u_m) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.33}$$

Then from (3.29) and (3.30),

$$||\Delta u_n - \Delta u_m||_p \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \qquad ||u_n - u_m||_p \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.34}$$

So

$$||u_n - u_m||_{2,p}^p \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.35}$$

Therefore $(u_n)_n$ is a Cauchy's sequence in $W^{2,p}(\Omega)$. This achieves the proof of the lemma. □

Set

$$\Gamma_k = \{K \subset \mathcal{M} : K \text{ is symmetric, compact and } \gamma(K) \geq k\}, \tag{3.36}$$

where $\gamma(K) = k$ is the genus of K, that is, the smallest integer k such that there exists an odd continuous map from K to $\mathbb{R}^k - \{0\}$.

Now, by the Ljusternick-Schnirelmann theory, see, for example, [9], we have our main result.

Theorem 3.3. *For any integer $k \in \mathbb{N}^*$,*

$$\lambda_k := \inf_{K \in \Gamma_k} \max_{u \in K} pA(u) \tag{3.37}$$

is a critical value of A restricted on $\mathcal{M}$. More precisely, there exists $u_k \in K_k \in \gamma_K$ such that

$$\lambda_k = pA(u_k) = \sup_{u \in K_k} pA(u) \tag{3.38}$$

and (λ_k, u_k) is a solution of (1.2) associated to the positive eigenvalue λ_k. Moreover,

$$\lambda_k \longrightarrow +\infty, \quad \text{as } k \longrightarrow +\infty. \tag{3.39}$$

Proof. We need only to prove that for any $k \in \mathbb{N}^*$, $\Gamma_k \neq \emptyset$ and the least assertion. Indeed, since $W^{2,p}(\Omega)$ is separable, there exists $(e_i)_{i \geq 1}$ linearly dense in $W^{2,p}(\Omega)$ such that $\operatorname{supp} e_i \cap \operatorname{supp} e_j = \emptyset$ if $i \neq$j. We can assume that $e_i \in \mathcal{M}$. Let $k \in \mathbb{N}^*$, denote $F_k = \operatorname{span}\{e_1, e_2, \ldots, e_k\}$. F_k is a vectorial subspace and $\dim F_k = k$.

If $v \in F_k$, then there exist $\alpha_1, \ldots, \alpha_k$ in $\mathbb{R}$ such that $v = \sum_{i=1}^k \alpha_i e_i$. Thus $B(v) = \sum_{i=1}^k |\alpha_i|^p B(e_i) = (1/p) \sum_{i=1}^k |\alpha_i|^p$. It follows that the map $v \mapsto (pB(v))^{1/p} := \|v\|$ defines a norm on F_k. Consequently, there is a constant $c > 0$ such that

$$c\|u\|_{2,p} \leq \|v\| \leq \frac{1}{c}\|u\|_{2,p}. \tag{3.40}$$

This implies that the set

$$V = F_k \cap \left\{ v \in W^{2,p}(\Omega) : B(v) \leq \frac{1}{p} \right\} \tag{3.41}$$

is bounded. Thus V is a symmetric bounded neighborhood of $0 \in F_k$. By [9, Proposition 2.3(f)], $\gamma\,(F_k \cap \mathcal{M}) = k$ because $F_k \cap \mathcal{M}$ is compact, and $\Gamma_k \neq \emptyset$.

Now we claim that $\lambda_k \to +\infty$, as $k \to +\infty$. Let $(e_n, e_j^*)_{n,j}$ be a biorthogonal system such that $e_n \in W^{2,p}(\Omega)$, $e_j^* \in (W^{2,p}(\Omega))'$, the e_n are linearly dense in $W^{2,p}(\Omega)$; and the e_j^* are total for $(W^{2,p}(\Omega))'$, see, for example, [9]. Set now, for $k \in \mathbb{N}^*$,

$$F_k = \operatorname{span}\{e_1, \ldots, e_k\}, \qquad F_k^{\perp} = \operatorname{span}\{e_{k+1, e_{k+2}, \ldots}\}. \tag{3.42}$$

By [9, Proposition 2.3(g)], we have for any $A \in \Gamma_k$, $A \cap F_{k-1}^{\perp} \neq \emptyset$. Thus,

$$t_k := \inf_{A \in \Gamma_k} \sup_{u \in A \cap F_{k-1}^{\perp}} pA(u) \longrightarrow +\infty. \tag{3.43}$$

