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ON UNIFORM KADEC-KLEE PROPERTIES AND ROTUNDITY IN GENERALIZED CESÀRO SEQUENCE SPACES

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We consider the generalized Cesàro sequence spaces defined by Suantai (2003) and consider it equipped with the Amemiya norm. The main purpose of this paper is to show that $\text{ces}_{(p)}$ equipped with the Amemiya norm is rotund and has uniform Kadec-Klee property.

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1. Introduction. In the whole paper, $\mathbb{N}$ and $\mathbb{R}$ stand for the sets of natural numbers and real numbers, respectively. Let $(X,\|\cdot\|)$ be a real normed space and $B(X)(S(X))$ the closed unit ball (the unit sphere) of X.

A point $x \in S(X)$ is called an *extreme point* if for any $y,z \in B(X)$ the equality $2x = y+z$ implies $y = z$.

A Banach space X is said to be *rotund* (abbreviated as (R)) if every point of $S(X)$ is an extreme point.

A Banach space X is said to have the *Kadec-Klee property* (or H-property) if every weakly convergent sequence on the unit sphere is convergent in norm.

Recall that a sequence $\{x_n\} \subset X$ is said to be *ε-separated sequence* for some $\varepsilon > 0$ if

$$\text{sep}(x_n) = \inf\{\|x_n - x_m\| : n \neq m\} > \varepsilon. \tag{1.1}$$

A Banach space is said to have the *uniform Kadec-Klee property* (abbreviated as (UKK)) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every sequence (x_n) in $S(X)$ with $\text{sep}(x_n) > \varepsilon$ and $x_n \xrightarrow{\omega} x$, we have $\|x\| < 1-\delta$. Every (UKK) Banach space has H-property (see [3]).

A Banach space is said to be *nearly uniformly convex* (abbreviated as (NUC)) if for every $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that for every sequence $(x_n) \subseteq B(X)$ with $\text{sep}(x_n) > \varepsilon$, we have

$$\text{conv}(x_n) \cap ((1-\delta)B(X)) \neq \emptyset. \tag{1.2}$$

Huff [3] proved that every (NUC) Banach space is reflexive and has H-property and he also proved that X is NUC if and only if X is reflexive and UKK.

A Banach space X is said to be *locally uniform rotund* (abbreviated as (LUR)) if for each $x \in S(X)$ and each sequence $(x_n) \subset S(X)$ such that $\lim_{n\to\infty}\|x_n + x\| = 2$ there holds $\lim_{n\to\infty}\|x_n - x\| = 0$.

A continuous function $\Phi : \mathbb{R} \to \mathbb{R}$ is called *convex* if

$$\Phi\left(\frac{u+v}{2}\right) \le \frac{\Phi(u)+\Phi(v)}{2} \tag{1.3}$$

for all $u,v \in \mathbb{R}$. If, in addition, the two sides of inequality (1.3) are not equal for all $u \neq v$, then we call Φ *strictly convex*.

For a real vector space X, a function $\varrho : X \to [0,\infty]$ is called a *modular* if it satisfies the following conditions:

(i) $\varrho(x) = 0$ if and only if $x = 0$;

(ii) $\varrho(\alpha x) = \varrho(x)$ for all scalar α with $|\alpha| = 1$;

(iii) $\varrho(\alpha x + \beta y) \le \varrho(x) + \varrho(y)$ for all $x,y \in X$ and all α, $\beta \ge 0$ with $\alpha+\beta = 1$.

The modular ϱ is called *convex* if

(iv) $\varrho(\alpha x + \beta y) \le \alpha\varrho(x) + \beta\varrho(y)$ for all $x,y \in X$ and all α, $\beta \ge 0$ with $\alpha+\beta = 1$.

For any modular ϱ on X, the space

$$X_\varrho = \{x \in X : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\} \tag{1.4}$$

is called the *modular space*. If ϱ is a convex modular, the functions

$$\begin{aligned} \|x\| &= \inf\left\{\lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \le 1\right\}, \\ \|x\|_0 &= \inf_{k>0} \frac{1}{k}\left(1+\varrho(kx)\right) \end{aligned} \tag{1.5}$$

are two norms on X_ϱ, which are called the *Luxemburg norm* and the *Amemiya norm*, respectively. In addition, $\|x\| \le \|x\|_0 \le 2\|x\|$ for all $x \in X_\varrho$ (see [6]).

