IJMMS 2003:7, 451–459
PII. S0161171203110198
http://ijmms.hindawi.com

ON THE EQUIVALENCE OF MANN AND ISHIKAWA ITERATION METHODS

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Received 19 October 2001

We show that certain Mann and Ishikawa iteration schemes are equivalent for various classes of functions.

2000 Mathematics Subject Classification: 47H10.

The Mann iterative scheme was invented in 1953, see [7], and was used to obtain convergence to a fixed point for many functions for which the Banach principle fails. For example, the first author in [8] showed that, for any continuous selfmap of a closed and bounded interval, the Mann iteration converges to a fixed point of the function.

In 1974, Ishikawa [5] devised a new iteration scheme to establish convergence for a Lipschitzian pseudocontractive map in a situation where the Mann iteration process failed to converge.

Let X be a Banach space. The Mann iteration is defined by

$$x_0 \in X, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0, \tag{1}$$

where the $\alpha_n \in (0,1)$, for all $n \geq 0$.

The Ishikawa iteration scheme is defined by

$$\begin{gathered} u_0 \in X, \quad u_{n+1} = (1-\alpha_n)u_n + \alpha_n T v_n, \\ v_n = (1-\beta_n)u_n + \beta_n T u_n, \quad n \geq 0, \end{gathered} \tag{2}$$

where

$$0 \leq \alpha_n \leq \beta_n \leq 1, \quad \forall n \geq 0. \tag{3}$$

In specific situations, additional conditions are placed on $(\alpha_n)_n$ and $(\beta_n)_n$. In [9], the first author modified the definition of Ishikawa by replacing (3) by

$$0 \leq \alpha_n, \beta_n \leq 1, \quad \forall n \geq 0. \tag{4}$$

During the past 25 years, a large literature has developed around the themes of establishing convergence of the Mann iteration for certain classes of functions, and then showing that the Ishikawa iteration, using (4), also converges.

Of course, having established the convergence of an Ishikawa method using (4), we obtain as a corollary the convergence of the corresponding Mann iteration method by setting each $\beta_n = 0$.

A reasonable conjecture is that, whenever T is a function for which Mann iteration converges, so does the Ishikawa iteration. Given the large variety of functions and spaces, such a global statement is, of course, not provable. However, in this paper, we do show that, for several classes of functions, Mann and Ishikawa iteration procedures are equivalent.

Picard iteration is defined by $p_0 \in X$ and $p_{n+1} = Tp_n$, where T is a selfmap of X.

Theorem 1. *Let X be a normed space and let B be a nonempty convex subset of X, $T : B \to B$, with T satisfying*

$$\|Tx - Ty\| \le k \max\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\} \tag{5}$$

for all $x, y \in B$, $0 \le k < 1$. Suppose that T possesses a fixed point $p \in B$. Then Picard iteration and certain Mann and Ishikawa iteration schemes converge strongly to p.

Proof. Let $p_0 \in B$ and define $p_{n+1} = Tp_n$, $n \ge 0$. From [3], it follows that $(p_n)_n$ is Cauchy in B. Hence, it converges to a point $x^* \in \overline{B}$. From (5),

$$\begin{aligned} ||p_{n+1} - p|| &= ||Tp_n - Tp|| \\ &\le k \max\{||p_n - p||, ||p_n - Tp_n||, \\ &\qquad ||p_n - Tp||, ||Tp_n - p||\} \\ &= k \max\{||p_n - p||, ||p_n - p_{n+1}||, \\ &\qquad ||p_n - p||, ||p_{n+1} - p||\}. \end{aligned} \tag{6}$$

Taking the limit as $n \to \infty$ yields $\|p - x^*\| \le k\|p - x^*\|$, which implies that $p = x^*$, that is, $(p_n)_n$ converges strongly to p.

In [9], it was noted that [8, Theorem 6] could be extended to maps satisfying (5), that is, Mann iteration of a T satisfying (5) with $\alpha_n \in (0,1)$ and bounded away from zero converges strongly to the unique fixed point p of T.

In [12], it was shown that, for T satisfying (5) with $\alpha_n \in (0,1)$, the Ishikawa method, with each $\alpha_n > 0$ and $\sum \alpha_n = \infty$, converges strongly to p. □

As shown in [10], inequality (5) is one of the most general contractive conditions for a single map.

We need the following lemma.

