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A FURI-PERA THEOREM IN HAUSDORFF TOPOLOGICAL SPACES FOR ACYCLIC MAPS

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We present new Furi-Pera theorems for acyclic maps between topological spaces.

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1. Introduction. In this paper, we present new Furi-Pera theorems [6, 7] for acyclic maps between Hausdorff topological spaces. The main result in our paper is based on a new Leray-Schauder alternative [1] for such maps which in turn is based on the notion of compactly null-homotopic.

We first recall some results and ideas from the literature. Let X and Z be subsets of Hausdorff topological spaces. We will consider maps $F: X \to K(Z)$; here $K(Z)$ denotes the family of nonempty compact subsets of Z. A nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial. Now $F: X \to K(Z)$ is *acyclic* if F is upper semicontinuous with acyclic values. Suppose X and Z are topological spaces. Given a class $\mathscr{X}$ of maps, $\mathscr{X}(X,Z)$ denotes the set of maps $F: X \to 2^Z$ (nonempty subsets of Z) belonging to $\mathscr{X}$, and $\mathscr{X}_c$ the set of finite compositions of maps in $\mathscr{X}$. We let

$$\mathscr{F}(\mathscr{X}) = \{W : \operatorname{Fix} F \neq \emptyset \ \forall F \in \mathscr{X}(W,W)\}, \tag{1.1}$$

where $\operatorname{Fix} F$ denotes the set of fixed points of F.

The class $\mathscr{U}$ of maps is defined by the following properties:

(i) $\mathscr{U}$ contains the class $\mathscr{C}$ of single-valued continuous functions;

(ii) each $F \in \mathscr{U}_c$ is upper semicontinuous and compact valued;

(iii) $B^n \in \mathscr{F}(\mathscr{U}_c)$ for all $n \in \{1,2,\dots\}$; here $B^n = \{x \in \mathbb{R}^n : \|x\| \le 1\}$.

Next we consider the class $\mathscr{U}_c^\kappa(X,Z)$ of maps $F: X \to 2^Z$ such that for each F and each nonempty compact subset K of X, there exists a map $G \in \mathscr{U}_c(K,Z)$ such that $G(x) \subseteq F(x)$ for all $x \in K$. Notice the Kakutani and acyclic maps are examples of $\mathscr{U}_c^\kappa$ maps (see [3, 4, 8] for other examples).

By a space, we mean a Hausdorff topological space. Let Q be a class of topological spaces. A space Y is an *extension space* for Q (written $Y \in \mathrm{ES}(Q)$) if for all $X \in Q$ and for all $K \subseteq X$ closed in X, any continuous function $f_0: K \to Y$ extends to a continuous function $f: X \to Y$.

For a subset K of a topological space X, we denote by $\mathrm{Cov}_X(K)$ the set of all coverings of K by open sets of X (usually we write $\mathrm{Cov}(K) = \mathrm{Cov}_X(K)$). Let Q be a class of topological spaces and Y a subset of a Hausdorff topological space. Given two maps $F,G : X \to 2^Y$ and $\alpha \in \mathrm{Cov}(Y)$, F and G are said to be α-close if for any $x \in X$, there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$, and $w \in G(x) \cap U_x$. A space Y is an *approximate extension space* for Q (written $Y \in \mathrm{AES}(Q)$) if for all $\alpha \in \mathrm{Cov}(Y)$, for all $X \in Q$, for all $K \subseteq X$ closed in X, and for any continuous function $f_0 : K \to Y$, there exists a continuous function $f : X \to Y$ such that $f|_K$ is α-close to f_0.

Let X be a uniform space. Then X is *Schauder admissible* if for every compact subset K of X and every covering $\alpha \in \mathrm{Cov}_X(K)$, there exists a continuous function (called the Schauder projection) $\pi_\alpha : K \to X$ such that

(i) π_α and $i : K \hookrightarrow X$ are α-close;

(ii) $\pi_\alpha(K)$ is contained in a subset $C \subseteq X$ with $C \in \mathrm{AES(compact)}$.