A modular ϱ is said to satisfy the Δ_2-condition ($\varrho \in \Delta_2$) if for any $\varepsilon > 0$ there exist constants $K \ge 2$ and $a > 0$ such that

$$\varrho(2x) \le K\varrho(x) + \varepsilon \tag{1.6}$$

for all $x \in X_\varrho$ with $\varrho(x) \le a$.

If ϱ satisfies the Δ_2-condition for all $a > 0$ with $K \ge 2$ dependent on a, we say that ϱ satisfies the *strong* Δ_2-condition ($\varrho \in \Delta_2^s$).

Let ℓ^0 be the space of all real sequences. The Musielak-Orlicz sequence space ℓ_Φ, where $\Phi = (\phi_i)_{i=1}^\infty$ is a sequence of Orlicz functions, is defined as

$$\ell_\Phi = \{x = (x(i))_{i=1}^\infty \in \ell^0 : \varrho_\Phi(\lambda x) < \infty \text{ for some } \lambda > 0\}, \tag{1.7}$$

where $\varrho_\Phi(x) = \sum_{i=1}^\infty \phi_i(x(i))$ is a convex modular on ℓ_Φ. Then ℓ_Φ is a Banach space with both Luxemburg norm $\|\cdot\|_{\ell_\Phi}$ and Amemiya norm $\|\cdot\|_{\ell_\Phi^0}$ (see [6]). In [2], Hudzik and Zbąszyniak proved that in the space ℓ_Φ endowed with the Amemiya norm, there exists $k \in \mathbb{R}$ such that

$$\|x\|_{\ell_\Phi^0} = \frac{1}{k}\left(1+\varrho_\Phi(kx)\right) \quad (x \in \ell_\Phi) \tag{1.8}$$

if $\phi_i(u)/u \to \infty$ as $u \to \infty$ for any $i \in \mathbb{N}$.

For $1 < p < \infty$, the Cesàro sequence space (ces_p) is defined by

$$\text{ces}_p = \left\{ x \in \ell^0 : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p < \infty \right\} \tag{1.9}$$

equipped with the norm

$$\|x\| = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p \right)^{1/p}. \tag{1.10}$$

This space was first introduced by Shue [8]. It is useful in the theory of matrix operator and others (see [4, 5]). Some geometric properties of the Cesàro sequence space ces_p were studied by many authors. Now, we introduce a generalized Cesàro sequence space.

Let $p = (p_n)$ be a sequence of positive real numbers with $p_n \geq 1$ for all $n \in \mathbb{N}$. The generalized Cesàro sequence space $\text{ces}_{(p)}$ is defined by

$$\text{ces}_{(p)} = \{ x \in l^0 : \rho(\lambda x) < \infty \text{ for some } \lambda > 0 \}, \tag{1.11}$$

where

$$\rho(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^{p_n} \tag{1.12}$$

is a convex modular on $\text{ces}_{(p)}$. To simplify the notations, we put $\text{ces}_{(p)} = (\text{ces}_{(p)}, \|\cdot\|)$ and $\text{ces}^0_{(p)} = (\text{ces}_{(p)}, \|\cdot\|_0)$.

For $\text{ces}_{(p)}$, Suantai [9] proved that $\text{ces}_{(p)}$ is LUR, hence it is R and has H-property where $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$.

In $\text{ces}^0_{(p)}$, the set of all k's, at which the infimum in the definition of $\|x\|_0$ for a fixed $x \in \text{ces}^0_{(p)}$ is attained, will be denoted by $K(x)$.

Throughout this paper, we let $p = (p_k)$ be a bounded sequence of positive real numbers.

2. Main results.

We first give an important fact for $\|x\|_0$ on $\text{ces}^0_{(p)}$.

Proposition 2.1. *For each $x \in \text{ces}^0_{(p)}$, there exists $k \in \mathbb{R}$ such that*

$$\|x\|_0 = \frac{1}{k}(1 + \rho(kx)). \tag{2.1}$$

Proof. First, we note that $\phi(t) = |t|^r$ $(r > 1)$ is an Orlicz function which satisfies $\phi_i(u)/u \to \infty$ as $u \to \infty$.