Lemma 2 (see [11]). *Let $(a_n)_n$ be a nonnegative sequence that satisfies the inequality*

$$a_{n+1} \le (1 - \lambda_n)a_n + \sigma_n, \tag{7}$$

where $\lambda_n \in (0,1)$ *for each* $n \in \mathbb{N}$*,* $\sum_{n=1}^{\infty} \lambda_n = \infty$*, and* $\sigma_n = \epsilon_n \lambda_n$*,* $\lim \epsilon_n = 0$*. Then* $\lim a_n = 0$.

We are able now to prove the following result.

Theorem 3. *Let* X *be a normed space,* K *a nonempty closed convex subset of* X*, and* T *a Lipschitzian selfmap of* K *with Lipschitz constant* $L \le 1$*. Suppose that* T *has a fixed point* $p \in B$*. Let* $x_0 = u_0 \in K$*, and define* x_n *and* u_n *by (1) and (2), with* α_n*,* β_n *satisfying (4),* (i) $\lim \alpha_n = \lim \beta_n = 0$*, and* (ii) $\sum \alpha_n = \infty$*. Then the following are equivalent:*

(a) *the Mann iteration converges strongly to* p*,*

(b) *the Ishikawa iteration converges strongly to* p*.*

Proof. That (b) implies (a) is obvious setting $\beta_n = 0$ in (2). We prove that (a) implies (b). From (1) and (2),

$$\begin{aligned}
||x_{n+1} - u_{n+1}|| &= ||(1-\alpha_n)(x_n - u_n) + \alpha_n (Tx_n - Tv_n)|| \\
&\le (1-\alpha_n)||x_n - u_n|| + \alpha_n L ||Tx_n - Tv_n|| \\
&= (1-\alpha_n)||x_n - u_n|| + \alpha_n L ||(1-\beta_n)u_n + \beta_n T u_n - x_n|| \\
&= (1-\alpha_n)||x_n - u_n|| + \alpha_n L ||(1-\beta_n)(u_n - x_n) + \beta_n (Tu_n - x_n)|| \\
&\le [1 - \alpha_n(1 - L(1-\beta_n))]||x_n - u_n|| + \alpha_n \beta_n M,
\end{aligned} \tag{8}$$

for some positive M since $(\|Tu_n - x_n\|)_n$ is bounded. This fact is obvious if we prove that $(\|u_n\|)_n$ is bounded. A simple induction lead us to

$$\begin{aligned}
||u_{n+1}|| &\le (1-\alpha_n)||u_n|| + \alpha_n ||Tv_n|| \\
&\le (1-\alpha_n)||u_n|| + \alpha_n L ||(1-\beta_n)u_n + \beta_n T u_n|| \\
&\le (1-\alpha_n)||u_n|| + \alpha_n L (1-\beta_n)||u_n|| + \alpha_n \beta_n L ||Tu_n|| \\
&= ||u_n|| \le \cdots \le ||u_0||.
\end{aligned} \tag{9}$$

With $a_n := \|x_n - u_n\|$, $\lambda_n := \alpha_n(1 - L(1-\beta_n)) \in (0,1)$, and $\sigma_n := \alpha_n \beta_n M$, for each $n \in \mathbb{N}$, the inequality of Lemma 2 is satisfied. Therefore,

$$\lim ||x_n - u_n|| = 0. \tag{10}$$

Since (a) is true, using (10),

$$||u_n - p|| \le ||x_n - p|| + ||x_n - u_n||, \tag{11}$$

which implies that $\lim \|u_n - p\| = 0$. □

Let X be an arbitrary Banach space and let J be the normalized duality map from X into 2^{X^*}. A map T with domain $D(T)$ and range $R(T)$ is called *strongly pseudocontractive* (pseudocontractive) if, for each $x,y \in D(T)$, there exists $j(x-y) \in J(x-y)$ and a $t > 1$ $(t = 1)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \frac{1}{t}\|x-y\|^2. \tag{12}$$

Equivalently, there exists a constant $t > 1$ such that

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \ge \frac{t-1}{t}\|x-y\|^2. \tag{13}$$

If we set $k = (t-1)/t$, then the above inequality can be written in the form

$$\langle (I-T-kI)x - (I-T-kI)y, j(x-y)\rangle \ge 0 \tag{14}$$

and, from a result of Kato [6],

$$\|x-y\| \le \left|\left|x-y+r[(I-T-kI)x-(I-T-kI)y]\right|\right|, \tag{15}$$

for all $x,y \in X$ and $r > 0$.

