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THE PSEUDODIFFERENTIAL OPERATOR $A(x,D)$

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The pseudodifferential operator (p.d.o.) $A(x,D)$, associated with the Bessel operator $d^2/dx^2 + (1-4\mu^2)/4x^2$, is defined. Symbol class $H^m_{\rho,\delta}$ is introduced. It is shown that the p.d.o. associated with a symbol belonging to this class is a continuous linear mapping of the Zemanian space H_μ into itself. An integral representation of p.d.o. is obtained. Using Hankel convolution $L^p_{\sigma,\alpha}$-norm continuity of the p.d.o. is proved.

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1. Introduction. The theory of the Hankel transformation

$$(h_\mu f)(y) = \int_0^\infty (xy)^{1/2} J_\mu(xy) f(x)\,dx, \quad \mu \geq -\frac{1}{2}, \tag{1.1}$$

where $0 < y < \infty$ and J_μ is the Bessel function of the first kind and of order μ, has been extended by Zemanian [14] to distributions belonging to H'_μ, the dual of the test function space H_μ, consisting of all complex-valued smooth functions ϕ defined on $I = (0,\infty)$ and satisfying

$$\gamma^\mu_{m,k}(\phi) = \sup_{0<x<\infty} \left| x^m \left(x^{-1}\frac{d}{dx}\right)^k \left[x^{-\mu-1/2}\phi(x)\right]\right| < \infty \tag{1.2}$$

for each pair of nonnegative integers m and k. Pseudodifferential operators associated with a numerical valued symbol $a(x,y)$ were investigated by R. S. Pathak and Pandey [8, 9] and by R. S. Pathak and S. Pathak [10].

We will use the notation and terminology of [4, 7, 11, 12, 14]. The differential operators N_μ, M_μ, and S_μ are defined by

$$N_\mu = N_{\mu,x} = x^{\mu+3/2}\left(x^{-1}\frac{d}{dx}\right)x^{-\mu-1/2}, \tag{1.3}$$

$$M_\mu = M_{\mu,x} = x^{-\mu+1/2}\left(x^{-1}\frac{d}{dx}\right)x^{\mu+1/2}, \tag{1.4}$$

$$\begin{aligned} S_\mu = S_{\mu,x} = M_\mu N_\mu &= (2\mu+2)x^{\mu+1/2}\left(x^{-1}\frac{d}{dx}\right)x^{-\mu-1/2} \\ &\quad + x^{\mu+5/2}\left(x^{-1}\frac{d}{dx}\right)^2 x^{-\mu-1/2}. \end{aligned} \tag{1.5}$$

From [14, page 139] and [5, page 948] we know the following relations for any $\phi \in H_\mu$:

$$h_{\mu+1}(N_\mu\phi) = -y h_\mu \phi, \tag{1.6}$$

$$h_\mu(S_\mu\phi) = -y^2 h_\mu \phi, \tag{1.7}$$

$$\left(x^{-1}\frac{d}{dx}\right)^k (x^{-\mu-1/2}\theta\phi) = \sum_{v=0}^{k} \binom{k}{v} \left(x^{-1}\frac{d}{dx}\right)^v \theta \left(x^{-1}\frac{d}{dx}\right)^{k-v} (x^{-\mu-1/2}\phi), \tag{1.8}$$

$$S^r_{\mu,x}\phi(x) = \sum_{j=0}^{r} b_j x^{2j+\mu+1/2} \left(x^{-1}\frac{d}{dx}\right)^{r+j} (x^{-\mu-1/2}\phi(x)), \tag{1.9}$$

where the b_j are constants depending only on μ.

Next, we define the space $L^p_\sigma(I)$, $1 \le p < \infty$, as the space of those real-valued measurable functions on I for which

$$\|f\|_{L^p_\sigma} = \left[\int_0^\infty |f(x)|^p d\sigma(x)\right]^{1/p} < \infty, \tag{1.10}$$

where

$$d\sigma(x) = \left(2^\mu\Gamma(\mu+1)\right)^{-1} x^{2\mu+1} dx. \tag{1.11}$$

Let $\triangle(x,y,z)$ denote the area of a triangle with sides x, y, and z, if such a triangle exists. For fixed $\mu \ge -1/2$, set

$$D(x,y,z) = 2^{3\mu-1/2}(\pi)^{-1/2}[\Gamma(\mu+1)]^2\left[\Gamma\left(\mu+\frac{1}{2}\right)\right]^{-1}(xyz)^{-2\mu}[\triangle(x,y,z)]^{2\mu-1}, \tag{1.12}$$

if $\triangle$ exists, and zero otherwise. Then $D(x,y,z) \ge 0$ and that $D(x,y,z)$ is symmetric in x, y, z. From [13, page 411], we know that

$$j(xt)j(yt) = \int_0^\infty j(zt)D(x,y,z)d\sigma(z), \tag{1.13}$$

where

$$j(x) = 2^\mu\Gamma(\mu+1)x^{-\mu}J_\mu(x). \tag{1.14}$$

From (1.13) it follows that

$$J_\mu(xt)J_\mu(yt) = \left(2^\mu\Gamma(\mu+1)\right)^{-1}\int_0^\infty (xyt)^\mu z^{-\mu} J_\mu(zt)D(x,y,z)d\sigma(z). \tag{1.15}$$