Now, observe that for each $x = (x(i))_{i=1}^{\infty} \in \text{ces}^0_{(p)}$ we have

$$x' = \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)_{n=1}^{\infty} \in \ell_{\Phi}, \tag{2.2}$$

where $\Phi = (\phi_i)_{i=1}^{\infty}$ and $\phi_i(t) = |t|^{p_i}$ for each $i \in \mathbb{N}$. Moreover, $\|x\|_0 = \|x'\|_{\ell_\Phi^0}$ and by (1.8) there exists $k \in \mathbb{R}$ such that

$$\begin{aligned} \|x\|_0 = \|x'\|_{\ell_\Phi^0} &= \frac{1}{k}\left(1+\varrho_\Phi(kx')\right) \\ &= \frac{1}{k}\left(1+\sum_{n=1}^{\infty}\left(\frac{k}{n}\sum_{i=1}^{n}|x(i)|\right)^{p_n}\right) = \frac{1}{k}\left(1+\rho(kx)\right). \end{aligned} \tag{2.3}$$

□

Proposition 2.2. *For a modular space X_ϱ, convergence in norm and convergence in modular are equivalent if and only if $\varrho \in \Delta_2$.*

Proof. See [1]. □

Proposition 2.3. *Suppose that $\{x_n\}$ is a bounded sequence in $\text{ces}_{(p)}^0$ with $p_k > 1$ for all $k \in \mathbb{N}$ and $x_n \xrightarrow{w} x$ for some $x \in \text{ces}_{(p)}^0$. If $k_n \in K(x_n)$ and $k_n \to \infty$, then $x = 0$.*

Proof. For each $n \in \mathbb{N}$, $\eta > 0$, put $G_{(n,\eta)} = \{i \in \mathbb{N} : (1/i)\sum_{j=1}^{i}|x_n(j)| \geq \eta\}$. First, we claim that for each $\eta > 0$, $G_{(n,\eta)} = \emptyset$ for all large $n \in \mathbb{N}$. If not, without loss of generality, we may assume that $G_{(n,\eta)} \neq \emptyset$ for all $n \in \mathbb{N}$ for some $\eta > 0$. Then,

$$\|x_n\|_0 = \frac{1}{k_n}\left(1+\rho(k_n x_n)\right) \geq \frac{(k_n\eta)^{p_i}}{k_n} \quad (i \in G_{(n,\eta)}). \tag{2.4}$$

Applying the fact $|t|^r/t \to \infty$ as $t \to \infty$, where $r > 1$, we obtain $\|x_n\|_0 \to \infty$ which contradicts the fact that $\{x_n\}$ is bounded, hence we have the claim. By the claim, we have $(1/i)\sum_{j=1}^{i}|x_n(j)| \to 0$ as $n \to \infty$ for all $i \in \mathbb{N}$. Hence, we obtain by induction that $x_n(i) \to 0$ as $n \to \infty$ for all $i \in \mathbb{N}$. Since $x_n \xrightarrow{w} x$, we have $x_n(i) \to x(i)$ for all $i \in \mathbb{N}$, so it must be $x(i) = 0$ for all $i \in \mathbb{N}$. □

Theorem 2.4. *The space $\text{ces}_{(p)}^0$ is R if each $p_k > 1$.*

Proof. Let $x \in S(\text{ces}_{(p)}^0)$ and suppose $y, z \in S(\text{ces}_{(p)}^0)$ with $y + z = 2x$. Take $k' \in K(y)$, $k'' \in K(z)$ and define $k = k'k''/(k'+k'')$. Then by convexity of $u \mapsto |u|^{p_n}$ for every $n \in \mathbb{N}$, we have

$$\begin{aligned} 2 = \|y\|_0 + \|z\|_0 &= \frac{k'+k''}{k'k''}\left[1+\frac{k''}{k'+k''}\rho(k'y)+\frac{k'}{k'+k''}\rho(k''z)\right] \\ &\geq \frac{1}{k}\left[1+\rho(ky+kz)\right] = \frac{2}{2k}\left[1+\rho(2kx)\right] \geq 2\|x\|_0 = 2. \end{aligned} \tag{2.5}$$

This implies

$$\frac{k''}{k'+k''}\left(\frac{k'}{n}\sum_{i=1}^{n}|y(i)|\right)^{p_n} + \frac{k'}{k'+k''}\left(\frac{k''}{n}\sum_{i=1}^{n}|z(i)|\right)^{p_n} = \left(\frac{2k}{n}\sum_{i=1}^{n}|x(i)|\right)^{p_n} \tag{2.6}$$

for all $n \in \mathbb{N}$.