Theorem 4. *Let K be a closed convex subset of an arbitrary Banach space X and let T be a Lipschitzian strongly pseudocontractive selfmap of K. Let $x_0 = u_0 \in K$, and x_n and u_n be defined by (1) and (2), with α_n, β_n, satisfying (4) and conditions (i) and (ii) of Theorem 3. Let p be the fixed point of T. Then the following are equivalent:*

(a) *the Mann iterative scheme converges to p,*

(b) *the Ishikawa iteration scheme converges to p.*

Proof. The existence of a fixed point p comes from [4, Corollary 1] which holds in an arbitrary Banach space. That (b) implies (a) is obvious setting $\beta_n = 0$.

Without loss of generality, we may assume that the Lipschitz constant L of T is greater than or equal to 1. If $L \in (0,1]$, then the result follows from Theorem 3.

To prove that (a) implies (b), it is necessary to express $\|u_{n+1} - x_{n+1}\|$ in terms of (15). Using (1), (2), and the identity which appears as [2, formula (10), page 782], we obtain

$$\begin{aligned}
||u_n - x_n|| &= ||u_{n+1} + \alpha_n u_n - \alpha_n T v_n - x_{n+1} - \alpha_n x_n + \alpha_n T x_n|| \\
&= ||(1+\alpha_n)u_{n+1} + \alpha_n (I-T-kI)u_{n+1} - (1-k)\alpha_n u_n \\
&\quad + (2-k)\alpha_n^2 (u_n - T v_n) + \alpha_n (T u_{n+1} - T v_n) \\
&\quad - (1+\alpha_n)x_{n+1} - \alpha_n (I-T-kI)x_{n+1} + (1-k)\alpha_n x_n
\end{aligned}$$

$$
\begin{aligned}
&- (2-k)\alpha_n^2(x_n - Tx_n) - \alpha_n(Tx_{n+1} - Tx_n)\| \\
&= \|(1+\alpha_n)(u_{n+1} - x_{n+1}) + \alpha_n[(I-T-kI)u_{n+1} - (I-T-kI)x_{n+1}] \\
&\quad - (1-k)\alpha_n(u_n - x_n) + (2-k)\alpha_n^2(u_n - Tv_n - x_n + Tx_n) \\
&\quad + \alpha_n(Tu_{n+1} - Tv_n - Tx_{n+1} + Tx_n)\|.
\end{aligned}
\tag{16}
$$

Using the triangular inequality and (15),

$$
\begin{aligned}
&\|u_n - x_n\| \\
&\quad \geq (1+\alpha_n)\|(u_{n+1} - x_{n+1}) + \frac{\alpha_n}{1+\alpha_n}[(I-T-kI)u_{n+1} - (I-T-kI)x_{n+1}]\| \\
&\qquad - (1-k)\alpha_n\|u_n - x_n\| - (2-k)\alpha_n^2\|u_n - Tv_n - x_n + Tx_n\| \\
&\qquad - \alpha_n\|Tu_{n+1} - Tv_n - Tx_{n+1} + Tx_n\| \\
&\quad \geq (1+\alpha_n)\|u_{n+1} - x_{n+1}\| - (1-k)\alpha_n\|u_n - x_n\| \\
&\qquad - (2-k)\alpha_n^2\|u_n - Tv_n - x_n + Tx_n\| - \alpha_n\|Tu_{n+1} - Tv_n - Tx_{n+1} + Tx_n\|.
\end{aligned}
\tag{17}
$$

Solving the above inequality for $\|u_{n+1} - x_{n+1}\|$ gives

$$
\begin{aligned}
\|u_{n+1} - x_{n+1}\| &\leq \frac{[1+(1-k)\alpha_n]}{1+\alpha_n}\|u_n - x_n\| \\
&\quad + \frac{(2-k)\alpha_n^2}{1+\alpha_n}\|u_n - Tv_n - x_n + Tx_n\| \\
&\quad + \frac{\alpha_n}{1+\alpha_n}\|Tu_{n+1} - Tv_n - Tx_{n+1} + Tx_n\| \\
&\leq \frac{[1+(1-k)\alpha_n]}{1+\alpha_n}\|u_n - x_n\| + (2-k)\alpha_n^2\|u_n - Tv_n\| \\
&\quad + (2-k)\alpha_n^2\|Tx_n - x_n\| + \alpha_n\|Tu_{n+1} - Tv_n\| \\
&\quad + \alpha_n\|Tx_{n+1} - Tx_n\|,
\end{aligned}
\tag{18}
$$

$$
\|u_n - Tv_n\| \leq \|u_n - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Tv_n\|. \tag{19}
$$