Let $f \in L^1_\sigma(I)$, then its associated function $f(x,y)$ is defined by

$$f(x,y) = \int_0^\infty f(z)D(x,y,z)d\sigma(z), \quad 0 < x,y < \infty, \tag{1.16}$$

and the Hankel convolution of f and g is defined by

$$f\#g(x) = \int_0^\infty f(x,y)g(y)d\sigma(y), \quad 0 < x < \infty. \tag{1.17}$$

From [9, page 102] we know that if $f(x) \in L_\sigma^q(I)$, $g(x) \in L_\sigma^p(I)$ with $p > 1$, $q > 1$, and $1/p + 1/q > 1$, then

$$\|f\#g\|_r \le \|f\|_q \|g\|_p, \tag{1.18}$$

where

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 > 0. \tag{1.19}$$

Many algebraic and topological properties of the generalized Hankel convolution have been given by Betancor and his associates [1, 2, 3, 6].

We also note the differentiation formula

$$\left(x^{-1}\frac{d}{dx}\right)^m x^{-\mu} J_\mu(xt) = (-t)^m x^{-\mu-m} J_{\mu+m}(xt). \tag{1.20}$$

In this paper, a general class $H_{\rho,\delta}^m$ of symbols associated with the differentiation formula (1.20) and similar to the Hörmander class $S_{\rho,\delta}^m$ is introduced. For $\rho = 1/2$ and $\delta = 0$, the class $H_{\rho,\delta}^m$ reduces to the symbol class H_0^m studied by R. S. Pathak and Pandey [8]. An integral representation for $A(x,D)$ is given when the symbol belongs to the class $H_{\rho,\delta}^m$. It is shown that $A(x,D)$ is a continuous linear mapping of Zemanian's space H_μ into itself. The space $L_{\sigma,\alpha}^p$ is defined and an $L_{\sigma,\alpha}^p$-boundedness result is also obtained.

2. The pseudodifferential operator $A(x,D)$

Definition 2.1. Let $a(x,\xi)$ be a complex-valued smooth function belonging to the space $C^\infty(I \times I)$, where $I = (0,\infty)$, and let its derivatives satisfy certain growth conditions such as (2.3). The pseudodifferential operator (p.d.o.) $A(x,D)$, associated with the symbol $a(x,\xi)$, is defined by

$$A(x,D)f(x) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi) a(x,\xi)(h_\mu f)(\xi)d\xi, \tag{2.1}$$

where

$$(h_\mu f)(\xi) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi) f(x)dx, \quad \mu \ge -\frac{1}{2}. \tag{2.2}$$

Definition 2.2. The function $a(x,\xi) : C^\infty(I \times I) \to \mathbb{C}$ belongs to the symbol class $H_{\rho,\delta}^m$, $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$, if and only if for all $\alpha, \beta \in \mathbb{N}_0$, there exists $C_{\alpha,\beta,m} > 0$ such that

$$\left|\left(\xi^{-1}\frac{d}{d\xi}\right)^\alpha \left(x^{-1}\frac{d}{dx}\right)^\beta a(x,\xi)\right| \le C_{\alpha,\beta,m}(1+\xi^2)^{m/2-\rho\alpha-\delta\beta}. \tag{2.3}$$

A few examples of elements of $H^m_{\rho,\delta}$ are given as

(i) $(1+x^2+\xi^2)^{m/2}$, $m<0$,

(ii) $\sin x^2(1+x^2+\xi^2)^{m/2}$, $m<0$,

(iii) $e^{-x^2}(1+x^2+\xi^2)^{m/2}$, $m<0$.

We give a proof for example (i). We have

$$\begin{gathered} a(x,\xi)=(1+x^2+\xi^2)^{m/2}, \quad m<0, \\ \left(x^{-1}\frac{d}{dx}\right)^{\beta}(1+x^2+\xi^2)^{m/2}=D_{m,\beta}(1+x^2+\xi^2)^{m/2-\beta}, \end{gathered} \tag{2.4}$$

so that

$$\begin{aligned} &\left|\left(\xi^{-1}\frac{d}{d\xi}\right)^{\alpha}\left(x^{-1}\frac{d}{dx}\right)^{\beta}(1+x^2+\xi^2)^{m/2}\right| \\ &\qquad = \left|D_{m,\alpha,\beta}(1+x^2+\xi^2)^{m/2-\beta-\alpha}\right| \le D_{m,\alpha,\beta}(1+\xi^2)^{m/2-\rho\alpha-\delta\beta} \quad (m<0) \end{aligned} \tag{2.5}$$

for $0\le\delta\le 1$ and $0\le\rho\le 1$.

