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ON THE DIFFERENCE OF VALUES OF THE KERNEL FUNCTION AT CONSECUTIVE INTEGERS

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For each positive integer n, set $\gamma(n) = \prod_{p|n} p$. Given a fixed integer $k \neq \pm 1$, we establish that if the ABC-conjecture holds, then the equation $\gamma(n+1) - \gamma(n) = k$ has only finitely many solutions. In the particular cases $k = \pm 1$, we provide a large family of solutions for each of the corresponding equations.

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1. Introduction. Comparing values of an arithmetic function at consecutive integers is a common problem in number theory. For example, in 1952, Erdős and Mirsky [4] asked if there are infinitely many integers n such that $d(n) = d(n+1)$ (here $d(n)$ stands for the number of divisors of n), a question which was answered in the affirmative when Heath-Brown [8] proved in 1984 that the number of positive integers $n \leq x$ such that $d(n) = d(n+1)$ is $\gg x/\log^7 x$, a lower bound which was later improved by Hildebrand [9] and thereafter by Erdős et al. [6].

An apparently more difficult problem seems to be that of establishing that the equations $\phi(n) = \phi(n+1)$ (where ϕ is Euler's function) and $\sigma(n) = \sigma(n+1)$ (where $\sigma(n)$ stands for the sum of the divisors of n) each has infinitely many solutions n; see Erdős et al. [5] for developments concerning this problem.

The distribution of the values of the *kernel function* $\gamma(n) := \prod_{p|n} p$ (also called the *core function* or the *algebraic radical* of n) is the source of a variety of open problems, many of them tied in with the famous ABC-conjecture. For instance, in 1999, Cochrane and Dressler [2] showed that, assuming the ABC-conjecture, if two positive integers have the same prime factors, they cannot be too close; more precisely, they proved that if the ABC-conjecture is true, then given any $\varepsilon > 0$, there exists a positive constant $C = C(\varepsilon)$ such that if $\gamma(n) = \gamma(n+k)$, then $k \geq Cn^{1/2-\varepsilon}$. No easier is the conjecture of P. Erdős and A. Woods which asserts that there exists an integer $k \geq 3$ such that if m and n are positive integers satisfying $\gamma(m+i) = \gamma(n+i)$ for all $1 \leq i \leq k$, then $m = n$. Although it remains unsolved, this conjecture has been extensively studied and generalized; see, for instance, Langevin [10], Balasubramanian et al. [1], as well as Langevin [11].

Many more results regarding the kernel function and the ABC-conjecture have been published; for some recent ones, see Mitrinović et al. [12], Granville [7], or Cutter et al. [3].

Perhaps an even more difficult problem seems to be the one of comparing the values of the kernel function at consecutive integers. In this note, we look at the values of the function $\gamma(n+1)-\gamma(n)$ for positive integers n.

2. Preliminary observations and statement of the main results. We first make a few observations. Note that it is always the case that one of the numbers n and $n+1$ is even and the other one is odd. In particular, $\gamma(n+1)-\gamma(n)$ is always an odd number, and therefore the equation $\gamma(n+1)-\gamma(n)=k$ has no solutions when k is a fixed even positive integer. From now on, we assume that k is a fixed odd positive integer. When $k=1$ and both n and $n+1$ are square-free, we certainly have that $\gamma(n+1)-\gamma(n)=(n+1)-n=1=k$. Since, for a large positive real number x, there are only $(1-6/\pi^2+o(1))x$ positive integers $n<x$ for which n is *not* square-free, it follows that there are at most $(2-12/\pi^2+o(1))x$ positive integers $n<x$ such that one of n or $n+1$ is *not* square-free. In particular, the number of numbers $n<x$ for which both n and $n+1$ are square-free is at least

$$x-\left(2-\frac{12}{\pi^2}+o(1)\right)x=\left(\frac{12}{\pi^2}-1+o(1)\right)x>0.215x. \tag{2.1}$$

Thus, the solutions n of the equation $\gamma(n+1)-\gamma(n)=1$ form a subset of all the positive integers of positive lower asymptotic density.

From now on, we look for positive integer solutions n of the equation $\gamma(n+1)-\gamma(n)=k$ such that $n(n+1)$ is not square-free. Here is a "parametric family" of solutions for $k=1$. Let $r>1$ be an integer and assume that both $2^{r-1}-1$ and 2^r-1 are square-free. Put $n=2^{r+1}(2^{r-1}-1)$. Then $n+1=2^{r+1}(2^{r-1}-1)+1=2^{2r}-2^{r+1}+1=(2^r-1)^2$. It is clear that $n+1$ is not square-free, and if $r\geq 3$, then n is not square-free either. Since both $2^{r-1}-1$ and 2^r-1 are square-free, we have $\gamma(n+1)=2^r-1$, $\gamma(n)=2(2^{r-1}-1)=2^r-2$, which implies that $\gamma(n+1)-\gamma(n)=(2^r-1)-(2^r-2)=1$. It is not even known if there are infinitely many r such that 2^r-1 is square-free, and therefore it is not known if there are infinitely many solutions n of the above form to the equation $\gamma(n+1)-\gamma(n)=1$. However, computations revealed that there are 106 values of the positive integer $r<200$ having the property that both $2^{r-1}-1$ and 2^r-1 are square-free; all these values are listed in Section 5. A similar type of "parametric solution" can be found when $k=-1$. In this case, if $r>1$ is such that both $2^{r-1}+1$ and 2^r+1 are square-free, then taking $n=2^{r+1}(2^{r-1}+1)$, we have $n+1=(2^r+1)^2$, in which case $\gamma(n+1)-\gamma(n)=(2^r+1)-2(2^{r-1}+1)=-1$. The list of those $r<200$ such that both $2^{r-1}+1$ and 2^r+1 are square-free is also included in Section 5.

