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THE ROOTS OF THE THIRD JACKSON q-BESSEL FUNCTION

L. D. ABREU, J. BUSTOZ, and J. L. CARDOSO

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We derive analytic bounds for the zeros of the third Jackson q-Bessel function $J_\nu^{(3)}(z;q)$.

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1. Introduction. There are three known q-analogs of classical Bessel functions [6, 5] that are due to Jackson [7]. Following the notation of Ismail [6, 5], these are designated by $J_\nu^{(k)}(z;q)$, $k = 1,2,3$.

The parameter q is taken to satisfy $0 < q < 1$. The third Jackson q-Bessel function $J_\nu^{(3)}(z;q)$ is defined as

$$J_\nu^{(3)}(z;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} z^\nu {}_1\Phi_1\left[\begin{matrix} 0 \\ q^{\nu+1} \end{matrix} ;q,qz^2\right]. \tag{1.1}$$

This function is also known as the Hahn-Exton q-Bessel function [8, 9]. The notation ${}_1\Phi_1$ in (1.1) is the standard in use for q-hypergeometric series [4]. The function $J_\nu^{(3)}(z;q)$ satisfies a linear q-difference equation and it is known that $J_\nu^{(3)}(z;q)$ has an infinite number of simple real zeros [8]. In this paper, we will give lower and upper bounds for these zeros. The roots of these functions are of interest for several reasons. Firstly, it is intrinsically interesting to provide information about the roots of a function such as (1.1), which is an entire function of order zero. Also, the roots of $J_\nu^{(2)}(z;q)$ and $J_\nu^{(3)}(z;q)$ figure prominently in expansions in terms of "q-Fourier series" [2, 3]. Lastly, if we denote the roots of $J_\nu^{(3)}(z;q)$ by $j_{n,\nu}^{(3)}$, then the mass points of the orthogonality measure for a q-analog of Lommel polynomials are located at the points $1/j_{n,\nu}^{(3)}$. Furthermore, although the function defined in (1.1) is of a simpler character than the remaining Jackson q-Bessel functions, it is hoped that the results given here for $J_\nu^{(3)}(z;q)$ may be extended in the future to $J_\nu^{(k)}(z;q)$, $k = 1,2$.

2. The roots of $J_\nu^{(3)}(z;q)$. We prove two lemmas stating the existence of an odd number of roots in a certain interval and then we prove that $J_\nu^{(3)}(z;q)$ has only one root in such an interval.

First we apply the following transformation to (1.1):

$$(c;q)_\infty {}_1\Phi_1\left[\begin{matrix}0\\c\end{matrix};q,z\right]=(z;q)_\infty {}_1\Phi_1\left[\begin{matrix}0\\z\end{matrix};q,c\right]. \tag{2.1}$$

This produces

$$J_\nu^{(3)}(z;q)=\frac{(qz^2;q)_\infty}{(q;q)_\infty}z^\nu {}_1\Phi_1\left[\begin{matrix}0\\qz^2\end{matrix};q,q^{\nu+1}\right]. \tag{2.2}$$

This last relation in a series form gives

$$\begin{aligned}J_\nu^{(3)}(z;q)&=\frac{z^\nu}{(q;q)_\infty}\sum_{k=0}^{\infty}(-1)^k\frac{(qz^2;q)_\infty}{(qz^2;q)_k(q;q)_k}q^{k(k+2\nu+1)/2}\\&=\frac{z^\nu}{(q;q)_\infty}\sum_{k=0}^{\infty}(-1)^k\frac{(q^{k+1}z^2;q)_\infty}{(q;q)_k}q^{k(k+2\nu+1)/2}.\end{aligned} \tag{2.3}$$

This representation will be critical in the proof of the next two lemmas.

Lemma 2.1. *If* $q^{\nu+1}<(1-q)^2$, *then* $\operatorname{sgn}[J_\nu^{(3)}(q^{-m/2};q)]=(-1)^m$, $m=1,2,\ldots$.