Theorem 2.3. *Let the symbol $a(x,\xi)\in H^m_{\rho,\delta}$. Then for $\mu\ge -1/2$, the p.d.o. $A(x,D)$ is a continuous linear mapping of H_μ into H_μ.*

Proof. Let $\Phi(x)=A(x,D)f(x)$, $f\in H_\mu$. Then using definitions (1.3) and (2.1), we have

$$N_\mu\Phi(x)=x^{\mu+3/2}\left(x^{-1}\frac{d}{dx}\right)x^{-\mu-1/2}\Phi(x). \tag{2.6}$$

By induction, we get

$$\begin{aligned} &N_{\mu+k-1}\cdots N_\mu\Phi(x) \\ &\quad = x^{\mu+k+1/2}\left(x^{-1}\frac{d}{dx}\right)^{k}x^{-\mu-1/2}\Phi(x) \\ &\quad = x^{\mu+k+1/2}\left(x^{-1}\frac{d}{dx}\right)^{k}x^{-\mu-1/2}\int_0^\infty (x\xi)^{1/2}J_\mu(x\xi)a(x,\xi)(h_\mu f)(\xi)d\xi. \end{aligned} \tag{2.7}$$

Using formula (1.8), we get

$$\begin{aligned} &N_{\mu+k-1}\cdots N_\mu\Phi(x) \\ &\quad = \sum_{r=0}^{k}\binom{k}{r}x^{\mu+k+1/2}\int_0^\infty \left(x^{-1}\frac{d}{dx}\right)^{k-r}x^{-\mu}J_\mu(x\xi) \\ &\qquad\qquad \times\left(x^{-1}\frac{d}{dx}\right)^{r}a(x,\xi)\xi^{1/2}(h_\mu f)(\xi)d\xi. \end{aligned} \tag{2.8}$$

Next, applying (1.20), we have

$$\begin{aligned} N_{\mu+k-1}\cdots N_\mu\Phi(x) &= \sum_{r=0}^{k}\binom{k}{r}x^{\mu+k+1/2}\int_0^\infty(-\xi)^{k-r}x^{-\mu-k+r}J_{\mu+k-r}(x\xi) \\ &\qquad\times\left(x^{-1}\frac{d}{dx}\right)^r a(x,\xi)\xi^{1/2}(h_\mu f)(\xi)d\xi \\ &= \sum_{r=0}^{k}\binom{k}{r}\int_0^\infty x^r(x\xi)^{1/2}J_{\mu+k-r}(x\xi) \\ &\qquad\times\left(x^{-1}\frac{d}{dx}\right)^r a(x,\xi)(-\xi)^{k-r}(h_\mu f)(\xi)d\xi. \end{aligned} \tag{2.9}$$

Using formula (1.6), we get

$$\begin{aligned} &(-x)(N_{\mu+k-1}\cdots N_\mu\Phi(x)) \\ &\quad= A_0\int_0^\infty(x\xi)^{1/2}J_{\mu+k+1}(x\xi)N_{\mu+k}[a(x,\xi)(-\xi)^k(h_\mu f)(\xi)]d\xi \\ &\qquad+A_1\int_0^\infty x^{3/2}\xi^{1/2}J_{\mu+k}(x\xi)N_{\mu+k-1} \\ &\qquad\qquad\times\left[\left(x^{-1}\frac{d}{dx}\right)a(x,\xi)(-\xi)^{k-1}(h_\mu f)(\xi)\right]d\xi \\ &\qquad+A_2\int_0^\infty x^{5/2}\xi^{1/2}J_{\mu+k-1}(x\xi)N_{\mu+k-2} \\ &\qquad\qquad\times\left[\left(x^{-1}\frac{d}{dx}\right)^2 a(x,\xi)(-\xi)^{k-2}(h_\mu f)(\xi)\right]d\xi+\cdots \\ &\qquad+A_k\int_0^\infty x^{k+1/2}\xi^{1/2}J_{\mu+1}(x\xi)N_\mu\left[\left(x^{-1}\frac{d}{dx}\right)^k a(x,\xi)(h_\mu f)(\xi)\right]d\xi, \\ &(-x^2)(N_{\mu+k-1}\cdots N_\mu\Phi(x)) \\ &\quad= A_0\int_0^\infty(x\xi)^{1/2}J_{\mu+k+2}(x\xi)N_{\mu+k+1}N_{\mu+k}[a(x,\xi)(-\xi)^k(h_\mu f)(\xi)]d\xi \\ &\qquad+A_1\int_0^\infty x^{3/2}\xi^{1/2}J_{\mu+k+1}(x\xi)N_{\mu+k}N_{\mu+k-1} \\ &\qquad\qquad\times\left[\left(x^{-1}\frac{d}{dx}\right)a(x,\xi)(-\xi)^{k-1}(h_\mu f)(\xi)\right]d\xi \\ &\qquad+A_2\int_0^\infty x^{5/2}\xi^{1/2}J_{\mu+k}(x\xi)N_{\mu+k-1}N_{\mu+k-2} \\ &\qquad\qquad\times\left[\left(x^{-1}\frac{d}{dx}\right)^2 a(x,\xi)(-\xi)^{k-2}(h_\mu f)(\xi)\right]d\xi+\cdots \\ &\qquad+A_k\int_0^\infty x^{k+1/2}\xi^{1/2}J_{\mu+2}(x\xi)N_{\mu+1}N_\mu\left[\left(x^{-1}\frac{d}{dx}\right)^k a(x,\xi)(h_\mu f)(\xi)\right]d\xi. \end{aligned} \tag{2.10}$$

