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MEASURES OF CONCORDANCE DETERMINED BY D_4-INVARIANT COPULAS

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A continuous random vector (X,Y) uniquely determines a copula $C:[0,1]^2 \to [0,1]$ such that when the distribution functions of X and Y are properly composed into C, the joint distribution function of (X,Y) results. A copula is said to be D_4-invariant if its mass distribution is invariant with respect to the symmetries of the unit square. A D_4-invariant copula leads naturally to a family of measures of concordance having a particular form, and all copulas generating this family are D_4-invariant. The construction examined here includes Spearman's rho and Gini's measure of association as special cases.

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1. Introduction. Let $I = [0,1]$ and $I^2 = [0,1]\times[0,1]$. μ is a doubly stochastic measure on I^2 if it is a probability measure on the Borel sets of I^2 such that $\mu(A\times I) = \mu(I\times A) = \lambda(A)$, where A is a Borel set of I and λ is the one-dimensional Lebesgue measure. A copula (more precisely a 2-copula) is a function $C: I^2 \to I$ that is related to some doubly stochastic measure, μ, by $C(x,y) = \mu([0,x]\times[0,y])$ (see [3]). There is a one-to-one correspondence between copulas and doubly stochastic measures.

Besides being associated with a doubly stochastic measure, a copula can be uniquely determined by a continuous random vector. By Sklar's theorem, for any continuous random vector, (X,Y), with marginals, F_X and F_Y, respectively, and joint distribution function, $F_{X,Y}$, there exists a unique copula, C, such that $F_{X,Y}(x,y) = C(F_X(x),F_Y(y))$ (see [1, 3]).

The simplest examples are as follows. If Y is an almost surely increasing function of X, then its associated copula is $M(x,y) = \min(x,y)$. If Y is an almost surely decreasing function of X, then its associated copula is $W(x,y) = \max(x+y-1,0)$. Finally, if X and Y are independent, then their associated copula is $\Pi(x,y) = xy$ (again see [3]).

When considering two random variables, it can be useful to know how much large values of one random variable correspond to large values of the other. More formally, for any two observations, (X_1,Y_1) and (X_2,Y_2), from a continuous random vector, (X,Y), the two observations are said to be concordant if either $X_1 < X_2$ and $Y_1 < Y_2$, or $X_2 < X_1$ and $Y_2 < Y_1$. Similarly, the two observations are said to be discordant if either $X_1 < X_2$ and $Y_2 < Y_1$, or $X_2 < X_1$ and $Y_1 < Y_2$. The properties of concordance and discordance can be gauged by a measure of concordance, a concept developed by Scarsini [4] and presented in [3].

A measure of concordance associates to a continuous random vector, (X,Y), a real number, $\kappa_{X,Y}$. As developed by Scarsini, it can be shown that this value depends only on the copula C, uniquely associated with (X,Y). Because of this, we sometimes write κ_C instead of $\kappa_{X,Y}$. The following definition of a measure of concordance can be found in [3].

Definition 1.1. Let C be the copula associated with the continuous random vector, (X,Y). Let $\kappa_{X,Y}$ be a functional mapping the set of all copulas to $\mathbb{R}$. $\kappa_{X,Y}$ (which can also be denoted κ_C if C is the copula for (X,Y)) is a measure of concordance if the following conditions are satisfied:

(1) κ_C is defined for every copula, C,
(2) $-1 \le \kappa_C \le 1$,
(3) $\kappa_{X,X} = 1$,
(4) $\kappa_{-X,X} = -1$,
(5) $\kappa_{-X,Y} = \kappa_{X,-Y} = -\kappa_{X,Y}$,
(6) $\kappa_{X,Y} = \kappa_{Y,X}$,
(7) if X and Y are independent, then $\kappa_{X,Y} = 0$,
(8) if C_1 and C_2 are copulas, where $C_1 \le C_2$ pointwise, then $\kappa_{C_1} \le \kappa_{C_2}$,
(9) if C_n is a sequence of copulas, where $C_n \to C$ pointwise, then $\kappa_{C_n} \to \kappa_C$.