Since the function $u \mapsto |u|^{p_n}$ is strictly convex function for all $n \in \mathbb{N}$, it implies that

$$k'\left[\frac{1}{n}\sum_{i=1}^{n}|y(i)|\right] = k''\left[\frac{1}{n}\sum_{i=1}^{n}|z(i)|\right] = 2k\left[\frac{1}{n}\sum_{i=1}^{n}|x(i)|\right] \tag{2.7}$$

for each $n \in \mathbb{N}$. This gives $k'|y(i)| = k''|z(i)|$ for all $i \in \mathbb{N}$, and it follows that $k' = \|k'y\|^\circ = \|k''z\|^\circ = k''$, hence $|y(i)| = |z(i)|$ for all $i \in \mathbb{N}$. To complete the proof, it suffices to show that $y(i) = z(i)$ for all $i \in \mathbb{N}$. If not, let $i_o \in \mathbb{N}$ be the first coordinate such that $y(i_o) \neq z(i_o)$, so $y(i_o) = -z(i_o)$ and hence $2x(i_o) = y(i_o) + z(i_o) = 0$. Since $k' = k'' = 2k$, we have

$$\begin{gathered}\left[\frac{1}{i_o-1}\sum_{i=1}^{i_o-1}|z(i)|\right] = \left[\frac{1}{i_o-1}\sum_{i=1}^{i_o-1}|x(i)|\right],\\ \left[\frac{1}{i_o}\sum_{i=1}^{i_o}|z(i)|\right] = \left[\frac{1}{i_o}\sum_{i=1}^{i_o}|x(i)|\right]\end{gathered} \tag{2.8}$$

which implies $z(i_o) = 0$, which is a contradiction. Hence $y = z$. □

Theorem 2.5. *The space* $\mathrm{ces}^0_{(p)}$ *is UKK if each* $p_k > 1$.

Proof. For a given $\varepsilon > 0$, by Proposition 2.2 there exists $\delta \in (0,1)$ such that $\|y\|_0 \geq \varepsilon/4$ implies $\rho(y) \geq 2\delta$. Given $x_n \in B(\mathrm{ces}^0_{(p)})$, $x_n \to x$ weakly, and $\|x_n - x_m\|_0 \geq \varepsilon$ $(n \neq m)$, we will complete the proof by showing that $\|x\|_0 \leq 1-\delta$. Indeed, if $x = 0$, then we have nothing to show. So, we assume that $x \neq 0$. In this case, by Proposition 2.3 we have that $\{k_n\}$ is bounded, where $k_n \in K(x_n)$. Passing to a subsequence, if necessary we may assume that $k_n \to k$ for some $k > 0$. Next, we select a finite subset I of $\mathbb{N}$ such that $\|x_{|_I}\|_0 \geq \|x\|_0 - \delta$, say $I = \{1,2,3,\dots,j\}$; since the weak convergence of $\{x_n\}$ implies that $x_n \to x$ coordinatewise, we deduce that $x_n \to x$ uniformly on I. Consequently, there exists $n_o \in \mathbb{N}$ such that

$$||(x_n - x_m)_{|_I}||_0 \leq \frac{\varepsilon}{2} \quad \forall n,m \geq n_o, \tag{2.9}$$

which implies

$$||(x_n - x_m)_{|_{\mathbb{N}\setminus I}}||_0 \geq \frac{\varepsilon}{2} \quad \forall n,m \geq n_o,\ m \neq n. \tag{2.10}$$