By induction, we have

$$\begin{aligned}(-x)^n\big(N_{\mu+k-1}\cdots N_\mu\Phi(x)\big)\\ &= A_0\int_0^\infty (x\xi)^{1/2}J_{\mu+k+n}(x\xi)\\ &\qquad\times N_{\mu+k+n-1}\cdots N_{\mu+k}\big[a(x,\xi)(-\xi)^k(h_\mu f)(\xi)\big]d\xi\\ &\quad+A_1\int_0^\infty x^{3/2}\xi^{1/2}J_{\mu+k+n-1}(x\xi)N_{\mu+k+n-2}\cdots N_{\mu+k-1}\\ &\qquad\times\left[\left(x^{-1}\frac{d}{dx}\right)a(x,\xi)(-\xi)^{k-1}(h_\mu f)(\xi)\right]d\xi+\cdots\\ &\quad+A_k\int_0^\infty x^{k+1/2}\xi^{1/2}J_{\mu+n}(x\xi)N_{\mu+n-1}\cdots N_\mu\\ &\qquad\times\left[\left(x^{-1}\frac{d}{dx}\right)^k a(x,\xi)(h_\mu f)(\xi)\right]d\xi.\end{aligned}\tag{2.11}$$

Therefore,

$$\begin{aligned}(-x)^n\big(N_{\mu+k-1}\cdots N_\mu\Phi(x)\big)\\ &=\sum_{r=0}^{k}\binom{k}{r}\int_0^\infty x^{r+1/2}\xi^{1/2}J_{\mu+k-r+n}(x\xi)N_{\mu+k-r+n-1}\cdots N_{\mu+k-r}\\ &\qquad\times\left[\left(x^{-1}\frac{d}{dx}\right)^r a(x,\xi)(-\xi)^{k-r}(h_\mu f)(\xi)\right]d\xi,\end{aligned}\tag{2.12}$$

which in view of formula (2.7) can be written as

$$\begin{aligned}(-x)^n x^{\mu+k+1/2}\left(x^{-1}\frac{d}{dx}\right)^k x^{-\mu-1/2}\Phi(x)\\ &=\sum_{r=0}^{k}\binom{k}{r}(-1)^{k-r}\int_0^\infty x^{r+1/2}\xi^{1/2}J_{\mu+k-r+n}(x\xi)\xi^{\mu+k-r+n+1/2}\\ &\qquad\times\left(\xi^{-1}\frac{d}{d\xi}\right)^n\left[\xi^{-\mu-1/2}\left(x^{-1}\frac{d}{dx}\right)^r a(x,\xi)(h_\mu f)(\xi)\right]d\xi\\ &=\sum_{r=0}^{k}\binom{k}{r}(-1)^{k-r}\int_0^\infty (x\xi)^{-\mu-k+r}J_{\mu+k-r+n}(x\xi)x^{\mu+k+1/2}\\ &\qquad\times\xi^{2(\mu+k-r)+n+1}\left(\xi^{-1}\frac{d}{d\xi}\right)^n\\ &\qquad\times\left[\xi^{-\mu-1/2}\left(x^{-1}\frac{d}{dx}\right)^r a(x,\xi)(h_\mu f)(\xi)\right]d\xi.\end{aligned}\tag{2.13}$$

Putting $\mu+k-r=\lambda$, we get

$$\begin{aligned}(-x)^n\left(x^{-1}\frac{d}{dx}\right)^k x^{-\mu-1/2}\Phi(x) &= \sum_{r=0}^{k}\binom{k}{r}(-1)^{k-r}\int_0^\infty (x\xi)^{-\lambda}J_{\lambda+n}(x\xi)\xi^{2\lambda+n+1}\left(\xi^{-1}\frac{d}{d\xi}\right)^n \\ &\quad\times\left[\xi^{-\mu-1/2}\left(x^{-1}\frac{d}{dx}\right)^r a(x,\xi)(h_\mu f)(\xi)\right]d\xi.\end{aligned}\tag{2.14}$$

Now an application of formula (1.8) yields

$$\begin{aligned}&\left|x^n\left(x^{-1}\frac{d}{dx}\right)^k x^{-\mu-1/2}\Phi(x)\right| \\ &\quad\le\sum_{r=0}^{k}\binom{k}{r}\int_0^\infty \left|(x\xi)^{-\lambda}J_{\lambda+n}(x\xi)\right|\xi^{2\lambda+n+1} \\ &\qquad\times\sum_{v=0}^{n}\binom{n}{v}\left|\left(\xi^{-1}\frac{d}{d\xi}\right)^v\left(x^{-1}\frac{d}{dx}\right)^r a(x,\xi)\right| \\ &\qquad\times\left|\left(\xi^{-1}\frac{d}{d\xi}\right)^{n-v}\xi^{-\mu-1/2}(h_\mu f)(\xi)\right|d\xi.\end{aligned}\tag{2.15}$$