Spearman's rho, ρ, and Gini's measure of association, γ, are two examples of measures of concordance. Spearman's rho can be expressed as $\rho_C = 12\int_{I^2} C\,d\Pi - 3$, where $\Pi(x,y) = xy$, while Gini's measure of association can be expressed as $\gamma_C = 8\int_{I^2} C\,d((M+W)/2) - 2$, where $M(x,y) = \min(x,y)$ and $W(x,y) = \max(x+y-1,0)$ [3, 5]. Note that each is of the form $\kappa_C = \alpha\int_{I^2} C\,dA - \beta$, where A is a fixed copula and $\alpha,\beta \in \mathbb{R}$.

Definition 1.2. A *copular* measure of concordance is one of the form $\kappa_C = \alpha\int_{I^2} C\,dA - \beta$, where A is a fixed copula and $\alpha,\beta \in \mathbb{R}$.

Definition 1.3. A copula A, is *copular generating* if there exist $\alpha,\beta \in \mathbb{R}$ such that $C \mapsto \alpha\int_{I^2} C\,dA - \beta$ is a measure of concordance.

When you are dealing with copular measures of concordance you are in effect dealing with an expression where the difference of the probabilities of concordance and discordance are taken. Namely, for any continuous random vectors, (X_1,X_2) and (Y_1,Y_2), respectively, associated with a copula C and a fixed, copular-generating copula A, we are dealing with $P((X_1-Y_1)(X_2-Y_2) > 0) - P((X_1-Y_1)(X_2-Y_2) < 0)$. For more details on this matter, one may refer to [3].

The standard notation for the group of symmetries on the unit square I^2 is D_4. We have $D_4 = \{e,r,r^2,r^3,h,hr,hr^2,hr^3\}$, where e is the identity, h is the reflection about $x = 1/2$, and r is a 90° counterclockwise rotation.

For $d \in D_4$, a new copula, C^d, can be formed, where $C^d(x,y) = \mu_{C^d}([0,x]\times[0,y]) = \mu_C(d([0,x]\times[0,y]))$ gives the amount of probabilistic mass contained in the rectangle $d([0,x]\times[0,y])$ as determined by the doubly stochastic measure associated with C.

Definition 1.4. A copula is D_4-*invariant* if for every $d \in D_4$, $C(x,y) = C^d(x,y)$ for all $(x,y) \in I^2$.

TABLE 2.1. Symmetries of copulas on I^2 and their associated random vectors.

D_4	Copula	Random vector
e	$C(x,y)$	(X,Y)
r	$C^r(x,y)=x-C(1-y,x)$	$(-Y,X)$
r^2	$C^{r^2}(x,y)=x+y-1+C(1-x,1-y)$	$(-X,-Y)$
r^3	$C^{r^3}(x,y)=y-C(y,1-x)$	$(Y,-X)$
h	$C^h(x,y)=y-C(1-x,y)$	$(-X,Y)$
hr	$C^{hr}(x,y)=C(y,x)$	(Y,X)
hr^2	$C^{hr^2}(x,y)=x-C(x,1-y)$	$(X,-Y)$
hr^3	$C^{hr^3}(x,y)=x+y-1+C(1-y,1-x)$	$(-Y,-X)$

While it might not always be obvious that a copula C is D_4-invariant, it is certainly easy to construct one from C since $C^* = (1/8)\sum_{d\in D_4} C^d$ is D_4-invariant. For example, while M is not D_4-invariant, $M^* = (M+W)/2$ is D_4-invariant.

It is when the properties $\kappa_{-X,Y} = \kappa_{X,-Y} = -\kappa_{X,Y}$ and $\kappa_{X,Y} = \kappa_{Y,X}$ are considered in terms of κ_{C^d} for $d \in D_4$ that the principles behind the main theorem take shape, giving a nice theoretical characterization and providing a way to construct many measures of concordance. The theorem states that *a copula is copular generating if and only if it is D_4-invariant.*

In the second section, some background information is given, where measures of concordance are considered entirely in terms of copulas and their symmetries. Also included in the section are some helpful lemmas with their proofs. The third and final section includes the formulation and proof of the main result, in addition to some remarks we think may be of some interest.

