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# MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS

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Let $\sum^*(\alpha,\beta,k)$ be a class of meromorphic functions $f(z)$ with positive coefficients in $D = \{0<|z|<1\}$. The aim of the present note is to prove some properties for the class $\sum^*(\alpha,\beta,k)$.

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**1. Introduction.** Let $\sum$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the punctured disc $D = \{z : 0 < |z| < 1\}$ with a simple pole at the origin with residue 1. A function $f(z) \in \sum$ is said to be meromorphically starlike of order $\alpha$ if it satisfies the following:

$$\operatorname{Re}\left(-\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in D) \tag{1.2}$$

for some $\alpha$ $(0 \le \alpha < 1)$. We say that $f(z)$ is in the class $\sum^*(\alpha)$ of such functions. Similarly a function $f(z)$ in $\sum$ is said to be meromorphically convex of order $\alpha$ if it satisfies the following:

$$\operatorname{Re}\left(-1-\frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in D) \tag{1.3}$$

for some $\alpha$ $(0 \le \alpha < 1)$. We say that $f(z)$ is in the class $\sum_C(\alpha)$ of such functions.

The class $\sum^*(\alpha)$ and various other subclasses of $\sum$ have been studied rather extensively by Nehari and Netanyahu [9], Clunie [4], Pommerenke [11, 12], Miller [7], Royster [13], and others (cf., e.g., Bajpai [2], Goel and Sohi [6], Mogra et al. [8], Uralegaddi and Ganigi [15], Cho et al. [3], Aouf [1], and Uralegaddi and Somanatha [16, 17]; see also Duren [5, pages 29 and 137], and Srivastava and Owa [14, pages 86 and 429]).

Owa and Pascu [10] obtained some coefficient properties for the class $\sum^*(\alpha,k)$ which satisfies

$$\left|\frac{zf'(z)}{f(z)}+k\right| < \left|\frac{zf'(z)}{f(z)}+(2\alpha-k)\right| \tag{1.4}$$

for some $\alpha$ $(\alpha > 0)$, $k$ $(0 \le k \le 1)$, and for all $z \in D$.

In this note, the above definition is extended and we obtain the extended coefficient inequalities.

The extended class of functions is defined as follows.

**Definition 1.1.** A function $f(z) \in \sum$ is said to be a member of the class $\sum^*(\alpha,\beta,k)$ if it satisfies

$$\frac{1}{\beta}\left|\frac{zf'(z)}{f(z)}+k\right| < \left|\frac{zf'(z)}{f(z)}+(2\alpha-k)\right| \tag{1.5}$$

for some $\alpha$ $(\alpha > 0)$, $\beta$ $(0<\beta\le 1)$, $k$ $(0\le k\le 1)$, and for all $z\in D$.

## 2. Coefficient inequalities for functions.

For the class $\sum^*(\alpha,k)$, Owa and Pascu [10] showed the following theorem.

**Theorem 2.1.** *Let the function $f(z)$ be defined by (1.1). If*

$$\sum_{n=2}^{\infty}(n+k+|2\alpha+n-k|)|a_n|r^{n+1} \le 2(1-\alpha) \tag{2.1}$$

*for some $k$ $(0\le k\le 1)$ and $\alpha$ $(0\le\alpha<1)$, then $f(z)\in\sum^*(\alpha,k)$.*

Our first result for functions $f(z)\in\sum^*(\alpha,\beta,k)$ is given as the following theorem.

**Theorem 2.2.** *Let the function $f(z)$ be defined by (1.1). If*

$$\sum_{n=2}^{\infty}(n+k+\beta|2\alpha+n-k|)|a_n|r^{n+1} \le \beta(k+1-2\alpha)+1-k \tag{2.2}$$

*for some $k$ $(0\le k\le 1)$, $\alpha$ $(0\le\alpha<1)$, and $\beta$ $(0<\beta\le 1)$, then $f(z)\in\sum^*(\alpha,\beta,k)$.*

**Proof.** Using the same technique as in [10], we know that for $f\in\sum^*(\alpha,\beta,k)$,

$$\begin{aligned}&|zf'(z)+kf(z)|-\beta|zf'(z)+(2\alpha-k)f(z)|\\&\quad=\left|(k-1)\frac{1}{z}+\sum_{n=0}^{\infty}(n+k)a_nz^n\right|-\beta\left|(2\alpha-k-1)\frac{1}{z}+\sum_{n=0}^{\infty}(2\alpha+n-k)a_nz^n\right|.\end{aligned} \tag{2.3}$$