Let X be a Hausdorff topological space and let $\alpha \in \mathrm{Cov}(X)$. X is said to be *Schauder admissible α-dominated* if there exist a Schauder admissible space X_α and two continuous functions $r_\alpha : X_\alpha \to X$, $s_\alpha : X \to X_\alpha$ such that $r_\alpha s_\alpha : X \to X$ and $i : X \to X$ are α-close. X is said to be *almost Schauder admissible dominated* if X is Schauder admissible α-dominated for each $\alpha \in \mathrm{Cov}(X)$. In [2], we established the following result.

Theorem 1.1. *Let X be a uniform space and let X be almost Schauder admissible dominated. Also suppose $F \in \mathcal{U}_c^\kappa(X,X)$ is a compact upper semicontinuous map with closed values. Then F has a fixed point.*

In our next definitions, Y will be a completely regular topological space and U an open subset of Y.

Definition 1.2. $F \in \mathrm{AC}(\overline{U},Y)$ if $F : \overline{U} \to K(Y)$ is an acyclic compact map; here $\overline{U}$ denotes the closure of U in Y.

Definition 1.3. $F \in \mathrm{AC}_{\partial U}(\overline{U},Y)$ if $F \in \mathrm{AC}(\overline{U},Y)$ with $x \notin F(x)$ for $x \in \partial U$; here ∂U denotes the boundary of U in Y.

Definition 1.4. $F \in \mathrm{AC}(Y,Y)$ if $F : Y \to K(Y)$ is an acyclic compact map.

Definition 1.5. If $F \in \mathrm{AC}(Y,Y)$ and $p \in Y$, then $F \cong \{p\}$ in $\mathrm{AC}(Y,Y)$ if there exists an acyclic compact map $R : Y \times [0,1] \to K(Y)$ with $R_1 = F$ and $R_0 = \{p\}$ (here $R_t(x) = R(x,t)$).

The following three results were established in [1]. We note that Theorem 1.7 follows from Theorems 1.8, 1.1, and 1.6.

Theorem 1.6. *Let Y be a metrizable ANR, $p \in Y$, and $F \in \mathrm{AC}(Y,Y)$ with $F \cong \{p\}$ in* $\mathrm{AC}(Y,Y)$. *Then F has a fixed point.*

Theorem 1.7. *Let Y be a completely regular topological space, U an open subset of Y, $u_0 \in U$, and $F \in \mathrm{AC}_{\partial U}(\overline{U},Y)$. Suppose there exists an acyclic compact map $H : \overline{U} \times [0,1] \to K(Y)$ with $H_1 = F$, $H_0 = \{u_0\}$, and with $x \notin H_t(x)$ for $x \in \partial U$ and $t \in (0,1)$. In addition assume either of the following occurs:*

(A) *Y is a uniform space and Y is almost Schauder admissible dominated;*

(B) *Y is a metrizable ANR.*

Then F has a fixed point.

Theorem 1.8. *Let Y be a completely regular topological space, U an open subset of Y, $u_0 \in U$, and $F \in \mathrm{AC}_{\partial U}(\overline{U}, Y)$. Suppose there exists an acyclic compact map $H : \overline{U} \times [0,1] \to K(Y)$ with $H_1 = F$, $H_0 = \{u_0\}$, and with $x \notin H_t(x)$ for $x \in \partial U$ and $t \in (0,1)$. In addition, assume the following property holds:*

$$\begin{aligned} &\text{for any } G \in \mathrm{AC}(Y,Y) \text{ and any } p \in Y \text{ with } G \cong \{p\} \\ &\text{in } \mathrm{AC}(Y,Y),\ G \text{ has a fixed point in } Y. \end{aligned} \tag{1.2}$$

Then F has a fixed point in U.

Let Q be a subset of a Hausdorff topological space X. Then Q is called a *special retract* of X if there exists a continuous retraction $r : X \to Q$ with $r(x) \in \partial Q$ for $x \in X \backslash Q$.