2. Background and lemmas. Here and in all that follows, we assume that we are dealing with continuous random vectors.

Observe Table 2.1 with regard to the correspondence between the copula C^d for each $d \in D_4$ and a random vector with which it is associated. Note that $C^{d_1d_2} = (C^{d_1})^{d_2}$, where $d_1, d_2 \in D_4$.

When considering the copulas M, W, and Π as well as Table 2.1, Definition 1.1 may be rewritten.

DEFINITION 2.1. Let C be the copula associated with the continuous random vector, (X,Y). Let κ_C be a functional mapping the set of all copulas to $\mathbb{R}$. κ_C is a measure of concordance if the following conditions are satisfied:

(1) κ_C is defined for every copula C,
(2) $-1 \le \kappa_C \le 1$,
(3) $\kappa_M = 1$,
(4) $\kappa_W = -1$,
(5) $\kappa_{C^h} = \kappa_{C^{hr^2}} = -\kappa_C$,
(6) $\kappa_C = \kappa_{C^{hr}}$,
(7) $\kappa_\Pi = 0$,
(8) if C_1 and C_2 are copulas, where $C_1 \le C_2$ pointwise, then $\kappa_{C_1} \le \kappa_{C_2}$,
(9) if C_n is a sequence of copulas, where $C_n \to C$ pointwise, then $\kappa_{C_n} \to \kappa_C$.

Lemma 2.2. *For any copulas A and B, $\int_{I^2} AdB = \int_{I^2} BdA$.*

Proof. Let (X_1, Y_1) and (X_2, Y_2) be independent, continuous random vectors associated with A and B, respectively. Since $\int_{I^2} AdB = P(X_1 < X_2, Y_1 < Y_2)$ [3], the proof is quite brief,

$$\begin{aligned}\int_{I^2} AdB &= P(X_1 < X_2, Y_1 < Y_2)\\ &= P(X_1 < X_2) - P(X_1 < X_2, Y_2 < Y_1)\\ &= P(X_1 < X_2) - P(Y_2 < Y_1) + P(X_2 < X_1, Y_2 < Y_1)\\ &= \frac{1}{2} - \frac{1}{2} + P(X_2 < X_1, Y_2 < Y_1)\\ &= \int_{I^2} BdA.\end{aligned} \tag{2.1}$$

□

Lemma 2.3. *Let $G = \{e, r^2, hr, hr^3\}$ and $hG = \{h, hr^2, r, r^3\}$. Given copulas A and B, the following are true:*

(1) $\int_{I^2} A^d dB = \int_{I^2} AdB^d$ *for every* $d \in G$,
(2) $\int_{I^2} A^d dB + \int_{I^2} AdB^d = 1/2$ *for every* $d \in hG$.

Proof. Let (X_1, Y_1) and (X_2, Y_2) be independent, continuous random vectors associated with A and B, respectively.

For $d = h$,

$$\begin{aligned}&\int_{I^2} A^h dB + \int_{I^2} AdB^h\\ &\quad= P(-X_1 < X_2, Y_1 < Y_2) + P(X_1 < -X_2, Y_1 < Y_2) = P(Y_1 < Y_2) = \frac{1}{2}.\end{aligned} \tag{2.2}$$

For $d = r^2$ using Lemma 2.2,

$$\begin{aligned}&\int_{I^2} A^{r^2} dB\\ &\quad= P(-X_1 < X_2, -Y_1 < Y_2) = P(-X_2 < X_1, -Y_2 < Y_1) = \int_{I^2} B^{r^2} dA = \int_{I^2} AdB^{r^2}.\end{aligned} \tag{2.3}$$

Noting that $hr = r^3 h$ and r^2 is in the center of D_4, we then have for $d = r$,

$$\begin{aligned}\int_{I^2} A^r dB = P(-Y_1 < X_2, X_1 < Y_2) &= \int_{I^2} A^{hr^2} dB^{hr} = \int_{I^2} A^{r^2 h} dB^{r^3 h}\\ &= \frac{1}{2} - \int_{I^2} A^{r^2} dB^{r^3} = \frac{1}{2} - \int_{I^2} AdB^{r^5} = \frac{1}{2} - \int_{I^2} AdB^r.\end{aligned} \tag{2.4}$$

Since our assertion holds for $d = r, r^2$, the case when $d = r^3$ is clear.