Then using inequality (2.3) and the boundedness property of the Bessel function, we find that the right-hand side of (2.15) is bounded by

$$\begin{aligned}&\sum_{r=0}^{k}\sum_{v=0}^{n}\binom{k}{r}\binom{n}{v}A_{\lambda,n}C_{v,r,m}\int_0^\infty \xi^{2\lambda+n+1}(1+\xi^2)^{m/2-\rho v-\delta r} \\ &\qquad\times\left|\left(\xi^{-1}\frac{d}{d\xi}\right)^{n-v}\xi^{-\mu-1/2}(h_\mu f)(\xi)\right|d\xi \\ &\quad\le\sum_{r=0}^{k}\sum_{v=0}^{n}\binom{k}{r}\binom{n}{v}A_{\lambda,n}C_{v,r,m}\int_0^\infty (1+\xi^2)^{\lambda+n/2+1/2+m/2-\rho v-\delta r} \\ &\qquad\times\left|\left(\xi^{-1}\frac{d}{d\xi}\right)^{n-v}\xi^{-\mu-1/2}(h_\mu f)(\xi)\right|d\xi.\end{aligned}\tag{2.16}$$

Let p be a positive integer greater than or equal to $\mu+k+m/2+n/2+1/2$, then we can write

$$\begin{aligned}&\left|x^n\left(x^{-1}\frac{d}{dx}\right)^k x^{-\mu-1/2}\Phi(x)\right| \\ &\quad\le\sum_{r=0}^{k}\sum_{v=0}^{n}\binom{k}{r}\binom{n}{v}A_{\lambda,n}C_{v,r,m} \\ &\qquad\times\int_0^\infty\frac{(1+\xi^2)^{p+1}}{(1+\xi^2)}\left|\left(\xi^{-1}\frac{d}{d\xi}\right)^{n-v}\xi^{-\mu-1/2}(h_\mu f)(\xi)\right|d\xi\end{aligned}$$

$$\leq \sum_{r=0}^{k}\sum_{v=0}^{n}\sum_{s=0}^{p+1}\binom{k}{r}\binom{n}{v}\binom{p+1}{s}A_{\lambda,n}C_{v,r,m}$$
$$\times\int_0^\infty \xi^{2s}\left|\left(\xi^{-1}\frac{d}{d\xi}\right)^{n-v}\xi^{-\mu-1/2}(h_\mu f)(\xi)\right|\frac{d\xi}{(1+\xi^2)}. \tag{2.17}$$

Thus

$$\gamma^{\mu}_{n,k}(\Phi) \leq \sum_{r=0}^{k}\sum_{v=0}^{n}\sum_{s=0}^{p+1}D_{\lambda,n,k,m}\gamma^{\mu}_{2s,n-v}(h_\mu f), \tag{2.18}$$

where $D_{\lambda,n,k,m}$ is a positive constant. From (2.18), the continuity of $A(x,D)$ follows. □

3. An integral representation. The function $a_x(\xi)$, associated with the symbol $a(x,\xi)$ and defined by

$$a_x(\xi) = \int_0^\infty (\eta\xi)^{1/2}J_\mu(\eta\xi)\left[(x\eta)^{1/2}J_\mu(x\eta)a(x,\eta)\right]d\eta, \tag{3.1}$$

will play a fundamental role in our investigation. An estimate for $a_x(\xi)$ is given by the following lemma.

Lemma 3.1. *Let the symbol $a(x,\eta) \in H^m_{\rho,\delta}$. Then the function $a_x(\xi)$ defined by (3.1) satisfies the inequality*

$$|a_x(\xi)| \leq E_{\mu,m,t}(1+\xi^2)^{-t}(1+x)^{\mu+4t+1/2}, \tag{3.2}$$

where $E_{\mu,m,t}$ is a positive constant, $\mu \geq -1/2$, and $m < -\mu-3/2-t$.

Proof. Using property (1.7), we can write

$$\begin{aligned} a_x(\xi) &= \int_0^\infty (\eta\xi)^{1/2}J_\mu(\eta\xi)\frac{(1-S_{\mu,\eta})^t}{(1+\xi^2)^t}\left[(x\eta)^{1/2}J_\mu(x\eta)a(x,\eta)\right]d\eta \\ &= \int_0^\infty (\eta\xi)^{1/2}J_\mu(\eta\xi)\sum_{r=0}^{t}(-1)^r\binom{t}{r}\frac{S^r_{\mu,\eta}}{(1+\xi^2)^t}\left[(x\eta)^{1/2}J_\mu(x\eta)a(x,\eta)\right]d\eta. \end{aligned} \tag{3.3}$$

