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A THIRD-ORDER NONLOCAL PROBLEM WITH NONLOCAL CONDITIONS

LAZHAR BOUGOFFA

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We study an equation with dominated lower-order terms and nonlocal conditions. Using the Riesz representation theorem and the Schauder fixed-point theorem, we prove the existence and uniqueness of a generalized solution.

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1. Introduction. Various problems arising in heat conduction, chemical engineering, plasma physics, thermoelasticity, and so forth, can be reduced to the nonlocal problems with integral conditions. This type of nonlocal boundary value problems has been investigated in [1, 2, 3, 4, 5, 6, 8] for parabolic equations and in [7, 10] for hyperbolic equations. However, some partial differential equations of higher order with dominated low terms and nonlocal conditions are encountered when studying models for certain natural and physical processes. An example of such type of equations is the equation of longitudinal waves in a thin elastic stem taking into account the effects of transversal inertia [9]:

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^4 u}{\partial t^2\partial x^2}=0. \tag{1.1}$$

Another example is the equation of moisture transfer:

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}+A\frac{\partial^2 u}{\partial x\partial t}\right), \tag{1.2}$$

where u is the concentration of moisture per unit, D is the coefficient of diffusivity, and $A>0$ is the varying coefficient of Hallaire. Motivated by this, we study the equation

$$lu=\frac{\partial^3 u}{\partial t\partial x^2}-\frac{\partial u}{\partial t}+a(t,x)\frac{\partial^2 u}{\partial x^2}+b(t,x)\frac{\partial u}{\partial x}+c(t,x)u=f(t,x) \tag{1.3}$$

in the rectangular domain $\Omega=(0,T)\times(0,1)$.

To (1.3), we attach the nonlocal conditions

$$\begin{gathered}\int_0^T u(t,x)\,dt=0,\quad \forall x\in(0,1),\\ u(t,0)=0,\quad \int_0^1 u(t,x)\,dx=0,\quad \forall t\in(0,T).\end{gathered} \tag{1.4}$$

We assume that the coefficients of l are smooth and bounded on Ω:

$$\begin{gathered} 0 < a(t,x) \leq a_0, \qquad 0 < b(t,x) \leq b_0 \frac{\sigma(x)}{\sqrt{2}}, \qquad 0 < c(t,x) \leq c_0, \\ \forall x \in (0,1), \quad \forall t \in (0,T), \quad \text{where } \sigma(x) = 1-x. \end{gathered} \tag{1.5}$$

2. Generalized solution. Define the operator l_1 by

$$l_1 u = -\frac{\partial u}{\partial t} + a(t,x)\frac{\partial^2 u}{\partial x^2} + b(t,x)\frac{\partial u}{\partial x} + c(t,x)u, \tag{2.1}$$

and

$$F(t,x,u) = f(t,x) - l_1 u. \tag{2.2}$$

Then (1.3) can be assumed to have the form

$$\frac{\partial^3 u}{\partial t \partial x^2} = F(t,x,u). \tag{2.3}$$

We introduce the function space

$$\begin{gathered} V = \left\{ v : v \in L_2(\Omega),\ \frac{\sigma(x)}{\sqrt{2}}\frac{\partial v}{\partial x} \in L_2(\Omega),\ \frac{\partial v}{\partial t} \in L_2(\Omega),\ \frac{\partial^2 v}{\partial x^2} \in L_2(\Omega), \right. \\ \left. \int_0^T v(t,x)\,dt = 0,\ v(t,0) = \int_0^1 v(t,x)\,dx = 0 \right\}. \end{gathered} \tag{2.4}$$