Therefore, applying the condition of the theorem, we have

$$\begin{aligned}&r|zf'(z)+kf(z)|-r\beta|zf'(z)+(2\alpha-k)f(z)|\\&\quad\le(k-1)+\sum_{n=0}^{\infty}(n+k)|a_n|r^{n+1}-\beta(k+1-2\alpha)\frac{1}{r}\\&\qquad+\sum_{n=0}^{\infty}\beta|2\alpha+n-k||a_n|r^{n+1}\\&\quad=k-1-\beta(k+1-2\alpha)+\sum_{n=0}^{\infty}\{(n+k)+\beta|2\alpha+n-k|\}|a_n|r^{n+1}\\&\quad\le 0,\end{aligned} \tag{2.4}$$

which shows that

$$\sum_{n=0}^{\infty} \{(n+k)+\beta|2\alpha+n-k|\}|a_n|r^{n+1} \le \beta(k+1-2\alpha)+1-k. \tag{2.5}$$

It follows from the above that

$$\frac{1}{\beta}\left|\frac{zf'(z)/f(z)+k}{zf'(z)/f(z)+(2\alpha-k)}\right| \le 1. \tag{2.6}$$

The function $f(z)$ given by

$$f_n(z) = \frac{1}{z}+a_0+\frac{\{\beta(k+1-2\alpha)+1-k\}z^n}{(n+k)+\beta|2\alpha+n-k|} \quad (n \ge 1) \tag{2.7}$$

belongs to the class $\sum^*(\alpha,\beta,k)$.

This completes the proof of the theorem. □

**Corollary 2.3.** *Let the function $f(z)$ be defined by (1.1) and let $f(z) \in \sum$. If $f \in \sum^*(\alpha,\beta,k)$, then*

$$a_n \le \frac{\beta(k+1-2\alpha)+1-k}{(n+k)+\beta|2\alpha+n-k|}, \quad n \ge 0. \tag{2.8}$$

*The result is sharp for functions $f_n(z)$ given by (2.7).*

**Remark 2.4.** If $f \in \sum^*(\alpha,\beta,k)$ with $a_0 = 0$, then Corollary 2.3 is true for some $\beta$ $(0 < \beta \le (k-1)/(k+1-2\alpha) \le 1)$ and $\alpha$ $(0 \le \alpha \le (k+1)/2 < 1)$.

If $\beta = 1$, we get the following corollary.

**Corollary 2.5** (see [10]). *Let the function $f(z)$ be defined by (1.1) and let $f(z) \in \sum$. If $f \in \sum^*(\alpha,1,k)$, then*

$$a_n \le \frac{1-\alpha}{n+\alpha}, \quad n \ge 0, \tag{2.9}$$

*for some $\alpha$ $(1/2 \le \alpha < 1)$.*

**3. Distortion theorem.** A distortion property for functions in the class $\sum^*(\alpha,\beta,k)$ is given as follows.

**Theorem 3.1.** *If the function $f(z)$ defined by (1.1) is in the class $\sum^*(\alpha,\beta,k)$, then for $0 < |z| = r < 1$,*

$$\begin{aligned}\frac{1}{r}-a_0-\frac{\{\beta(k+1-2\alpha)+1-k\}r}{(1+k)+\beta|2\alpha+1-k|}\\ \le |f(z)| \le \frac{1}{r}+a_0+\frac{\{\beta(k+1-2\alpha)+1-k\}r}{(1+k)+\beta|2\alpha+1-k|},\end{aligned} \tag{3.1}$$

*with equality for*

$$f_1(z) = \frac{1}{z} + a_0 + \frac{\{\beta(k+1-2\alpha)+1-k\}z}{(1+k)+\beta|2\alpha+1-k|} \quad (z = ir, r),$$

$$\begin{aligned} &\frac{1}{r^2} - a_0 - \frac{\beta(k+1-2\alpha)+1-k}{(1+k)+\beta|2\alpha+1-k|} \\ &\quad \le |f'(z)| \le \frac{1}{r^2} + a_0 + \frac{\beta(k+1-2\alpha)+1-k}{(1+k)+\beta|2\alpha+1-k|}, \end{aligned} \tag{3.2}$$

*with equality for*

$$f_1(z) = \frac{1}{z} + a_0 + \frac{\{\beta(k+1-2\alpha)+1-k\}z}{(1+k)+\beta|2\alpha+1-k|} \quad (z = \pm ir, \pm r). \tag{3.3}$$