Proof. Set $z=q^{-m/2}$ in (2.3) to obtain

$$\frac{(q;q)_\infty}{q^{-m\nu/2}}J_\nu^{(3)}(q^{-m/2};q)=\sum_{k=0}^{\infty}(-1)^k\frac{(q^{-m+k+1};q)_\infty}{(q;q)_k}q^{k(k+2\nu+1)/2}. \tag{2.4}$$

Now observing that $(q^{-m+k+1};q)_\infty=0$ if $k<m$, the series on the right-hand side of this last equality can be written as

$$\sum_{k=m}^{\infty}(-1)^k\frac{(q^{-m+k+1};q)_\infty}{(q;q)_k}q^{k(k+2\nu+1)/2}. \tag{2.5}$$

Setting $j=k-m$ in this last series yields

$$\frac{(q;q)_m}{q^{-m\nu/2}}J_\nu^{(3)}(q^{-m/2};q)=(-1)^m\sum_{j=0}^{\infty}(-1)^jA_j, \tag{2.6}$$

where

$$A_j=\frac{(q^{j+1};q)_\infty}{(q;q)_{j+m}}q^{(j+m)(j+m+2\nu+1)/2}. \tag{2.7}$$

Now we prove that $A_{j+1}<A_j$. A calculation shows that $A_{j+1}<A_j$ is equivalent to $q^{m+j+\nu+1}<(1-q^{m+j+1})(1-q^{j+1})$. But the left-hand side of this inequality is decreasing in m and j, while the right-hand side is increasing in m and j. So we only need to verify the case $j=m=0$, that is, $q^{\nu+1}<(1-q)^2$, but this is the hypothesis of the lemma. Clearly, since $A_{j+1} < A_j$, then

$$\operatorname{sgn}(-1)^m \sum_{j=0}^{\infty} (-1)^j A_j = (-1)^m. \tag{2.8}$$

□

Lemma 2.1 states that there exist an odd number of roots in the interval $(q^{-m/2+1/2}, q^{-m/2})$. The next lemma refines this statement.

Lemma 2.2. *Let $q^{\nu+1} < (1-q)^2$ and define*

$$\alpha_m^{(\nu)}(q) = \frac{\log\left(1 - q^{m+\nu}/(1-q^m)\right)}{\log q}. \tag{2.9}$$

Then

$$\operatorname{sgn}\left[J_\nu^{(3)}\left(q^{-m/2+\alpha_m^{(\nu)}(q)/2}; q\right)\right] = (-1)^{m-1}, \quad m = 1, 2, \ldots. \tag{2.10}$$

Proof. First observe that the function $\alpha_m^{(\nu)}(q)$ is well defined because if $q^{\nu+1} < (1-q)^2$, then $q^{\nu+1} < (1-q)$ and so, for positive integer m, $q^{m+\nu} < (1-q^m)$, that is, $1 - q^{m+\nu}/(1-q^m) > 0$. Being defined, it is clear that $\alpha_m^{(\nu)}(q)$ is positive because $1 - q^{m+\nu}/(1-q^m) < 1$ holds for any $q \in (0,1)$.

Observe also that

$$\alpha_m^{(\nu)}(q) < 1 \Longleftrightarrow \log\left(1 - \frac{q^{m+\nu}}{1-q^m}\right) > \log q \Longleftrightarrow q^{m+\nu} < (1-q)(1-q^m), \tag{2.11}$$

which is true if $q^{\nu+1} < (1-q)^2$. So, we have $0 < \alpha_m^{(\nu)}(q) < 1$.