Since r^2 is in the center of D_4 and our assertion holds for $d = h, r^2$, the case when $d = hr^2$ is readily seen.

For $d = hr$,

$$\int_{I^2} A^{hr} dB - \int_{I^2} AdB^{hr} = P(Y_1 < X_2, X_1 < Y_2) - P(X_1 < Y_2, Y_1 < X_2) = 0. \tag{2.5}$$

Finally, for $d = hr^3$,

$$\begin{aligned}\int_{I^2} A^{hr^3} dB - \int_{I^2} A dB^{hr^3} \\ &= P(-Y_1 < X_2, -X_1 < Y_2) - P(X_1 < -Y_2, Y_1 < -X_2) \\ &= P(-Y_1 < X_2) - P(X_1 < -Y_2) = \frac{1}{2} - \frac{1}{2} = 0.\end{aligned} \tag{2.6}$$

□

Consider a grid being placed on I^2 such that it is divided into square cells having the dimensions $1/n \times 1/n$. We construct a copula by assigning a constant mass density, $\delta_{i,k}$, to the cell in the ith column from the left and kth row from the bottom, where $\sum_{i=1}^{n} \delta_{i,k} = \sum_{k=1}^{n} \delta_{i,k} = n$. Such a notion is a particular instance of a *checkerboard copula* (see [1]).

The following concepts and notation will be used to construct the checkerboard copulas, $Q^1_{p,n}$ and $Q^2_{p,n}$, that depend on a fixed point $p \in (0,1)^2$ and $n \in \mathbb{N}$.

(i) Given $n \in \mathbb{N}$, for $1 \le i, k \le n$, let $J_{i,k}$ be the square $[(i-1)/n, i/n][(k-1)/n, k/n]$.

(ii) Choose p in the interior of I^2. There exists $N \in \mathbb{N}$ such that for $p = (p_1, p_2)$, $1/N < \min(p_1, p_2)$ and $Np_1, Np_2 \notin \mathbb{N}$. Let $\mathcal{N}$ be an infinite, increasing sequence of such N.

(iii) Let $Q^1_{p,n}$ and $Q^2_{p,n}$ be two $n \times n$ checkerboard copulas, where $n \in \mathcal{N}$, and having density $\delta^1_{i,k}$ and $\delta^2_{i,k}$, respectively, in cell $J_{i,k}$.

(iv) The cell containing p will be denoted J_{i^*,k^*}.

(v) Let the $Q^1_{p,n}$ have the following densities assigned to its cells:

$$\delta^1_{i,k} = \begin{cases} 0, & (i,k) = (1,k^*) \text{ or } (i^*,1), \\ 2, & (i,k) = (1,1) \text{ or } (i^*,k^*), \\ 1, & \text{otherwise}. \end{cases} \tag{2.7}$$

(vi) Let $Q^2_{p,n}$ have the following densities assigned to its cells:

$$\delta^2_{i,k} = \begin{cases} 2, & (i,k) = (1,k^*) \text{ or } (i^*,1), \\ 0, & (i,k) = (1,1) \text{ or } (i^*,k^*), \\ 1, & \text{otherwise}. \end{cases} \tag{2.8}$$

We make use of $Q^1_{p,n}$ and $Q^2_{p,n}$ in some of the following proofs.