**Proof.** Since $f \in \sum^*(\alpha,\beta,k)$, Theorem 2.2 yields the inequality

$$\sum_{n=0}^{\infty} a_n \le \frac{\{\beta(k+1-2\alpha)+1-k\}}{(1+k)+\beta|2\alpha+1-k|}, \quad n \ge 0. \tag{3.4}$$

Thus, for $0 < |z| = r < 1$, and making use of (3.4), we have

$$\begin{aligned} |f(z)| &\le \left|\frac{1}{z}\right| + a_0 + \sum_{n=1}^{\infty} a_n|z|^n \le \frac{1}{r} + a_0 + r\sum_{n=1}^{\infty} a_n \\ &\le \frac{1}{r} + a_0 + \frac{\{\beta(k+1-2\alpha)+1-k\}r}{(1+k)+\beta|2\alpha+1-k|}, \\ |f(z)| &\ge \left|\frac{1}{z}\right| - a_0 - \sum_{n=1}^{\infty} a_n|z|^n \ge \frac{1}{r} - a_0 - r\sum_{n=1}^{\infty} a_n \\ &\ge \frac{1}{r} - a_0 - \frac{\{\beta(k+1-2\alpha)+1-k\}r}{(1+k)+\beta|2\alpha+1-k|}. \end{aligned} \tag{3.5}$$

Also from Theorem 2.1 it follows that

$$\sum_{n=1}^{\infty} na_n \le a_0 + \frac{\{\beta(k+1-2\alpha)+1-k\}}{(1+k)+\beta|2\alpha+1-k|}. \tag{3.6}$$

Thus

$$\begin{aligned} |f'(z)| &\le \frac{1}{|z|^2} + a_0 + \sum_{n=1}^{\infty} na_n|z|^{n-1} \le \frac{1}{r^2} + a_0 + \sum_{n=1}^{\infty} na_n \\ &\le \frac{1}{r^2} + a_0 + \frac{\{\beta(k+1-2\alpha)+1-k\}}{(1+k)+\beta|2\alpha+1-k|}, \\ |f'(z)| &\ge \frac{1}{|z|^2} - a_0 - \sum_{n=1}^{\infty} na_n|z|^{n-1} \ge \frac{1}{r^2} - a_0 - \sum_{n=1}^{\infty} na_n \\ &\ge \frac{1}{r^2} - a_0 - \frac{\{\beta(k+1-2\alpha)+1-k\}}{(1+k)+\beta|2\alpha+1-k|}. \end{aligned} \tag{3.7}$$

Hence we complete the proof of Theorem 3.1. □

**4. Radii of starlikeness and convexity.** The radii of starlikeness and convexity for the class $\sum^*(\alpha,\beta,k)$ are given by the following theorem.

**Theorem 4.1.** *If the function $f(z)$ defined by (1.1) is in the class $\sum^*(\alpha,\beta,k)$, then $f(z)$ is meromorphically starlike of order $\rho$ $(0 \le \rho < 1)$ in the disk $|z| < r_1(\alpha,\beta,\rho)$, where $r_1(\alpha,\beta,\rho)$ is the largest value for which*

$$r_1 = r_1(\alpha,\beta,\rho) = \inf_{n\ge 0}\left(\frac{(1-\rho)\{(n+k)+\beta|2\alpha+n-k|\}}{(n+2-\rho)\{\beta(k+1-2\alpha)+1-k\}}\right)^{1/(n+1)}. \tag{4.1}$$

*The result is sharp for functions $f_n(z)$ given by (2.7).*

**Proof.** It suffices to show that

$$\left|\frac{zf'(z)}{f(z)}+1\right| \le 1-\rho \tag{4.2}$$

for $|z| \le r_1$. We have

$$\left|\frac{zf'(z)}{f(z)}+1\right| \le \frac{\sum_{n=0}^{\infty}(n+1)|a_n||z|^{n+1}}{1-\sum_{n=0}^{\infty}|a_n||z|^{n+1}} \le 1-\rho. \tag{4.3}$$

Hence (4.3) holds true if

$$\sum_{n=0}^{\infty}(n+1)|a_n||z|^{n+1} \le (1-\rho)\left(1-\sum_{n=0}^{\infty}|a_n||z|^{n+1}\right) \tag{4.4}$$

or

$$\sum_{n=0}^{\infty}\frac{(n+2-\rho)}{1-\rho}a_n|z|^{n+1} \le 1; \tag{4.5}$$

with the aid of (2.8), (4.5) is true if

$$\frac{(n+2-\rho)}{1-\rho}|z|^{n+1} \le \frac{(n+k)+\beta|2\alpha+n-k|}{\beta(k+1-2\alpha)+1-k}. \tag{4.6}$$