Indeed, if not, for k is large, there exists $u_k \in F_{k-1}^{\perp}$ with $\|u_k\|_p = 1$ such that

$$t_k \leq pA(u_k) \leq M \tag{3.44}$$

for some $M > 0$ independent of k. Therefore,

$$||u_k||_{2,p} \leq M. \tag{3.45}$$

This implies that $(u_k)_k$ is bounded in $W^{2,p}(\Omega)$. For a subsequence of $\{u_k\}$ if necessary, we can assume that $\{u_k\}$ converges weakly in $W^{2,p}(\Omega)$ and strongly in $L^p(\Omega)$. By our choice of $F_{k-1}^{\perp}$, we have $u_k \hookrightarrow 0$ weakly in $W^{2,p}(\Omega)$ because $\langle e_n^*, e_k\rangle = 0$, for all $k \geq n$.

This contradicts the fact that $\|u_k\|_p = 1$ for all k. Since $\lambda_k \geq t_k$, the claim is proved, which completes the proof. □

Corollary 3.4. (i) $\lambda_1 = \inf\{\|v\|_{2,p}^p,\ v \in W^{2,p}(\Omega);\ \int_{\partial\Omega}\rho(x)|v|^p dx = 1\}$.

(ii) $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to +\infty$.

Proof. (i) For $u \in \mathcal{M}$, we put $K_1 = \{u, -u\}$, $\gamma(K_1) = 1$. A is even, so

$$pA(u) = \max_{K_1} pA \geq \inf_{K\in\Gamma_1} \max_{K} pA. \tag{3.46}$$

Hence,

$$\inf_{u\in\mathcal{M}} pA(u) \geq \inf_{K\in\Gamma_1} \max_{K} pA = \lambda_1. \tag{3.47}$$

On the other hand, for all $K \in \Gamma_1$, for all $u \in K$,

$$\sup_{K} pA \geq pA(u) \geq \inf_{u\in\mathcal{M}} pA(u). \tag{3.48}$$

So,

$$\inf_{K\in\Gamma_1} \max_{K} pA = \lambda_1 \geq \inf_{u\in\mathcal{M}} pA(u). \tag{3.49}$$

Then,

$$\lambda_1 = \inf_{u\in\mathcal{M}} pA(u) = \inf\left\{\|v\|_{2,p}^p,\ v \in W^{2,p}(\Omega);\ \int_{\partial\Omega}\rho(x)|v|^p dx = 1\right\}. \tag{3.50}$$

(ii) For all $i, j \in \mathbb{N}^*$, $\Gamma_i \subset \Gamma_j$. From the definition of λ_i, $i \in \mathbb{N}^*$, we have $\lambda_i \geq \lambda_j$. $\lambda_n \to +\infty$ is already proved in Theorem 3.3. The proof is achieved. □

Corollary 3.5. *If it is supposed that* $|(\partial\Omega)_\rho^-| \neq 0$ *with* $\partial\Omega_\rho^- = \{x \in \partial\Omega;\ \rho(x) < 0\}$, *then (1.2) has a decreasing sequence of the negative eigenvalues* $(\lambda_{-n})(\rho)_{n\geq 0}$ *such that* $\lim_{n\to+\infty}\lambda_{-n} = -\infty$.

Proof. First of all, we remark that $(\partial\Omega)_\rho^- = (\partial\Omega)_{(-\rho)}^+$. So $|(\partial\Omega)_{(-\rho)}^+| = |(\partial\Omega)_\rho^-| \neq 0$. From Theorem 3.3, (1.2) has a nondecreasing sequence of the positive eigenvalues $\lambda_n(-\rho)$ such that $\lim_{n\to+\infty}\lambda_n(-\rho) = +\infty$.

$\lambda_n(-\rho)$ satisfies $-(\partial/\partial n)(|\Delta u|^{p-2}\Delta u) = \lambda_n(-\rho)(-\rho)|u|^{p-2}u = -\lambda_n(-\rho)\rho|u|^{p-2}u$, $u \in W^{2,p}(\Omega)$. We put

$$\lambda_{-n}(\rho) := -\lambda_n(-\rho). \tag{3.51}$$

$\lambda_n(-\rho)_{n\geq 0}$ is a nondecreasing positive sequence, so $(\lambda_{-n})(\rho)_{n\geq 0}$ is a negative decreasing sequence.

On the other hand, $\lim_{n\to+\infty}\lambda_n(-\rho) = +\infty$. So,

$$\lim_{n\to+\infty}\lambda_{-n}(\rho) = -\infty. \tag{3.52}$$

□

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