This gives $\|x_{n|_{\mathbb{N}\setminus I}}\|_0 \geq \varepsilon/4$ or $\|x_{m|_{\mathbb{N}\setminus I}}\|_0 \geq \varepsilon/4$, for all $m,n \geq n_o$, $m \neq n$, which yields that $\|x_{n|_{\mathbb{N}\setminus I}}\|_0 \geq \varepsilon/4$ for infinitely many $n \in \mathbb{N}$, hence $\rho(x_{n|_{\mathbb{N}\setminus I}}) \geq 2\delta$. Without loss of generality, we may assume that $\|x_{n|_{\mathbb{N}\setminus I}}\|_0 \geq \varepsilon/4$, for all $n \in \mathbb{N}$. By using the inequality $(a+b)^t \geq a^t + b^t$ $(a,b \geq 0,\ t \geq 1)$ combined with the fact that $k_n \geq 1$ and the convexity of function $t \mapsto |t|^{p_n}$, we have

$$
\begin{aligned}
1-2\delta &\geq \|x_n\|_0 - \rho(x_{n|_{\mathbb{N}\setminus I}}) \\
&\geq \|x_n\|_0 - \frac{1}{k_n}\rho(k_n x_{n|_{\mathbb{N}\setminus I}}) \\
&= \frac{1}{k_n} + \frac{1}{k_n}\left[\sum_{i=1}^{\infty}\left(\frac{k_n}{i}\sum_{r=1}^{i}|x_n(r)|\right)^{p_i}\right] - \frac{1}{k_n}\left[\sum_{i=j+1}^{\infty}\left(\frac{k_n}{i}\sum_{r=1}^{i-j}|x_n(j+r)|\right)^{p_i}\right] \\
&= \frac{1}{k_n} + \frac{1}{k_n}\left[\sum_{i=1}^{j}\left(\frac{k_n}{i}\sum_{r=1}^{i}|x_n(r)|\right)^{p_i}\right] \\
&\quad + \frac{1}{k_n}\left[\sum_{i=j+1}^{\infty}\left(\frac{k_n}{i}\sum_{r=1}^{i}|x_n(r)|\right)^{p_i} - \sum_{i=j+1}^{\infty}\left(\frac{k_n}{i}\sum_{r=1}^{i-j}|x_n(j+r)|\right)^{p_i}\right] \\
&= \frac{1}{k_n} + \frac{1}{k_n}\left[\sum_{i=1}^{j}\left(\frac{k_n}{i}\sum_{r=1}^{i}|x_n(r)|\right)^{p_i}\right] \\
&\quad + \frac{1}{k_n}\left[\sum_{i=j+1}^{\infty}\left(\frac{k_n}{i}\sum_{r=1}^{j}|x_n(r)| + \frac{k_n}{i}\sum_{r=1}^{i-j}|x_n(j+r)|\right)^{p_i}\right. \\
&\qquad \left. - \sum_{i=j+1}^{\infty}\left(\frac{k_n}{i}\sum_{r=1}^{i-j}|x_n(j+r)|\right)^{p_i}\right] \\
&\geq \frac{1}{k_n} + \frac{1}{k_n}\left[\sum_{i=1}^{j}\left(\frac{k_n}{i}\sum_{r=1}^{i}|x_n(r)|\right)^{p_i}\right] + \frac{1}{k_n}\left[\sum_{i=j+1}^{\infty}\left(\frac{k_n}{i}\sum_{r=1}^{j}|x_n(r)|\right)^{p_i}\right] \\
&= \frac{1}{k_n} + \frac{1}{k_n}\rho(k_n x_{n|_I}) \longrightarrow \frac{1}{k} + \frac{1}{k}\rho(k x_{|_I}) \geq \|x_{|_I}\|_0 \geq \|x\|_0 - \delta,
\end{aligned}
\tag{2.11}
$$

hence $\|x\|_0 \leq 1-\delta$. □

Since every (UKK) Banach space has H-property, the following result is obtained.

Corollary 2.6. *The space* $\text{ces}^0_{(p)}$ *possesses H-property if each* $p_k > 1$.

Corollary 2.7. *The space* $\text{ces}^0_{(p)}$ *possesses the property NUC if each* $p_k > 1$ *and* $\lim_{k\to\infty}\inf p_k$.

Proof. By [7], $\text{ces}_{(p)}$ is NUC, so it is reflexive. Since a Banach space X is NUC if and only if X is reflexive and UKK, the corollary follows from Theorem 2.5. □

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