Example 1.9. Let X be a Hilbert space and Q a nonempty closed convex subset of X. Then Q is a special retract of X since we may take $r(\cdot)$ to be $P_Q(\cdot)$ which is the nearest point projection on Q.

Example 1.10. Let Q be a nonempty closed convex subset of a locally convex topological vector space X. Then we know from Dugundji's extension theorem that there exists a continuous retraction $r : X \to Q$. If $\operatorname{int} Q = \emptyset$, then $\partial Q = Q$ so $r(x) \in \partial Q = Q$ if $x \in X$. Now suppose $\operatorname{int} Q \neq \emptyset$. Without loss of generality, assume $0 \in \operatorname{int} Q$. Now we may take

$$r(x) = \frac{x}{\max\{1, \mu(x)\}}, \quad x \in X, \tag{1.3}$$

where μ is the Minkowski functional on Q, that is, $\mu(x) = \inf\{\alpha > 0 : x \in \alpha Q\}$. Note, $r(x) \in \partial Q$ for $x \in X \backslash Q$, so Q is a special retract of X.

2. Fixed point theory.

In this section we present three Furi-Pera type theorems based on Theorems 1.1, 1.6–1.8.

Theorem 2.1. *Let $E = (E, d)$ be a metrizable space, Q a closed subset of E, $u_0 \in Q$, and, Q a special retract of E. Also assume $F \in \mathrm{AC}(Q, E)$ with E almost admissible dominated. In addition, suppose the following condition is satisfied:*

$$\begin{aligned} &\text{there exists an acyclic compact map } H : Q \times [0,1] \longrightarrow K(E) \\ &\text{with } H_1 = F,\ H_0 = \{u_0\} \text{ such that if } \{(x_j, \lambda_j)\}_{j \in \mathbb{N}} \\ &(\text{here } \mathbb{N} = \{1,2,\ldots\}) \text{ is a sequence in } \partial Q \times [0,1] \text{ converging} \\ &\text{to } (x, \lambda) \text{ with } x \in H(x,\lambda) \text{ and } 0 \le \lambda < 1, \text{ then} \\ &\{H(x_j, \lambda_j)\} \subseteq Q \text{ for } j \text{ sufficiently large.} \end{aligned} \tag{2.1}$$

Then F has a fixed point in Q.

Proof. Now since Q is a special retract of E, there exists a continuous retraction $r : E \to Q$ with $r(z) \in \partial Q$ if $z \in E \backslash Q$. Consider

$$B = \{x \in E : x \in Fr(x)\}. \tag{2.2}$$

Clearly $Fr: E \to K(E)$ is acyclic valued, upper semicontinuous, and compact. Thus $Fr \in \mathrm{AC}(E,E)$, so Theorem 1.1 guarantees that $B \neq \emptyset$. Also since Fr is upper semicontinuous we have that B is closed. In fact, B is compact since Fr is a compact map. It remains to show $B \cap Q \neq \emptyset$. To do this, we argue by contradiction. Suppose $B \cap Q = \emptyset$. Then since B is compact and Q is closed, there exists a $\delta > 0$ with $\mathrm{dist}(B,Q) > \delta$. Choose $m \in \mathbb{N} = \{1,2,\dots\}$ with $1 < \delta m$. Let

$$U_i = \left\{x \in E : d(x,Q) < \frac{1}{i}\right\} \quad \text{for } i \in \{m,m+1,\dots\}. \tag{2.3}$$