In view of (1.9), we have

$$a_x(\xi) = \sum_{r=0}^{t}(-1)^r\binom{t}{r}\frac{1}{(1+\xi^2)^t}$$
$$\times\int_0^\infty (\eta\xi)^{1/2}J_\mu(\eta\xi)\sum_{j=0}^{r}b_j\eta^{2j+\mu+1/2}\left(\eta^{-1}\frac{d}{d\eta}\right)^{r+j}$$
$$\times\left[\eta^{-\mu-1/2}a(x,\eta)(x\eta)^{1/2}J_\mu(x\eta)\right]d\eta$$

$$
\begin{aligned}
&= \sum_{r=0}^{t} \sum_{j=0}^{r} (-1)^r \binom{t}{r} b_j \frac{1}{(1+\xi^2)^t} \\
&\quad \times \int_0^\infty (\eta\xi)^{1/2} J_\mu(\eta\xi) \eta^{2j+\mu+1/2} \\
&\quad\quad \times \sum_{i=0}^{r+j} \binom{r+j}{i} \left(\eta^{-1}\frac{d}{d\eta}\right)^i a(x,\eta) \left(\eta^{-1}\frac{d}{d\eta}\right)^{r+j-i} \\
&\quad\quad\quad \times \eta^{-\mu} J_\mu(x\eta) x^{1/2} d\eta.
\end{aligned}
\tag{3.4}
$$

Again applying formula (1.20), we have

$$
\begin{aligned}
a_x(\xi) = \sum_{r=0}^{t} \sum_{j=0}^{r} \sum_{i=0}^{r+j} (-1)^r \binom{t}{r} \binom{r+j}{i} b_j \frac{1}{(1+\xi^2)^t} & \\
\times \int_0^\infty (\eta\xi)^{1/2} J_\mu(\eta\xi) \eta^{2j+\mu+1/2} \left(\eta^{-1}\frac{d}{d\eta}\right)^i & \\
\times a(x,\eta)(-1)^{r+j-i} x^{r+j-i} \eta^{-\mu-r-j+i} & \\
\times J_{\mu+r+j-i}(x\eta) x^{1/2} d\eta. &
\end{aligned}
\tag{3.5}
$$

Therefore,

$$
\begin{aligned}
|a_x(\xi)| \le \sum_{r=0}^{t} \sum_{j=0}^{r} \sum_{i=0}^{r+j} \binom{t}{r} \binom{r+j}{i} |b_j| \frac{x^{\mu+1/2+2(r+j-i)}}{(1+\xi^2)^t} & \\
\times \int_0^\infty |(\eta\xi)^{1/2} J_\mu(\eta\xi)| \eta^{2j+\mu+1/2} \left| \left(\eta^{-1}\frac{d}{d\eta}\right)^i a(x,\eta) \right| & \\
\times |(x\eta)^{-\mu-r-j+i} J_{\mu+r+j-i}(x\eta)| d\eta. &
\end{aligned}
\tag{3.6}
$$

Then, using inequality (2.3) and the boundedness property of the Bessel function, we have

$$
\begin{aligned}
|a_x(\xi)| &\le \sum_{r=0}^{t} \sum_{j=0}^{r} \sum_{i=0}^{r+j} \binom{t}{r} \binom{r+j}{i} |b_j| \frac{x^{\mu+1/2+2(r+j-i)}}{(1+\xi^2)^t} \\
&\quad \times A_\mu B_{\mu,i,j,r} C_{i,m} \int_0^\infty \frac{\eta^{2j+\mu+1/2}}{(1+\eta^2)^{-m/2+\rho_i}} d\eta \\
&\le \sum_{r=0}^{t} \sum_{j=0}^{r} \sum_{i=0}^{r+j} \binom{t}{r} \binom{r+j}{i} |b_j| A_\mu B_{\mu,i,j,r} C_{i,m} x^{\mu+2r+2j-2i+1/2} \\
&\quad \times (1+\xi^2)^{-t} B\left(j+\frac{\mu}{2}+\frac{3}{4}, -\frac{m}{2}-\frac{\mu}{2}-\frac{3}{4}+\rho i - j\right)
\end{aligned}
\tag{3.7}
$$

for $m < -\mu - 3/2 - t$. Therefore there exists a constant $E_{\mu,m,t}$ such that

$$|a_x(\xi)| \le E_{\mu,m,t}(1+\xi^2)^{-t}(1+x)^{\mu+4t+1/2}. \tag{3.8}$$

□

Theorem 3.2. *For any symbol $a(x,\xi) \in H^m_{\rho,\delta}$, the associated operator $A(x,D)$ can be represented by*

$$A(x,D)f(x) = \int_0^\infty \int_0^\infty (\eta\xi)^{1/2} J_\mu(\eta\xi) a_x(\xi)(h_\mu f)(\eta)\, d\eta\, d\xi, \quad f \in H_\mu(I), \tag{3.9}$$

where all the involved integrals are convergent for $\mu \ge -1/2$ and $m < -\mu - 3/2 - t$ with $t > 1/2$.

