

TWO VERSIONS OF NAKAYAMA LEMMA FOR MULTIPLICATION MODULES

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The aim of this note is to generalize the Nakayama lemma to a class of multiplication modules over commutative rings with identity. In this note, by considering the notion of multiplication modules and the product of submodules of them, we state and prove two versions of Nakayama lemma for such modules. In the first version we give some equivalent conditions for faithful finitely generated multiplication modules, and in the second version we give them for faithful multiplication modules with a minimal generating set.

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1. Introduction. Let R be a commutative ring with identity and let M be a unitary R -module. Then M is called a multiplication R -module provided for each submodule N of M there exists an ideal I of R such that $N = IM$, we call I a presentation ideal of N . Note that our definitions agree with [2, 3], but in [5], the term multiplication module is used in a different way. (In this note, an R -module M is multiplication if and only if every submodule of M is a multiplication module in the above sense.)

If N and K are multiplication submodules of M , then NK , the product of N and K , is defined as IJM , where I and J are presentation ideals of N and K , respectively [1]. For a module M , the radical of M , denoted by $\text{rad}(M)$, is the intersection of all maximal submodules of M if they exist, and M otherwise (see [3, 4]). We denote the Jacobson radical of the ring R , the intersection of all maximal ideals of R , by $J(R)$. Recently, multiplication modules have been studied in a number of papers, see, for example, [1, 2, 3, 5]. The author in [1] states and proves a version of Nakayama lemma for multiplication modules (see [1, Theorem 3.23]). But the proof of (ii) \Rightarrow (iii) of this theorem needs to be amended. In this note, first we modify the proof of this theorem, and then we obtain another version of Nakayama lemma for multiplication modules by replacing the finitely generated condition with a minimal generating set for them.

Throughout this note, R denotes a commutative ring with identity and all related modules are unitary R -modules. All definitions and notations follow from [1, 2, 3].

2. Main results

LEMMA 2.1. *Let M be a finitely generated R -module and $M = IM$ for some ideal I of R . Then $(1 - r)M = 0$ for some $r \in R$.*

PROOF. Let M be generated by elements x_1, x_2, \dots, x_n . Since $M = IM$, every x_i can be written as an I -linear combination of x_1, x_2, \dots, x_n . Thus we have

$$x_i = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{ii}x_i + \dots + a_{in}x_n, \quad 1 \leq i \leq n, \quad a_{i1}, \dots, a_{in} \in I. \tag{2.1}$$

Then

$$(1 - a_{ii})x_i = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{ii-1}x_{i-1} + a_{ii+1}x_{i+1} + \dots + a_{in}x_n, \quad 1 \leq i \leq n, \quad a_{i1}, \dots, a_{in} \in I. \tag{2.2}$$

Recursively from (2.2) we can find each x_i of the form $(1 - r_i)x_i = 0$ for some $r_i \in R$, $1 \leq i \leq n$. Then we have $(1 - r_1)(1 - r_2) \dots (1 - r_n) = 1 - r$ for some $r \in I$. Thus $(1 - r)x_i = 0$, for all i , $1 \leq i \leq n$. This completes the proof. \square

LEMMA 2.2. *If M is a faithful multiplication R -module, then $\text{rad}(M) = J(R)M$.*

PROOF. Let \mathcal{M} denote the collection of all maximal ideals of R . Set $P_2(M) = \{P \in \mathcal{M} \mid \text{ann}(M) \subseteq P\}$. Since M is faithful, $P_2(M) = \mathcal{M}$. Then, by [1, Theorem 2.13], $\text{rad}(M) = (\bigcap_{P \in P_2(M)} P)M = J(R)M$. \square

DEFINITION 2.3 (See [5]). Let M be an R -module. An element $u \in M$ is called a *unit element* if $\langle u \rangle = M$.

In the next result we modify the proof of [1, Theorem 3.23].

THEOREM 2.4 (first version of Nakayama lemma). *Let M be a faithful multiplication R -module. Then, for every submodule N of M , the following conditions are equivalent:*

- (i) N is contained in $\text{rad}(M)$,
- (ii) if u is a unit in M , then $u - rx$ is a unit for all $r \in R$ and for all $x \in N$,
- (iii) if M is a finitely generated R -module such that $NM = M$, then $M = 0$,
- (iv) if M is finitely generated and K, N are submodules of M such that $M = NM + K$, then $M = K$.

PROOF. (i) \Rightarrow (ii). The proof is the same as in the proof of [1, Theorem 3.23].