Let L denote the Lipschitz constant for T. Then,

$$
\|Tx_n - Tv_n\| \leq L\|x_n - v_n\|, \tag{20}
$$

$$
\begin{aligned}
\|v_n - x_n\| &= \|(1-\beta_n)u_n + \beta_n Tu_n - x_n\| \\
&\leq (1-\beta_n)\|u_n - x_n\| + \beta_n\|Tu_n - x_n\| \\
&\leq (1-\beta_n)\|u_n - x_n\| + \beta_n[\|Tu_n - Tx_n\| + \|Tx_n - x_n\|] \\
&\leq (1-\beta_n)\|u_n - x_n\| + \beta_n L\|u_n - x_n\| + \beta_n\|Tx_n - x_n\| \\
&= (1-\beta_n + \beta_n L)\|u_n - x_n\| + \beta_n\|Tx_n - x_n\|.
\end{aligned}
\tag{21}
$$

Note that, for $L \geq 1$, $1-\beta_n+\beta_n L \leq L$. Substituting (21) into (20) and then (20) into (19) gives

$$\begin{aligned}||u_n - Tv_n|| &\leq ||u_n - x_n|| + ||x_n - Tx_n|| + L[L||u_n - x_n|| + \beta_n||Tx_n - x_n||] \\ &= (1+L^2)||u_n - x_n|| + (1+L\beta_n)||Tx_n - x_n||,\end{aligned} \tag{22}$$

$$\begin{aligned}||Tu_{n+1} - Tv_n|| &\leq L||u_{n+1} - v_n|| = L||(1-\alpha_n)u_n + \alpha_n Tv_n - v_n|| \\ &\leq L[(1-\alpha_n)||u_n - v_n|| + \alpha_n||Tv_n - v_n||].\end{aligned} \tag{23}$$

Using (21),

$$\begin{aligned}||Tv_n - v_n|| &\leq ||Tv_n - Tx_n|| + ||Tx_n - x_n|| + ||x_n - v_n|| \\ &\leq (1+L)||x_n - v_n|| + ||Tx_n - x_n|| \\ &\leq (1+L)[L||x_n - u_n|| + \beta_n||Tx_n - x_n||] + ||Tx_n - x_n|| \\ &= (1+L)L||x_n - u_n|| + [(1+L)\beta_n + 1]||Tx_n - x_n||,\end{aligned} \tag{24}$$

$$\begin{aligned}||u_n - v_n|| &= ||u_n - (1-\beta_n)u_n - \beta_n Tu_n|| = \beta_n||(u_n - Tu_n)|| \\ &\leq \beta_n[||u_n - x_n|| + ||x_n - Tx_n|| + ||Tx_n - Tu_n||] \\ &\leq \beta_n[(1+L)||x_n - u_n|| + ||x_n - Tx_n||].\end{aligned} \tag{25}$$

Substituting (25) and (24) into (23), we obtain

$$\begin{aligned}||Tu_{n+1} - Tv_n|| &\leq L(1-\alpha_n)[\beta_n(1+L)||u_n - x_n|| + \beta_n||Tx_n - x_n||] \\ &\quad + \alpha_n L[(1+L)L||x_n - u_n|| + [(1+L)\beta_n + 1]||Tx_n - x_n||] \\ &\leq [L(1-\alpha_n)\beta_n(1+L) + \alpha_n L^2(1+L)]||x_n - u_n|| \\ &\quad + \{\beta_n L(1-\alpha_n) + \alpha_n L[(1+L)\beta_n + 1]\}||x_n - Tx_n||.\end{aligned} \tag{26}$$

Substituting (22) and (26) into (18) and using $(1+\alpha_n)^{-1} \leq 1-\alpha_n+\alpha_n^2$, yields

$$\begin{aligned}||x_{n+1} - u_{n+1}|| &\leq (1+(1-k)\alpha_n)(1-\alpha_n+\alpha_n^2)||x_n - u_n|| \\ &\quad + (2-k)\alpha_n^2[(1+L^2)||x_n - u_n|| + (1+L\beta_n)||Tx_n - x_n||] \\ &\quad + (2-k)\alpha_n^2||x_n - Tx_n|| \\ &\quad + \alpha_n[L(1-\alpha_n)\beta_n(1+L) + \alpha_n L^2(1+L)]||x_n - u_n|| \\ &\quad + \alpha_n\{\beta_n L(1-\alpha_n) + \alpha_n L[(1+L)\beta_n + 1]\}||x_n - Tx_n|| \\ &\quad + \alpha_n L||x_{n+1} - x_n|| \\ &\leq \gamma_n||x_n - u_n|| + \delta_n||x_n - Tx_n|| + \alpha_n||x_{n+1} - x_n||,\end{aligned} \tag{27}$$

where

$$\begin{aligned}\delta_n &= \alpha_n[(2-k)(2+L\beta_n)\alpha_n+[\beta_nL(1-\alpha_n)+\alpha_nL[(1+L)\beta_n+1]]],\\ \gamma_n &= [1+(1-k)\alpha_n](1-\alpha_n+\alpha_n^2)+(2-k)(1+L^2)\alpha_n^2\\ &\quad+\alpha_nL(1+L)[\beta_n(1-\alpha_n)+L\alpha_n].\end{aligned} \tag{28}$$