Proof. Since $a_x(\xi) = \int_0^\infty (\eta\xi)^{1/2} J_\mu(\eta\xi)[(x\eta)^{1/2} J_\mu(x\eta) a(x,\eta)]\, d\eta$, by inversion, formally, we have

$$\int_0^\infty (\eta\xi)^{1/2} J_\mu(\eta\xi) a_x(\xi)\, d\xi = (x\eta)^{1/2} J_\mu(x\eta) a(x,\eta). \tag{3.10}$$

Therefore,

$$\begin{aligned} A(x,D)f(x) &= \int_0^\infty (x\eta)^{1/2} J_\mu(x\eta) a(x,\eta)(h_\mu f)(\eta)\, d\eta \\ &= \int_0^\infty (h_\mu f)(\eta) \left[\int_0^\infty (\eta\xi)^{1/2} J_\mu(\eta\xi) a_x(\xi)\, d\xi \right] d\eta \\ &= \int_0^\infty \int_0^\infty (\eta\xi)^{1/2} J_\mu(\eta\xi) a_x(\xi)(h_\mu f)(\eta)\, d\eta\, d\xi, \end{aligned} \tag{3.11}$$

so that

$$|A(x,D)f(x)| \le A_\mu \int_0^\infty |a_x(\xi)| \left(\int_0^\infty |(h_\mu f)(\eta)|\, d\eta \right) d\xi. \tag{3.12}$$

Since $(h_\mu f)(\eta) \in H_\mu(I)$, we have

$$|(h_\mu f)(\eta)| \le C\eta^{\mu+1/2}(1+\eta)^{-l} \quad \forall l > 0. \tag{3.13}$$

Now using the above estimate and (3.2), we obtain

$$\begin{aligned} &|A(x,D)f(x)| \\ &\quad \le A_\mu \int_0^\infty \int_0^\infty E_{\mu,m,t}(1+\xi^2)^{-t}(1+x)^{\mu+4t+1/2} C\eta^{\mu+1/2}(1+\eta)^{-l}\, d\eta\, d\xi \\ &\quad \le E^1_{\mu,m,t}(1+x)^{\mu+4t+1/2} \int_0^\infty (1+\xi^2)^{-t}\, d\xi \int_0^\infty (1+\eta)^{\mu+1/2-l}\, d\eta. \end{aligned} \tag{3.14}$$

The above integrals are convergent since $\mu \ge -1/2$, and t can be chosen greater than $1/2$ and l sufficiently large. □

4. $L^p_{\sigma,\alpha}$-boundedness of $A(x,D)$. In the proof of the boundedness result, we will need the following estimate for the Hankel transform of $\eta^{\mu+1/2}a(x,\eta)$. We write

$$A_x(z) = h_\mu\left(\eta^{\mu+1/2}a(x,\eta)\right)(z). \tag{4.1}$$

Lemma 4.1. *For $1/2 < \rho < 1$ and $\mu+m+3/2 < 0$, there exists a constant $A_{\mu,m,\rho} > 0$ such that*

$$|A_x(z)| \le A_{\mu,m,\rho}(1+z^2)^{-1}. \tag{4.2}$$

Proof. We have

$$A_x(z) = \int_0^\infty (z\eta)^{1/2}J_\mu(z\eta)\eta^{\mu+1/2}a(x,\eta)\,d\eta. \tag{4.3}$$

Then using the property (1.7), we can write

$$A_x(z) = \int_0^\infty (z\eta)^{1/2}J_\mu(z\eta)\frac{(1-S_{\mu,\eta})}{(1+z^2)}\eta^{\mu+1/2}a(x,\eta)\,d\eta. \tag{4.4}$$

Using (1.5), we get

$$\begin{aligned} A_x(z) = \frac{1}{(1+z^2)}\Bigg[&\int_0^\infty (z\eta)^{1/2}J_\mu(z\eta)\eta^{\mu+1/2}a(x,\eta)\,d\eta \\ &-(2\mu+2)\int_0^\infty (z\eta)^{1/2}J_\mu(z\eta)\eta^{\mu+1/2}\left(\eta^{-1}\frac{d}{d\eta}\right)a(x,\eta)\,d\eta \\ &-\int_0^\infty (z\eta)^{1/2}J_\mu(z\eta)\eta^{\mu+5/2}\left(\eta^{-1}\frac{d}{d\eta}\right)^2 a(x,\eta)\,d\eta\Bigg], \end{aligned} \tag{4.5}$$

so that

$$\begin{aligned} |A_x(z)| \le \frac{1}{(1+z^2)}\Bigg[&\int_0^\infty A_\mu\eta^{\mu+1/2}(1+\eta^2)^{m/2}\,d\eta \\ &+(2\mu+2)\int_0^\infty A_\mu\eta^{\mu+1/2}(1+\eta^2)^{m/2-\rho}\,d\eta \\ &+\int_0^\infty A_\mu\eta^{\mu+5/2}(1+\eta^2)^{m/2-2\rho}\,d\eta\Bigg]. \end{aligned} \tag{4.6}$$

Since all the three integrals are convergent for $\mu+m+3/2<0$ and $1/2<\rho<1$, therefore there exists a constant $A_{\mu,m,\rho}$ such that

$$|A_x(z)| \le A_{\mu,m,\rho}(1+z^2)^{-1}. \tag{4.7}$$

□

We will use the following subspace of H_μ.