(i) \Rightarrow (ii). Since u is a unit in M , then $M = \langle u \rangle$ by [1, Theorem 3.19]. By contradiction let $N \not\subseteq \text{rad}(M)$. Then there exists a maximal submodule K of M such that $N \not\subseteq K$. Thus, there exists $x \in N \setminus K$, and hence $\langle x \rangle + K = M$ by the maximality of K . Thus, $u = rx + a$ for some $r \in R$ and $a \in K$. Then, by hypothesis, $a = u - rx$ is a unit, and hence $K = M$, which is a contradiction. Therefore, $N \subseteq \text{rad}(M)$.

(i) \Rightarrow (iii). Let I be a presentation ideal of N , that is, $N = IM$. Then, by the product of two submodules (see [1, Definition 3.3]) and the hypothesis, we have $M = NM = IM \cdot RM = IM$. On the other hand, by hypothesis, we have $N \subseteq \text{rad}(M)$, and from Lemma 2.2 we conclude that $M = J(R)M$. Thus, by Lemma 2.1, there exists an $r \in J(R)$ such that $(1 - r)M = 0$, and hence $M = 0$ since $1 - r$ is a unit in R .

(iii) \Rightarrow (iv). By [1, Corollary 3.22], K/N is a multiplication R -module. Now, it is easy to verify that $(K + N)/K(M/K) = M/K$, and hence, by (iii), we must have $M/K = 0$, and hence $M = K$.

(iv) \Rightarrow (i). Let K be any maximal submodule of M ; then $K \subseteq NM + K$. Consequently, $NM + M = M$ or $NM + M = K$ by the maximality of K . If $NM + K = M$, then, by (iv),

we must have $K = M$, a contradiction. Thus, $K = NM + K$, and hence $N \subseteq NM \subseteq K$. Therefore, N is contained in every maximal submodule of M . \square

DEFINITION 2.5. Let M be an R -module. A subset X of M is called a *minimal generating set* if $\langle X \rangle = M$ and no proper subset of X generates M .

THEOREM 2.6 (second version of Nakayama lemma). *Let M be a faithful multiplication R -module with a minimal generating set. Then, for every submodule N of M , the following conditions are equivalent:*

- (i) N is contained in $\text{rad}(M)$,
- (ii) if $NM = M$, then $M = 0$,
- (iii) if K is a submodule of M such that $M = NM + K$, then $M = K$.

PROOF. (i) \Rightarrow (ii). Let X be a minimal generating set of M . If $M \neq 0$, then consider m_1 , $m_1 \neq 0$, by the minimality of X . Now, let I be a presentation ideal of N . Then $NM = M$ implies that $N = IM \cdot M = M$, and since M is faithful, by Lemma 2.2, we have $N \subseteq \text{rad}(M) = J(R)M$, and hence $M = J(R)M$. Thus, $m_1 = j_1 m_1 + j_2 m_2 + \cdots + j_n m_n$, $j_i \in J(R)$, $m_i \in X$, whence $j_1 m_1 = m_1$. If $n = 1$, then $(1 - j_1)m_1 = 0$; since $1 - j_1$ is a unit in R , $m_1 = 0$, and for $n > 1$,

$$(1 - j_1)m_1 = j_2 m_2 + \cdots + j_n m_n. \quad (2.3)$$

Since $1 - j_1$ is a unit in R , $m_1 = (1 - j_1)^{-1} j_2 m_2 + \cdots + (1 - j_1)^{-1} j_n m_n$. Thus, for $n > 1$, m_1 is a linear combination of m_2, m_3, \dots, m_n . Consequently, $\{m_2, \dots, m_n\}$ generates M , which contradicts the choice of X .

The proofs of (ii) \Rightarrow (iii) and (iii) \Rightarrow (i) are the same as the proofs of (iii) \Rightarrow (iv) and (iv) \Rightarrow (i) of Theorem 2.4. \square

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REFERENCES

- [1] R. Ameri, *On the prime submodules of multiplication modules*, Int. J. Math. Math. Sci. **2003** (2003), no. 27, 1715–1724.
- [2] A. Barnard, *Multiplication modules*, J. Algebra **71** (1981), no. 1, 174–178.
- [3] Z. A. El-Bast and P. F. Smith, *Multiplication modules*, Comm. Algebra **16** (1988), no. 4, 755–779.
- [4] R. L. McCasland and M. E. Moore, *On radicals of submodules of finitely generated modules*, Canad. Math. Bull. **29** (1986), no. 1, 37–39.
- [5] S. Singh and F. Mehdi, *Multiplication modules*, Canad. Math. Bull. **21** (1969), 1057–1061.

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