The completion of this space, with respect to the norm

$$\|v\|_{1,2,\sigma}^2 = \int_\Omega \left[v^2 + \frac{\sigma(x)^2}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial^2 v}{\partial x^2}\right)^2 \right] dt\,dx, \tag{2.5}$$

is denoted by $\widetilde{H}_\sigma^{1,2}(\Omega)$. Notice that $\widetilde{H}_\sigma^{1,2}(\Omega)$ is a Hilbert space with

$$(u,v)_{\widetilde{H}_\sigma^{1,2}(\Omega)} = \int_\Omega \left[uv + \frac{\sigma^2(x)}{2}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} \right] dt\,dx. \tag{2.6}$$

For $v \in \widetilde{H}_\sigma^{1,2}(\Omega)$, define the operator M by

$$\begin{gathered} Mv = (1-x)\int_0^t \int_0^x v(\tau,\xi)\,d\xi\,d\tau - \int_0^t \frac{\partial^2 v}{\partial x^2}(\tau,x)\,d\tau + \frac{(x-1)^2}{2}\int_0^t v(\tau,x)\,d\tau \\ + \int_0^x Jv(t,\xi)\,d\xi, \quad \text{where } Jv = \int_0^x \frac{\partial v}{\partial t}(t,\xi)\,d\xi. \end{gathered} \tag{2.7}$$

Definition 2.1. A function $u \in \widetilde{H}_\sigma^{1,2}(\Omega)$ is called a generalized solution to problem (1.3)-(1.4) if

$$(u,v)_{\widetilde{H}_\sigma^{1,2}(\Omega)} = \big(F(t,x,u), Mv\big)_{L_2(\Omega)} \quad \text{for every } v \in \widetilde{H}_\sigma^{1,2}(\Omega). \tag{2.8}$$

3. Existence and uniqueness theorem. In this section, we prove the existence and uniqueness of a generalized solution for the problem (1.3)-(1.4). For this, we first study the subsidiary problem

$$l_0 u \equiv \frac{\partial^3 u}{\partial t \partial x^2} = F(t,x,0) \tag{3.1}$$

with integral conditions (1.4), where

$$F(t,x,0) = f(t,x). \tag{3.2}$$

Theorem 3.1. *Let $F(t,x,0) \in L_2(\Omega)$. Then there exists one and only one generalized solution u_0 of the subsidiary problem*

$$\begin{gathered} l_0 u \equiv \frac{\partial^3 u}{\partial t \partial x^2} = F(t,x,0), \\ \int_0^T u(t,x)\,dt = 0, \quad \forall x \in (0,1), \\ u(t,0) = 0, \quad \int_0^1 u(t,x)\,dx = 0, \quad \forall t \in (0,T), \end{gathered} \tag{3.3}$$

such that

$$c_1 \|u_0\|_{1,2,\sigma} \leq \|F\|_{L_2(\Omega)}, \tag{3.4}$$

where c_1 is a positive constant.

Proof. For $F(t,x,0) \in L_2(\Omega)$, $\Psi(v) = (F, Mv)_{L_2(\Omega)}$ is a bounded linear functional on $\tilde{H}_\sigma^{1,2}(\Omega)$.

Indeed,

$$\left| (F, Mv)_{L_2(\Omega)} \right| \leq \|F\|_{L_2(\Omega)} \|Mv\|_{L_2(\Omega)}. \tag{3.5}$$

By substituting the expression of Mv in (3.5) and using the Poincaré estimates

$$\begin{gathered} \int_\Omega v^2(t,x)\,dt\,dx \leq 4\int_\Omega (1-x)^2 \left(\frac{\partial v}{\partial x}\right)^2 dt\,dx, \quad v(t,0) = 0, \\ \int_\Omega \left[\int_0^t v(\tau,x)\,d\tau\right]^2 dt\,dx \leq 4\int_\Omega (1-x)^2 v^2(t,x)\,dt\,dx, \end{gathered} \tag{3.6}$$

we find that $|\Psi(v)| \leq 4\max\{2T^2, 4\} \|F\|_{L_2(\Omega)} \|v\|_{1,2,\sigma}$.