Fix $i \in \{m,m+1,\dots\}$. Now since $\mathrm{dist}(B,Q) > \delta$, then $B \cap \overline{U_i} = \emptyset$. Notice also that U_i is open, $u_0 \in U_i$, and $Fr : \overline{U_i} \to K(E)$ is an upper semicontinuous, acyclic valued, and compact map (i.e., $Fr \in \mathrm{AC}(\overline{U_i}, E)$). Let $H : Q \times [0,1] \to K(E)$ be an acyclic compact map with $H_1 = F$, $H_0 = \{u_0\}$ as described in (2.1). Now let $R : \overline{U_i} \times [0,1] \to K(E)$ be given by $R(x,t) = H(r(x),t)$. Clearly $R : \overline{U_i} \times [0,1] \to K(E)$ is an acyclic compact map with $R_1 = Fr$ and $R_0 = \{u_0\}$. Now $B \cap \overline{U_i} = \emptyset$, together with Theorem 1.7, guarantees that there exists

$$(y_i, \lambda_i) \in \partial U_i \times (0,1) \quad \text{with } y_i \in H(r(y_i), \lambda_i). \tag{2.4}$$

We can do this for each $i \in \{m,m+1,\dots\}$. Consequently,

$$\{H(r(y_j),\lambda_j)\} \nsubseteq Q \quad \text{for each } j \in \{m,m+1,\dots\}. \tag{2.5}$$

We now look at

$$D = \{x \in E : x \in R_\lambda(r(x)) \text{ for some } \lambda \in [0,1]\}. \tag{2.6}$$

Now $D \neq \emptyset$ is closed and in fact compact (so sequentially compact). This together with

$$d(y_j, Q) = \frac{1}{j}, \quad |\lambda_j| \leq 1 \quad \text{for } j \in \{m,m+1,\dots\} \tag{2.7}$$

implies that we may assume without loss of generality that

$$\lambda_j \longrightarrow \lambda^\star \in [0,1], \qquad y_j \longrightarrow y^\star \in \partial Q. \tag{2.8}$$

In addition $y_j \in H(r(y_j),\lambda_j)$ with R upper semicontinuous (so closed, [5, page 465]) guarantees that $y^\star \in H(r(y^\star),\lambda^\star)$. Now if $\lambda^\star = 1$, then $y^\star \in H(r(y^\star),1) = Fr(y^\star)$ which contradicts $B \cap Q = \emptyset$. Thus $0 \leq \lambda^\star < 1$. But then (2.1) with $x_j = r(y_j) \in \partial Q$ (note that Q is a special retract of E) and $x = y^\star = r(y^\star)$ implies $\{H(r(y_j),\lambda_j)\} \subseteq Q$ for j sufficiently large. This contradicts (2.5). Thus $B \cap Q \neq \emptyset$, so there exists $x \in Q$ with $x \in Fr(x) = F(x)$. □

Remark 2.2. We can remove the assumption that Q is a special retract of E provided we assume that

$$\text{there exists a retraction } r : E \longrightarrow Q, \tag{2.9}$$

and (2.1) is replaced by the following:

$$\begin{aligned}&\text{there exists an acyclic compact map } H: Q\times[0,1]\longrightarrow K(E)\\ &\text{with } H_1=F,\ H_0=\{u_0\} \text{ such that if } \{(x_j,\lambda_j)\}_{j\in\mathbb{N}}\\ &(\text{here } \mathbb{N}=\{1,2,\dots\}) \text{ is a sequence in } Q\times[0,1] \text{ converging}\\ &\text{to } (x,\lambda) \text{ with } x\in H(x,\lambda) \text{ and } 0\le\lambda<1, \text{ then}\\ &\{H(x_j,\lambda_j)\}\subseteq Q \text{ for } j \text{ sufficiently large.}\end{aligned} \tag{2.10}$$

Theorem 2.3. *Let $E=(E,d)$ be a metrizable space, Q a closed subset of E, $u_0\in Q$, and Q a special retract of E. Also assume $F\in \mathrm{AC}(Q,E)$ with E an ANR. In addition, assume (2.1) is satisfied and that the following condition holds:*

$$\begin{aligned}&\text{for any } G\in\mathrm{AC}(E,E) \text{ and any } p\in E, \text{ there exists}\\ &\text{an acyclic compact map } \Phi: E\times[0,1]\longrightarrow K(E) \text{ with}\\ &\Phi_1=G \text{ and } \Phi_0=\{p\}\ \big(\text{here } \Phi_t(x)=\Phi(t,x)\big).\end{aligned} \tag{2.11}$$

Then F has a fixed point in Q.