Now, set $z = q^{-m/2+\alpha_m^{(\nu)}(q)/2}$ in (2.3). The substitution gives

$$\begin{aligned}
&\frac{(q;q)_\infty}{q^{-m\nu/2+\nu\alpha_m^{(\nu)}(q)/2}} J_\nu^{(3)}\left(q^{-m/2+\alpha_m^{(\nu)}(q)/2}; q\right) \\
&\quad = \sum_{k=0}^{m-2} (-1)^k \frac{\left(q^{-m+\alpha_m^{(\nu)}(q)+k+1}; q\right)_\infty}{(q;q)_k} q^{k(k+2\nu+1)/2} \\
&\qquad + \sum_{k=m-1}^{\infty} (-1)^k \frac{\left(q^{-m+\alpha_m^{(\nu)}(q)+k+1}; q\right)_\infty}{(q;q)_k} q^{k(k+2\nu+1)/2}.
\end{aligned} \tag{2.12}$$

Denote the first sum above by S_1 and the second by S_2. If $0 \le k \le m-2$, then $0 < \alpha_m^{(\nu)}(q) < 1$ implies that $\operatorname{sgn}(q^{-m+\alpha_m^{(\nu)}(q)+k+1}; q)_\infty = (-1)^{m-k-1}$. Thus $\operatorname{sgn} S_1 = (-1)^{m-1}$, $m = 1, 2, \ldots$.

In S_2, set $j = k - m + 1$ to obtain

$$S_2 = (-1)^{m-1} \sum_{j=0}^{\infty} (-1)^j A_j, \tag{2.13}$$

where

$$A_j = \frac{\left(q^{\alpha_m^{(\nu)}(q)+j};q\right)_\infty}{(q;q)_{j+m-1}} q^{(j+m-1)(j+m+2\nu)/2}. \tag{2.14}$$

A calculation shows that $A_{j+1} < A_j$ is reduced to $q^{j+m+\nu} < (1-q^{\alpha_m^{(\nu)}(q)+j})(1-q^{m+j})$, which holds because $q^{m+\nu} = (1-q^{\alpha_m^{(\nu)}(q)})(1-q^m)$ and because the left-hand side of the last inequality is decreasing in j and the right-hand side is increasing in j. The infinite series is thus positive and therefore

$$\operatorname{sgn} S_2 = (-1)^{m-1} = \operatorname{sgn} S_1. \tag{2.15}$$

□

From Lemmas 2.1 and 2.2, we know that $J_\nu^{(3)}(z;q)$ has an odd number of roots in the interval $(q^{-m/2+\alpha_m^{(\nu)}(q)}, q^{-m/2})$. The next theorem proves that there is exactly one root in each such interval and that there are no other roots.

Theorem 2.3. *If $q^{\nu+1} < (1-q)^2$ and if $w_k^{(\nu)}(q)$ are the positive roots of $J_\nu^{(3)}(z;q)$, ordered increasingly in k, then $w_k^{(\nu)}(q) = q^{-k/2+\epsilon_k(\nu)}$, with $0 < \epsilon_k(\nu) < \alpha_k^{(\nu)}(q)$, $k = 1,2,\ldots$.*

Proof. From the preceding lemmas, we know that $J_\nu^{(3)}(z;q)$ has roots of the form $w_k^{(\nu)} = q^{-k/2+\epsilon_k}$, with $0 < \epsilon_k < \alpha_k^{(\nu)}(q)$.

To simplify the notation, we set

$$F(z) = \frac{(q;q)_\infty}{(q^{\nu+1};q)_\infty z^\nu} J_\nu^{(3)}(z;q). \tag{2.16}$$

We prove that the only positive roots of $F(z)$ inside the disk $|z| < q^{-m/2}$ are $w_k^{(\nu)}(q)$, $k = 1,2,\ldots,m$.

Suppose there are other roots $\pm\lambda_k$, $k = 1,2,\ldots,P_m$; $\lambda_k > 0$. By Jensen's theorem [1], we can write

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} &\log\left|F(q^{-m/2}e^{i\theta})\right| d\theta \\
&= 2\sum_{k=1}^{m} \log\frac{q^{-m/2}}{w_k} + 2\sum_{k=1}^{P_m} \log\frac{q^{-m/2}}{\lambda_k} \\
&= 2\sum_{k=1}^{m} \log q^{-m/2+k/2-\epsilon_k} + 2\sum_{k=1}^{P_m} \log\frac{q^{-m/2}}{\lambda_k} \\
&= \frac{-m^2+m}{2}\log q - 2\log q \sum_{k=1}^{m}\epsilon_k + 2\sum_{k=1}^{P_m}\log\frac{q^{-m/2}}{\lambda_k}.
\end{aligned} \tag{2.17}$$