Solving (4.6) for $|z|$, we obtain

$$|z| \le \left(\frac{(1-\rho)\{(n+k)+\beta|2\alpha+n-k|\}}{(n+2-\rho)\{\beta(k+1-2\alpha)+1-k\}}\right)^{1/(n+1)} \quad (n\ge 0). \tag{4.7}$$

This completes the proof of Theorem 4.1. □

**Theorem 4.2.** *If the function $f(z)$ defined by (1.1) is in the class $\sum^*(\alpha,\beta,k)$, then $f(z)$ is meromorphically convex of order $\rho$ $(0 \le \rho < 1)$ in the disk $|z| < r_2(\alpha,\beta,\rho)$, where $r_2(\alpha,\beta,\rho)$ is the largest value for which*

$$r_2 = r_2(\alpha,\beta,\rho) = \inf_{n\ge 0}\left(\frac{(1-\rho)\{(n+k)+\beta|2\alpha+n-k|\}}{n(n+2-\rho)\{\beta(k+1-2\alpha)+1-k\}}\right)^{1/(n+1)}. \tag{4.8}$$

*The result is sharp for functions $f_n(z)$ given by (2.7).*

**Proof.** We omit the details of the proof as they are very tedious. It suffices to show that

$$\left|\frac{zf''(z)}{f'(z)}+2\right| \le 1-\rho \tag{4.9}$$

for $|z| \le r_2$, with the aid of Theorem 2.2. Thus we have the assertion of Theorem 4.2. □

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## References

[1] M. K. Aouf, *On a certain class of meromorphic univalent functions with positive coefficients*, Rend. Mat. Appl. (7) **11** (1991), no. 2, 209–219.

[2] S. K. Bajpai, *A note on a class of meromorphic univalent functions*, Rev. Roumaine Math. Pures Appl. **22** (1977), no. 3, 295–297.

[3] N. E. Cho, S. H. Lee, and S. Owa, *A class of meromorphic univalent functions with positive coefficients*, Kobe J. Math. **4** (1987), no. 1, 43–50.

[4] J. Clunie, *On meromorphic schlicht functions*, J. London Math. Soc. **34** (1959), 215–216.

[5] P. L. Duren, *Univalent Functions*, Grundlehren der mathematischen Wissenschaften, vol. 259, Springer-Verlag, New York, 1983.

[6] R. M. Goel and N. S. Sohi, *On a class of meromorphic functions*, Glas. Mat. Ser. III **17(37)** (1982), no. 1, 19–28.

[7] J. Miller, *Convex meromorphic mappings and related functions*, Proc. Amer. Math. Soc. **25** (1970), 220–228.

[8] M. L. Mogra, T. R. Reddy, and O. P. Juneja, *Meromorphic univalent functions with positive coefficients*, Bull. Austral. Math. Soc. **32** (1985), no. 2, 161–176.

[9] Z. Nehari and E. Netanyahu, *On the coefficients of meromorphic schlicht functions*, Proc. Amer. Math. Soc. **8** (1957), 15–23.

[10] S. Owa and N. N. Pascu, *Coefficient inequalities for certain classes of meromorphically starlike and meromorphically convex functions*, JIPAM. J. Inequal. Pure Appl. Math. **4** (2003), no. 1, Article 17, 1–6.

[11] Ch. Pommerenke, *Über einige Klassen meromorpher schlichter Funktionen*, Math. Z. **78** (1962), 263–284 (German).

[12] ———, *On meromorphic starlike functions*, Pacific J. Math. **13** (1963), 221–235.

[13] W. C. Royster, *Meromorphic starlike multivalent functions*, Trans. Amer. Math. Soc. **107** (1963), 300–308.

[14] H. M. Srivastava and S. Owa (eds.), *Current Topics in Analytic Function Theory*, World Scientific Publishing, New Jersey, 1992.

[15] B. A. Uralegaddi and M. D. Ganigi, *A certain class of meromorphically starlike functions with positive coefficients*, Pure Appl. Math. Sci. **26** (1987), no. 1-2, 75–81.

[16] B. A. Uralegaddi and C. Somanatha, *Certain differential operators for meromorphic functions*, Houston J. Math. **17** (1991), no. 2, 279–284.

[17] ———, *New criteria for meromorphic starlike univalent functions*, Bull. Austral. Math. Soc. **43** (1991), no. 1, 137–140.

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