Note that

$$\begin{aligned}(1+(1-k)\alpha_n)(1-\alpha_n+\alpha_n^2) &= 1-k\alpha_n+k\alpha_n^2+(1-k)\alpha_n^3\\ &\le 1-k\alpha_n+k\alpha_n^2+(1-k)\alpha_n^2\\ &= 1-k\alpha_n+\alpha_n^2.\end{aligned} \tag{29}$$

Therefore,

$$\begin{aligned}\gamma_n &\le 1-k\alpha_n+\alpha_n\{2\alpha_n+(2-k)(1+L^2)\alpha_n\\ &\qquad+L(1+L)[\beta_n(1-\alpha_n)+L\alpha_n]\}\\ &= 1-k\alpha_n+\alpha_n\{[2+(2-k)(1+L^2)+L^2(1+L)]\alpha_n\\ &\qquad+L(1+L)(1-\alpha_n)\beta_n\}\\ &\le 1-k\alpha_n+\alpha_nM(\alpha_n+\beta_n),\end{aligned} \tag{30}$$

where $M=2+(2-k)(1+L^2)+L^2(1+L)$.

Since α_n and β_n satisfy (i), there exists an integer N such that $M(\alpha_n+\beta_n)\le k(1-k)$ for all $n\ge N$.

Thus,

$$\begin{aligned}||x_{n+1}-u_{n+1}|| &\le (1-k^2\alpha_n)||x_n-u_n||\\ &\quad+\alpha_n\{[(2-k)(2+L\beta_n)\alpha_n\\ &\qquad+[\beta_nL(1-\alpha_n)+\alpha_nL((1+L)\beta_n+1)]]||x_n-Tx_n||\\ &\qquad+L||x_{n+1}-x_n||\}.\end{aligned} \tag{31}$$

With $\lambda_n := k^2\alpha_n$, $a_n=\|x_n-u_n\|$, and $\epsilon_n=$ the quantity in braces, we have

$$a_{n+1}\le(1-\lambda_n)a_n+\epsilon_n\lambda_n. \tag{32}$$

Since $x_n\to p$ and T is Lipschitzian, then is continuous. Therefore, $\lim Tx_n = Tp = p$ and $\|Tx_n-x_n\|\to 0$ as $n\to\infty$. Also $\lim\|x_{n+1}-x_n\|=0$. Thus, the conditions of Lemma 2 are satisfied and $\lim\|x_n-u_n\|=0$.

Consequently,

$$\|u_n - p\| \le \|u_n - x_n\| + \|x_n - p\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{33}$$

and the Ishikawa method converges. □

Using the argument in [2], it follows that we also have corresponding theorem for Lipschitz strictly hemicontractive operators.

Let $S : X \to X$ be a Lipschitz operator with $L > 1$. It is well known that the operator $S : X \to X$ is strongly accretive if and only if $(I - S)$ is strongly pseudocontractive operator, and conversely. Consider the equation $Sx = f$, where $f \in X$ is given and S is a strongly accretive operator. A fixed point for $Tx = f + (I - S)x$ will be the solution of $Sx = f$, and conversely. If we consider in (1) and (2) the operator $Tx = f + (I - S)x$, then T will be strongly pseudocontractive. Theorem 4 assures the equivalence between the convergencies of Mann and Ishikawa iteration. We consider equation $x + Sx = f$, with S an accretive operator, that is, $(I - S)$ is a pseudocontractive operator. A solution for $x + Sx = f$ is a fixed point for $Tx = f - Sx$, which is a strongly pseudocontractive operator. Replacing (1) and (2), we obtain the equivalence between the convergencies of Mann and Ishikawa iteration for an accretive operator. The solutions existences in the above two equations hold as in [2].

All our results hold for multivalued operators provided that this admit appropriate single-valued selections.

It has been shown in [1] that there exists a Lipschitzian pseudocontractive map defined on a compact subset of $\mathbb{R}^2$ for which an Ishikawa method, with $\alpha_n \le \beta_n$, converges to a fixed point, but for which no Mann iterative method converges. Therefore, it is not possible to extend Theorem 4 to pseudocontractive maps.

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