Definition 4.2. The space $L^p_{\sigma,\alpha}(0,\infty)$, $\alpha\in\mathbb{R}$, $1\le p\le\infty$, is defined as the set of all those elements $f\in H_\mu(I)$ which satisfy

$$\|f\|_{L^p_{\sigma,\alpha}} = \left\|\xi^{-\alpha}f(\xi)\right\|_{L^p_\sigma}. \tag{4.8}$$

Theorem 4.3. *Pseudodifferential operator $A(x,D)$ is a continuous linear mapping from $L^p_{\sigma,\mu+1/2}$ into $L^r_{\sigma,\mu+1/2}$, where $1 \le p \le \infty$, $q > (2\mu+2)/(\mu+5/2)$, and $1/r = 1/p + 1/q - 1$.*

Proof. From (3.9), we have

$$A(x,D)f(x) = \int_0^\infty a_x(\xi)\left(\int_0^\infty (\eta\xi)^{1/2}J_\mu(\eta\xi)(h_\mu f)(\eta)d\eta\right)d\xi = \int_0^\infty a_x(\xi)f(\xi)d\xi. \tag{4.9}$$

Using (3.1), this can be written as

$$A(x,D)f(x) = \int_0^\infty\left(\int_0^\infty (\eta\xi)^{1/2}J_\mu(\eta\xi)(x\eta)^{1/2}J_\mu(x\eta)a(x,\eta)d\eta\right)f(\xi)d\xi. \tag{4.10}$$

Next using relation (1.15), we can express it in the form

$$\begin{aligned} A(x,D)&f(x) \\ &= \int_0^\infty\left(\int_0^\infty (\eta\xi)^{1/2}(x\eta)^{1/2}\left(2^\mu\Gamma(\mu+1)\right)^{-1}\right. \\ &\qquad\left.\times\int_0^\infty (x\xi\eta)^\mu z^{-\mu}J_\mu(z\eta)D(x,\xi,z)d\sigma(z)a(x,\eta)d\eta\right)f(\xi)d\xi \\ &= \int_0^\infty\int_0^\infty\left(\int_0^\infty (z\eta)^{1/2}J_\mu(z\eta)\eta^{\mu+1/2}a(x,\eta)d\eta\right) \\ &\qquad\times x^{\mu+1/2}\xi^{-\mu-1/2}f(\xi)z^{-\mu-1/2}D(x,\xi,z)d\sigma(\xi)d\sigma(z) \\ &= x^{\mu+1/2}\int_0^\infty\int_0^\infty A_x(z)\xi^{-\mu-1/2}f(\xi)z^{-\mu-1/2}D(x,\xi,z)d\sigma(\xi)d\sigma(z). \end{aligned} \tag{4.11}$$

An application of inequality (4.2) yields

$$\begin{aligned} &\left|x^{-\mu-1/2}A(x,D)f(x)\right| \\ &\quad\le A_{\mu,m,\rho}\int_0^\infty\int_0^\infty\left|\xi^{-\mu-1/2}f(\xi)\right|\left|z^{-\mu-1/2}\right|\left(1+z^2\right)^{-1}D(x,\xi,z)d\sigma(\xi)d\sigma(z). \end{aligned} \tag{4.12}$$

In view of definitions (1.16) and (1.17), the last expression can be expressed as a convolution, and for $F(\xi) = |\xi^{-\mu-1/2}f(\xi)|$ and $G(z) = |z^{-\mu-1/2}|(1+z^2)^{-1}$, we have

$$\left|x^{-\mu-1/2}A(x,D)f(x)\right| \le A_{\mu,m,\rho}(F\#G)(x), \tag{4.13}$$

so that

$$\left(\int_0^\infty\left|x^{-\mu-1/2}A(x,D)f(x)\right|^r d\sigma(x)\right)^{1/r} \le A_{\mu,m,\rho}\left(\int_0^\infty (F\#G)^r(x)d\sigma(x)\right)^{1/r}. \tag{4.14}$$

We note that for $q > (2\mu+2)/(\mu+5/2)$, $G(z) \in L^q_\sigma(I)$, and $F(z) \in L^p_\sigma(I)$, $1 \le p \le \infty$, and using (1.18) finally we obtain

$$\left\|x^{-\mu-1/2}A(x,D)f(x)\right\|_{L^r_\sigma} \le \|F\#G\|_{L^r_\sigma} \le \|F\|_{L^p_\sigma}\|G\|_{L^q_\sigma}, \tag{4.15}$$

where $1/r = 1/p+1/q-1 > 0$. Therefore, in view of Definition 4.2, we get

$$\left\|A(x,D)f(x)\right\|_{L^r_\sigma,\mu+1/2} \le \|f\|_{L^p_\sigma,\mu+1/2}\left\|\left(1+z^2\right)^{-1}\right\|_{L^q_\sigma,\mu+1/2}. \tag{4.16}$$

□

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