Consider the scalar product $(l_0, Mv)_{L_2(\Omega)} = \int_\Omega l_0 u \cdot Mv\,dt\,dx$; employing integration by parts and taking account of $v \in \tilde{H}_\sigma^{1,2}(\Omega)$, we obtain

$$\left(\frac{\partial^3 u}{\partial t \partial x^2}, Mv\right)_{L_2(\Omega)} = (u,v)_{\tilde{H}_\sigma^{1,2}(\Omega)}. \tag{3.7}$$

Thus, by the Riesz representation theorem, there exists a unique solution

$$u_0 \in \tilde{H}_\sigma^{1,2}(\Omega) : \Psi(v) = (F, Mv)_{L_2(\Omega)} = (u_0, v)_{\tilde{H}_\sigma^{1,2}(\Omega)}, \quad \forall v \in \tilde{H}_\sigma^{1,2}(\Omega). \tag{3.8}$$

Hence, $(u,v)_{\widetilde{H}_{\sigma}^{1,2}(\Omega)} = (u_0,v)_{\widetilde{H}_{\sigma}^{1,2}(\Omega)}$, that is, u_0 is a generalized solution. Letting $1/c_1 = 4\max\{2T^2,4\}$, we obtain inequality (3.4). □

Lemma 3.2. *The operator $l_1 : \widetilde{H}_{\sigma}^{1,2}(\Omega) \to L_2(\Omega)$ is bounded, that is, there exists a positive constant c_2 such that $\|l_1u\|_{L_2(\Omega)} \le c_2\|u\|_{1,2,\sigma}$.*

Proof. By using conditions (1.5), we directly obtain

$$||l_1u||^2_{L_2(\Omega)} \le 4\left(\left\|\frac{\partial u}{\partial t}\right\|^2_{L_2} + a_0^2\left\|\frac{\partial^2 u}{\partial x^2}\right\|^2_{L_2} + b_0^2\left\|\frac{\partial u}{\partial x}\right\|^2_{L_{2,\sigma}} + c_0^2\|u\|^2_{L_2}\right), \tag{3.9}$$

where $\|\partial u/\partial x\|^2_{L_{2,\sigma}} = \int_{\Omega}(\sigma^2(x)/2)(\partial u/\partial x)^2 dt\,dx$.

Hence, $\|l_1u\|^2_{L_2(\Omega)} \le c_2^2\|u\|^2_{1,2,\sigma}$, where $c_2^2 = 4\max\{1,a_0^2,b_0^2,c_0^2\}$. □

Since l_1 is linear, then $l_1(\sqrt{2}\mu u) = \sqrt{2}\mu l_1(u)$ for an arbitrary μ.

Let $l_{1,\mu}(w) = l_1(\sqrt{2}\mu w)$ for $\mu > 1/c_1$.

Now, consider the general case. The idea in the proof is to derive the results for the equation $lu = f$ with integral conditions (1.4).

Theorem 3.3. *Let $f(t,x) \in L_2(\Omega)$ and $|f(t,x)| \le \lambda/\sqrt{2}$, where λ is a constant. Then there exists at least one generalized solution $u_0 \in \widetilde{H}_{\sigma}^{1,2}(\Omega)$ to problem (1.3)-(1.4). Furthermore, the solution is uniquely determined if $c_2 < c_1$.*

Proof. Let $W = \{l_{1,\mu}w : l_{1,\mu}w \in L_2(\Omega),\ \|l_1u\|^2_{L_2(\Omega)} \le \lambda^2T/\kappa^2\}$ be a closed ball, where $\kappa^2 = c_1^2 - 1/\mu^2$.

It is clear that

$$|F(t,x,w)| \le |f(t,x)| + \sqrt{\frac{c_1^2-k^2}{2}}\,|l_{1,\mu}(w)|, \tag{3.10}$$

and we have $\|F(t,x,w)\|^2_{L_2(\Omega)} \le c_1^2\lambda^2T/\kappa^2$ for all $l_{1,\mu}w \in W$.