Lemma 2.4. *If for copulas A and B, $\int_{I^2} A\,dC = \int_{I^2} B\,dC$ for every copula C, then $A = B$.*

Proof. For convenience, we write $\int_{I^2}(A-B)dC = 0$ for every copula C. Since A and B are copulas, $A(p) = B(p)$ for any p on the boundary of I^2. Thus, only p in the interior of I^2 needs to be considered. Using $Q^1_{p,n}$ and $Q^2_{p,n}$ as choices for C yields

$$\begin{aligned} 0 &= \int_{I^2} (A-B)\, d(Q^1_{p,n} - Q^2_{p,n}) \\ &= 2\left\{ \int_{J_{1,1} \cup J_{i^*,k^*}} (A-B)\, d\Pi - \int_{J_{1,k^*} \cup J_{i^*,1}} (A-B)\, d\Pi \right\}. \end{aligned} \tag{2.9}$$

By the mean value theorem, there exists $p_{a,b} \in J_{a,b}$ such that

$$\int_{J_{a,b}} (A-B)\,d\Pi = \frac{A(p_{a,b}) - B(p_{a,b})}{n^2}. \tag{2.10}$$

Hence, $0 = (A-B)(p_{1,1}) + (A-B)(p_{i^*,k^*}) - (A-B)(p_{1,k^*}) - (A-B)(p_{i^*,1})$. Letting $n \to \infty$, since either the x coordinate, y coordinate, or both coordinates of $p_{1,1}$, $p_{i^*,1}$, and p_{1,j^*} will go to 0, it follows from the continuity of A and B that

$$(A-B)(p_{1,1}),\ (A-B)(p_{i^*,1}),\ (A-B)(p_{1,k^*}) \longrightarrow 0, \tag{2.11}$$

while

$$(A-B)(p_{i^*,k^*}) \longrightarrow (A-B)(p). \tag{2.12}$$

Therefore, $A = B$. □

Lemma 2.5. *For a fixed copula A and $\alpha, \beta \in \mathbb{R}$, let $\kappa_C = \alpha \int_{I^2} C\,dA - \beta$, where $\alpha \neq 0$. If $\kappa_C = -\kappa_{C^h}$ and $\kappa_C = \kappa_{C^{hr}}$, then A is D_4-invariant.*

Proof. Note that since A, A^h, and A^{hr} are all copulas, $A(p) = A^h(p) = A^{hr}(p)$ for every p on the boundary of I^2. Because of this, only p in the interior of I^2 needs to be considered. Using $Q^1_{p,n}$ and $Q^2_{p,n}$ as choices for C yields $\kappa_{Q^l_{p,n}} = -\kappa_{(Q^l_{p,n})^h} = \kappa_{(Q^l_{p,n})^{hr}}$, for $l = 1, 2$. So,

$$\begin{aligned} \alpha \int_{I^2} Q^l_{p,n}\,dA - \beta &= -\alpha \int_{I^2} (Q^l_{p,n})^h\,dA + \beta, \\ \alpha \int_{I^2} Q^l_{p,n}\,dA - \beta &= \alpha \int_{I^2} (Q^l_{p,n})^{hr}\,dA - \beta. \end{aligned} \tag{2.13}$$

By subtraction and application of Lemmas 2.2 and 2.3, we have

$$\begin{aligned} \int_{I^2} A\,d(Q^1_{p,n} - Q^2_{p,n}) &= \int_{I^2} A^h\,d(Q^1_{p,n} - Q^2_{p,n}), \\ \int_{I^2} A\,d(Q^1_{p,n} - Q^2_{p,n}) &= \int_{I^2} A^{hr}\,d(Q^1_{p,n} - Q^2_{p,n}) \end{aligned} \tag{2.14}$$

so that

$$\begin{aligned} \int_{I^2} (A - A^h)\,d(Q^1_{p,n} - Q^2_{p,n}) &= 0, \\ \int_{I^2} (A - A^{hr})\,d(Q^1_{p,n} - Q^2_{p,n}) &= 0. \end{aligned} \tag{2.15}$$

Finally, by using the same argument as in Lemma 2.4, the results $A = A^h$ and $A = A^{hr}$ are attained. Since h and hr generate D_4, we know A is D_4-invariant. □

3. A characterization of copular generating copulas and remarks

Theorem 3.1. *A copula is copular generating if and only if it is D_4-invariant.*

Proof. Suppose that A is copular generating. Note that $\alpha \neq 0$ since a measure of concordance is not constant. Therefore, by Lemma 2.5, A is D_4-invariant.