Proof. Let r and B be as in the proof of Theorem 2.1. Notice $Fr\in\mathrm{AC}(E,E)$. Fix $p\in E$. Now (2.11) guarantees that there exists an acyclic compact map $\Psi: E\times[0,1]\to K(E)$ with $\Psi_1=Fr$ and $\Psi_0=\{p\}$. This together with Theorem 1.6 guarantees that $B\neq\emptyset$. Essentially the same reasoning as in Theorem 2.1 establishes the result. □

Remark 2.4. In Theorem 2.3, we can replace "Q is a special retract of E" provided we assume (2.9) and replace (2.1) with (2.10).

Remark 2.5. From the proof of Theorem 2.3, we can see immediately that (2.11) could be replaced by the following:

$$\begin{aligned}&\text{there exist } p\in E \text{ and an acyclic compact map}\\ &\Phi: E\times[0,1]\longrightarrow K(E) \text{ with } \Phi_1=Fr \text{ and } \Phi_0=\{p\}.\end{aligned} \tag{2.12}$$

Our next result is a generalization of Theorem 2.3.

Theorem 2.6. *Let $E=(E,d)$ be a metrizable space, Q a closed subset of E, $u_0\in Q$, and Q a special retract of E. Also assume $F\in\mathrm{AC}(Q,E)$ and that (2.1) and (2.11) are satisfied. In addition, suppose the following condition holds:*

$$\begin{aligned}&E \text{ is such that for any } G\in\mathrm{AC}(E,E) \text{ and any}\\ &p\in E \text{ with } G\cong\{p\} \text{ in } \mathrm{AC}(E,E),\\ &G \text{ has a fixed point.}\end{aligned} \tag{2.13}$$

Then F has a fixed point in Q.

Proof. Let r and B be as in the proof of Theorem 2.1. The argument in Theorem 2.3 guarantees that $B\neq\emptyset$. Also of course B is closed and compact. Suppose $B\cap Q=\emptyset$. Then there exists a $\delta>0$ with $\mathrm{dist}(B,Q)>\delta$. Choose $m\in\mathbb{N}=\{1,2,\dots\}$ with $1<\delta m$ and let U_i ($i\in\{m,m+1,\dots\}$) be as in Theorem 2.1. Fix $i\in\{m,m+1,\dots\}$. Note $B\cap\overline{U_i}=\emptyset$ and $Fr\in\mathrm{AC}(\overline{U_i},E)$. Let $H: Q\times[0,1]\to K(E)$ be an acyclic compact map with $H_1 = F$, $H_0 = \{u_0\}$ as described in (2.1) and let $R : \overline{U_i} \times [0,1] \to K(E)$ be given by $R(x,t) = H(r(x),t)$. Clearly $R : \overline{U_i} \times [0,1] \to K(E)$ is an acyclic compact map with $R_1 = Fr$ and $R_0 = \{u_0\}$. Now $B \cap \overline{U_i} = \emptyset$, (2.13), and Theorem 1.8 guarantee that there exists $(y_i, \lambda_i) \in \partial U_i \times (0,1)$ with $y_i \in H(r(y_i), \lambda_i)$. We can do this for each $i \in \{m, m+1, \dots\}$. Consequently $\{H(r(y_j), \lambda_j)\} \notin Q$ for each $j \in \{m, m+1, \dots\}$. Essentially the same reasoning as in Theorem 2.1 from (2.5) onwards establishes the result. □

Remark 2.7. In Theorem 2.6, we can replace "Q is a special retract of E" provided we assume (2.9) and replace (2.1) with (2.10).

Remark 2.8. In Theorem 2.6, note (2.11) could be replaced by (2.12).

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