From Theorem 3.1, there exists a unique generalized solution of the problem

$$\frac{\partial^3 u}{\partial t\partial x^2} = F(t,x,w) \tag{3.11}$$

with integral conditions (1.4), so that

$$(u,v)_{\widetilde{H}_{\sigma}^{1,2}(\Omega)} = (F,Mv)_{L_2(\Omega)}. \tag{3.12}$$

Define an operator $S : l_1w \in W \to u = Sl_1w \in \widetilde{H}_{\sigma}^{1,2}(\Omega)$, $S(W) \subset W$.

Notice that S is completely continuous. To show this, let $(l_1w)_n, (l_1w)_0 \in W$ and $\|(l_1w)_n - (l_1w)_0\|^2_{L_2(\Omega)} \to 0$, as $n \to \infty$.

Then, for $u_n = S(l_1w)_n$, $u_0 = S(l_1w)_0$, we have

$$\begin{aligned}(u_n-u_0,v)_{\widetilde{H}_{\sigma}^{1,2}(\Omega)} &= (F(t,x,(w)_n) - F(t,x,(w)_0),Mv)\\ &= \left((l_1w)_n - (l_1w)_0,Mv\right)_{L_2(\Omega)} \quad \text{for every } v \in \widetilde{H}_{\sigma}^{1,2}(\Omega).\end{aligned} \tag{3.13}$$

Now, from Theorem 3.1,

$$c_1||u_n - u_0||_{1,2,\sigma} \leq ||(l_1 w)_n - (l_1 w)_0||_{L_2(\Omega)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.14}$$

Again, taking a sequence $\{(l_1 w)_n\} \subset W$, $\|(l_1 w)_n\|^2_{L_2(\Omega)} \leq \lambda^2 T/\kappa^2$. For $u_n = S(l_1 w)_n$, we have $\|u_n\|^2_{L_2(\Omega)} \leq \lambda^2 T/\kappa^2$, so a sequence $\{u_n\}$ is bounded in $\tilde{H}^{1,2}_\sigma(\Omega)$; therefore there exists a subsequence weakly convergent in $\tilde{H}^{1,2}_\sigma(\Omega)$.

Since any bounded set in $\tilde{H}^{1,2}_\sigma(\Omega)$ is compact in $L_2(\Omega)$, then there exists a subsequence, which we also denote by $\{u_n\}$, strongly convergent in $L_2(\Omega)$ to u_0, as $n \to \infty$. As l_1 is a bounded operator, S is completely continuous, and so Sl_1 is completely continuous. Thus, from Schauder's fixed-point theorem, there exists at least one fixed point $u_0 \in W$ such that $u_0 = Sl_1 u_0$ and $(u_0, v)_{\tilde{H}^{1,2}_\sigma(\Omega)} = (F(t,x,u_0), Mv)_{L_2(\Omega)}$ for every $v \in \tilde{H}^{1,2}_\sigma(\Omega)$.

Now, assume that u_1, u_2 are distinct generalized solutions, then $(u_1 - u_2, v)_{\tilde{H}^{1,2}_\sigma(\Omega)} = (F(t,x,u_1) - F(t,x,u_2), Mv)_{L_2(\Omega)}$ for all $v \in \tilde{H}^{1,2}_\sigma(\Omega)$.

From (3.4) and Lemma 3.2, we have

$$||u_1 - u_2||_{1,2,\sigma} \leq \frac{1}{c_1}||l_1 u_1 - l_1 u_2|| \leq \frac{c_2}{c_1}||u_1 - u_2||_{1,2,\sigma}. \tag{3.15}$$

Thus, if $c_2 < c_1$, then it gives a contradiction; therefore $u_1 = u_2$. □

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Lazhar Bougoffa: Department of Mathematics, Faculty of Science, King Khalid University, P. O. Box 9004, Abha, Saudi Arabia

E-mail address: abogafah@kku.edu.sa