Now, we assume that A is D_4-invariant. Setting

$$\kappa_C = \alpha\left(\int_{I^2} C\,dA - \frac{1}{4}\right), \tag{3.1}$$

where $\alpha = (\int_{I^2} M\,dA - 1/4)^{-1}$, we will show that κ is a measure of concordance.

It needs to be shown that $\int_{I^2} M\,dA - 1/4 \neq 0$ in order for κ_C to be defined for every copula C. Noting by the D_4-invariance of A that $A(1-x,1-x) = 1-2x+A(x,x)$ and $\int_0^{1/2} A(x,x)\,dx > 0$, we have

$$\begin{aligned}\int_{I^2} M\,dA = \int_{I^2} A\,dM &= \int_0^{1/2} A(x,x)\,dx + \int_{1/2}^1 A(x,x)\,dx \\ &= \int_0^{1/2} A(x,x)\,dx + \int_0^{1/2} A(1-x,1-x)\,dx \\ &= \frac{1}{4} + 2\int_0^{1/2} A(x,x)\,dx > \frac{1}{4}.\end{aligned} \tag{3.2}$$

By the chosen form of κ, it is clear that $\kappa_M = 1$.

By the D_4-invariance of A and Lemma 2.3,

$$\kappa_C = \alpha\left(\int_{I^2} C\,dA - \frac{1}{4}\right) = \alpha\left(\int_{I^2} C^{hr}\,dA - \frac{1}{4}\right) = \kappa_{C^{hr}}. \tag{3.3}$$

It is similarly attained that $\kappa_{C^h} = \kappa_{C^{hr^2}} = -\kappa_C$. In particular, noting that $M^h = W$ and $\Pi^h = \Pi$, we see that $\kappa_W = -1$ and $\kappa_\Pi = 0$.

Recall from (3.2) that $\alpha > 0$. Since $\int_{I^2} C_1\,dA \le \int_{I^2} C_2\,dA$ whenever $C_1 \le C_2$ pointwise, it is also true that $\kappa_{C_1} \le \kappa_{C_2}$. Furthermore, since $W \le C \le M$ (see [2, 3]) for every copula C, $\kappa_W \le \kappa_C \le \kappa_M$, or more precisely, $-1 \le \kappa_C \le 1$.

Finally, since every sequence of copulas converging to a copula pointwise does so uniformly (see [3]), it follows that if $C_n \to C$ pointwise, then $\int_{I^2} C_n\,dA \to \int_{I^2} C\,dA$. Hence, $\kappa_{C_n} \to \kappa_C$. □

Remark 3.2. By Theorem 3.1, any D_4-invariant copula and only a D_4-invariant copula generates a copular measure of concordance. For example, one may generate a copular measure of concordance using the copula associated with the circular uniform distribution which is presented in [3]:

$$A(x,y) = \begin{cases} M(x,y), & |x-y| > \dfrac{1}{2}, \\ W(x,y), & |x+y-1| > \dfrac{1}{2}, \\ \dfrac{x+y}{2} - \dfrac{1}{4}, & \text{otherwise}. \end{cases} \tag{3.4}$$

Remark 3.3. There is a uniqueness among copular measures of concordance. In other words, for any two copular measures of concordance,

$$\hat{\kappa}_C = \hat{\alpha}\int_{I^2} C\,d\hat{A} - \hat{\beta}, \qquad \kappa_C = \alpha\int_{I^2} C\,dA - \beta, \tag{3.5}$$

where $\hat{A}$ and A are copular generating and $\hat{\alpha},\alpha,\hat{\beta},\beta\in\mathbb{R}$, if $\hat{\kappa}_C=\kappa_C$ for every copula C, then $\hat{\alpha}=\alpha$, $\hat{\beta}=\beta$, and $\hat{A}=A$. Here is a verification.