On the other hand, by the definition of the q-Bessel function, we have

$$\begin{aligned} F(q^{-m/2}e^{i\theta}) &= \sum_{k=0}^{\infty}(-1)^k \frac{q^{k(k+1)/2-mk}}{(q^{\nu+1};q)_k(q;q)_k}e^{2ik\theta} \\ &= (-1)^m \frac{q^{(-m^2+m)/2}}{(q^{\nu+1};q)_m(q;q)_m}e^{2im\theta} \\ &\quad \times \sum_{k=-m}^{\infty}(-1)^k \frac{q^{k(k+1)/2}}{(q^{m+\nu+1};q)_k(q^{m+1};q)_k}e^{2ik\theta}. \end{aligned} \tag{2.18}$$

Then we have

$$\begin{aligned} \log\left|F(q^{-m/2}e^{i\theta})\right| &= \frac{-m^2+m}{2}\log q - \log(q^{\nu+1};q)_m - \log(q;q)_m \\ &\quad + \log\left|\sum_{k=-m}^{\infty}(-1)^k \frac{q^{k(k+1)/2}}{(q^{m+\nu+1};q)_k(q^{m+1};q)_k}e^{2ik\theta}\right| \end{aligned} \tag{2.19}$$

so that

$$\begin{aligned} &\lim_{m\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\log\left|F(q^{-m/2}e^{i\theta})\right|d\theta \\ &\quad = \lim_{m\to\infty}\frac{-m^2+m}{2}\log q - \log(q^{\nu+1};q)_\infty - \log(q;q)_\infty \\ &\qquad + \lim_{m\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\log\left|\sum_{k=-m}^{\infty}(-1)^k \frac{q^{k(k+1)/2}}{(q^{m+\nu+1};q)_k(q^{m+1};q)_k}e^{2ik\theta}\right|d\theta. \end{aligned} \tag{2.20}$$

Now observe that

$$\lim_{m\to\infty}\sum_{k=-m}^{\infty}(-1)^k \frac{q^{k(k+1)/2}}{(q^{m+\nu+1};q)_k(q^{m+1};q)_k}e^{2ik\theta} = \sum_{k=-\infty}^{\infty}(-1)^k q^{k(k+1)/2}e^{2ik\theta}. \tag{2.21}$$

The above limit is uniform in θ. By the Jacobi triple product identity [4],

$$\sum_{k=-\infty}^{\infty}\left(-q^{1/2}e^{2i\theta}\right)^k\left(q^{1/2}\right)^{k^2} = (q;qe^{2i\theta};e^{-2i\theta};q)_\infty. \tag{2.22}$$

Using the uniform convergence to interchange the limit and the integral,

$$
\begin{aligned}
&\lim_{m\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\log\left|\sum_{k=-m}^{\infty}(-1)^k\frac{q^{k(k+1)/2}}{(q^{m+\nu+1};q)_k(q^{m+1};q)_k}e^{2ik\theta}\right|d\theta\\
&\quad=\frac{1}{2\pi}\int_0^{2\pi}\log\left|(q;qe^{2i\theta};e^{-2i\theta};q)_\infty\right|d\theta\\
&\quad=\log(q;q)_\infty+\frac{1}{2\pi}\int_0^{2\pi}\log\left|(qe^{2i\theta};q)_\infty\right|d\theta\\
&\qquad+\frac{1}{2\pi}\int_0^{2\pi}\log\left|(e^{-2i\theta};q)_\infty\right|d\theta\\
&\quad=\log(q;q)_\infty+\sum_{j=0}^{\infty}\frac{1}{2\pi}\int_0^{2\pi}\log\left|(1-q^{j+1}e^{2i\theta})_\infty\right|d\theta\\
&\qquad+\sum_{j=0}^{\infty}\frac{1}{2\pi}\int_0^{2\pi}\log\left|(1-q^{j}e^{-2i\theta})_\infty\right|d\theta\\
&\quad=\log(q;q)_\infty.
\end{aligned}
\tag{2.23}
$$

The integrals in the third equality above vanish because of the mean value theorem for harmonic functions.

We have thus concluded that

$$
\begin{aligned}
&\lim_{m\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\log\left|F(q^{-m/2}e^{i\theta})\right|d\theta\\
&\quad=\lim_{m\to\infty}\frac{-m^2+m}{2}\log q-\log(q^{\nu+1};q)_\infty.
\end{aligned}
\tag{2.24}
$$

From (2.17), we can write

$$
\lim_{m\to\infty}\left(2\sum_{k=1}^{P_m}\log\frac{q^{-m/2}}{\lambda_k}\right)-\lim_{m\to\infty}\left(2\log q\sum_{k=1}^{m}\epsilon_k\right)=-\log(q^{\nu+1};q)_\infty. \tag{2.25}
$$

But, as can be seen by the Taylor expansion of $\alpha_k^{(\nu)}(q)$, $\epsilon_k=O(q^k)$, and this implies $\sum_{k=1}^{\infty}\epsilon_k<\infty$.

Also

$$
\sum_{k=1}^{P_m}\log\frac{q^{-m/2}}{\lambda_k}>\log\frac{q^{-m/2}}{\lambda_1}\longrightarrow\infty\quad\text{as } m\longrightarrow\infty. \tag{2.26}
$$

So the identity (2.25) can only hold if the first sum is empty, and the only roots are thus $\pm\, w_k^{(\nu)}(q)$, $k=1,2,\ldots$. □

Remark 2.4. It follows from (2.25) that $\sum_{k=0}^{\infty}\epsilon_k = \log(q^{\nu+1};q)_\infty/2\log q$.

Remark 2.5. Observe that for fixed j, we can always choose m sufficiently large so that

$$\begin{aligned} q^{m+j+\nu+1} &< (1-q^{m+j+1})(1-q^{j+1}), \\ q^{j+m+\nu} &< (1-q^{\alpha_m^\nu(q)+j})(1-q^{m+j}). \end{aligned} \tag{2.27}$$

Thus, we have the asymptotic behaviour

$$w_k \backsim q^{-m/2} \quad \text{when } m \longrightarrow \infty \tag{2.28}$$

without the restriction $q^{\nu+1} < (1-q)^2$. In [5], Ismail conjectured that

(1) $\lim_{m\to\infty} q^{m/2} w_m^{(\nu)}(q) = 1$,

(2) $\lim_{m\to\infty} w_{m+1}^{(\nu)}(q^2)/w_m^{(\nu)}(q^2) = 1/q$.

The asymptotic relation (2.28) establishes these conjectures.

Remark 2.6. Lemmas 2.1 and 2.2 and Theorem 2.3 state that the roots $w_k^{(\nu)}(q)$ satisfy the inequalities

$$q^{-m/2+\alpha_m^{(\nu)}(q)} < w_k^{(\nu)}(q) < q^{-m/2}. \tag{2.29}$$

These bounds are quite accurate. This is evident if we estimate the length of the interval containing the roots. A somewhat tedious calculation with Taylor series shows that

$$q^{-m/2} - q^{-m/2+\alpha_m^{(\nu)}(q)} = q^{m/2+\nu}O(1). \tag{2.30}$$

Clearly, for fixed q satisfying the conditions of the theorem, the bounds become increasingly accurate as either k or ν increases.

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L. D. Abreu: Department of Mathematics, University of Coimbra, Coimbra 3000, Portugal
E-mail address: daniel@mat.uc.pt

J. Bustoz: Department of Mathematics, Arizona State University, Tempe, AZ 85287-1804, USA
E-mail address: bustoz@asu.edu

J. L. Cardoso: Department of Mathematics, University of Tras-os-Montes and Alto Douro, Vila Real 5000-410, Portugal
E-mail address: jluis@utad.pt