Since $\hat{\kappa}_{C^h}=-\hat{\kappa}_C$ and $\kappa_{C^h}=-\kappa_C$, we know that

$$\hat{\alpha}\left[\int_{I^2}Cd\hat{A}+\int_{I^2}C^hd\hat{A}\right]=2\hat{\beta},\qquad \alpha\left[\int_{I^2}CdA+\int_{I^2}C^hdA\right]=2\beta. \tag{3.6}$$

Then, from the D_4-invariance of $\hat{A}$ and A we have by Lemma 2.3,

$$\hat{\alpha}\cdot\frac{1}{2}=2\hat{\beta},\qquad \alpha\cdot\frac{1}{2}=2\beta, \tag{3.7}$$

which gives us $\hat{\beta}=\hat{\alpha}/4$ and $\beta=\alpha/4$. Choosing $p\in(0,1)^2$, and copulas $Q^1_{p,n}$ and $Q^2_{p,n}$, one has

$$\hat{\alpha}\left(\int_{I^2}Q^l_{p,n}d\hat{A}-\frac{1}{4}\right)=\alpha\left(\int_{I^2}Q^l_{p,n}dA-\frac{1}{4}\right) \tag{3.8}$$

for $l=1,2$. Subtraction then yields

$$\hat{\alpha}\int_{I^2}(Q^1_{p,n}-Q^2_{p,n})d\hat{A}=\alpha\int_{I^2}(Q^1_{p,n}-Q^2_{p,n})dA. \tag{3.9}$$

Thus, by Lemma 2.2, $\int_{I^2}(\hat{\alpha}\hat{A}-\alpha A)d(Q^1_{p,n}-Q^2_{p,n})=0$. Using the same argument as in Lemma 2.4 results in $\hat{\alpha}\hat{A}(p)=\alpha A(p)$ for any $p\in(0,1)^2$. For $p=(p_1,p_2)$, letting $p_1\to 1$ or $p_2\to 1$, the uniform margins and continuity of $\hat{A}$ and A force $\hat{\alpha}=\alpha$ and consequently, $\hat{\beta}=\beta$. Thus, $\int_{I^2}\hat{A}dC=\int_{I^2}AdC$ for every copula C, which shows that $\hat{A}=A$ by Lemma 2.4.

Remark 3.4. Not all measures of concordance are copular. For example, Kendall's tau, $\tau_C=4\int_{I^2}CdC-1$ [3, 5], though a measure of concordance, is not a copular measure of concordance.

To see this, first note that the convex sum of any copulas C_1 and C_2, is also a copula. Assume there exists κ_C, a copular measure of concordance, such that $\kappa_C=\tau_C$ for every copula C. Notice that if $p,q\geq 0$ and $p+q=1$, then $\kappa_{pC_1+qC_2}=p\kappa_{C_1}+q\kappa_{C_2}$ and $\tau_{pC_1+qC_2}=p^2\tau_{C_1}+q^2\tau_{C_2}+2pq(4\int_{I^2}C_1dC_2-1)$. By hypothesis, one has $\kappa_{p\Pi+qM}=\tau_{p\Pi+qM}$, but $\tau_{p\Pi+qM}=q^2+2pq(4\int_{I^2}\Pi dM-1)=q^2+(2/3)pq=q(q+(2/3)p)<q=\kappa_{p\Pi+qM}$.

Remark 3.5. A probabilistic interpretation can be made for any copular measure of concordance. Fix a copula A which is copular generating. A is associated with some continuous random vector, say (W,Z). Choose any copula C. It is associated with some continuous random vector, say (X,Y). Let (X_1,Y_1) and (X_2,Y_2) be independent observations of (X,Y),

$$\kappa_C=\alpha\left(\int_{I^2}CdA-\frac{1}{4}\right)=\alpha\left(\int_{I^2}CdA-\int_{I^2}\Pi dA\right) \tag{3.10}$$

and by independence of (X_1, Y_1) and (X_2, Y_2),

$$\kappa_C = \alpha\big(P(X_1 < W, Y_1 < Z) - P(X_1 < W, Y_2 < Z)\big), \tag{3.11}$$

where $\alpha = (\int_{I^2} M\,dA - 1/4)^{-1}$.

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