We conjecture that for every fixed value of $k \neq \pm 1$, the equation $\gamma(n+1) - \gamma(n) = k$ has only finitely many positive integer solutions n; the solutions $n < 10^9$ of this equation, when $1 < |k| < 100$, are given in Section 4. We also conjecture that when $k = \pm 1$, the equation $\gamma(n+1) - \gamma(n) = k$ has only finitely many positive integer solutions n, which are not of one of the forms specified above.

In this note, we prove that our conjectures are implied by the ABC-conjecture. In fact, assuming the ABC-conjecture, we prove a much stronger statement which implies the above conjectures.

We first recall that the ABC-conjecture is the following statement.

The ABC-conjecture. For every $\varepsilon > 0$, there exists a constant $K := K(\varepsilon)$ such that whenever A, B, C are three coprime nonzero integers with $A + B = C$, then

$$\max\{|A|, |B|, |C|\} < K\gamma(ABC)^{1+\varepsilon}. \tag{2.2}$$

We will choose to write the above inequality as

$$\max\{|A|, |B|, |C|\} \ll_{\varepsilon} \gamma(ABC)^{1+\varepsilon}. \tag{2.3}$$

Theorem 2.1. (i) *Let $\varepsilon > 0$ be given. Then the ABC-conjecture implies that the inequality*

$$|\gamma(n+1) - \gamma(n)| \gg_{\varepsilon} n^{1/10-\varepsilon} \tag{2.4}$$

holds whenever $|\gamma(n+1) - \gamma(n)| > 1$. In particular, if $k > 1$ is any fixed positive integer, then the equation $|\gamma(n+1) - \gamma(n)| = k$ has only finitely positive integer solutions n.

(ii) *The ABC-conjecture implies that there are only finitely many positive solutions n to the equation $\gamma(n+1) - \gamma(n) = 1$ such that $n(n+1)$ is not square-free, and such that n is not of the form $n = 2^{r+1}(2^{r-1} - 1)$ with some integer $r > 1$ such that both $2^{r-1} - 1$ and $2^r - 1$ are square-free.*

(iii) *The ABC-conjecture implies that there are only finitely many positive solutions n to the equation $\gamma(n+1) - \gamma(n) = -1$ for which n is not of the form $n = 2^{r+1}(2^{r-1} + 1)$ with some integer $r > 1$ such that both $2^{r-1} + 1$ and $2^r + 1$ are square-free.*

Remark 2.2. The above result implies that the only cluster points of the sequence $\{|\gamma(n+1) - \gamma(n)|\}_{n \geq 1}$ are 1 and infinity, and that

$$\liminf_{\substack{n \to \infty \\ n \in \mathcal{A}}} \frac{\log|\gamma(n+1) - \gamma(n)|}{\log n} \geq \frac{1}{10}, \tag{2.5}$$

where $\mathscr{A}$ is the set of all positive integers n such that $n(n+1)$ is not square-free, and n is not of the form $2^{r+1}(2^{r-1}\pm 1)$.

The following result is a more general version of Theorem 2.1.

Theorem 2.3. (i) *Let $\varepsilon > 0$ be given. Then the ABC-conjecture implies that the inequality*

$$\left|\gamma(m)-\gamma(n)\right| \gg_{\varepsilon} |m-n|^{1/15-\varepsilon} \tag{2.6}$$

holds for all coprime positive integers m and n.

(ii) *Let $\varepsilon > 0$ be given and let $j > 1$ be a fixed integer. Then the ABC-conjecture implies that the inequality*

$$\left|\gamma(n+j)-\gamma(n)\right| \gg_{j,\varepsilon} n^{1/10-\varepsilon} \tag{2.7}$$

holds for all positive integers n coprime to j such that $|\gamma(n+j)-\gamma(n)| > j$.

(iii) *Let $j > 1$ be a fixed integer. Then the ABC-conjecture implies that all but finitely many solutions in positive integers n coprime to j of the inequality*

$$\left|\gamma(n+j)-\gamma(n)\right| \le j \tag{2.8}$$

have the property that $n(n+j)$ is square-free, unless $j = j_0^2$ is a perfect square, in which case all the other solutions of inequality (2.8) are of the form $n = 2^r(2^r+2\eta j_0)$, for some $\eta \in \{\pm 1\}$ and some nonnegative integer r such that both $2^r+2\eta j_0$ and $2^r+\eta j_0$ are square-free.

3. The proof of Theorem 2.1. We let $\varepsilon > 0$ be some small number. Now let k be an odd integer and n a positive integer such that $\gamma(n+1)-\gamma(n) = k$. Furthermore, let a and b be the two square-free integers given by $a := \gamma(n+1)$ and $b := \gamma(n)$. Assume first that $\max\{a,b\} \le 2|k|$. In this case, the *ABC*-conjecture applied to the equation

$$(n+1)-n = 1 \tag{3.1}$$

yields

$$n \le n+1 \ll_{\varepsilon} \left(\gamma(n)\gamma(n+1)\right)^{1+\varepsilon} \ll_{\varepsilon} (ab)^{1+\varepsilon} \ll_{\varepsilon} \left(2|k|\right)^{2(1+\varepsilon)} \ll_{\varepsilon} |k|^{2(1+\varepsilon)}, \tag{3.2}$$

leading to

$$|k| \gg_{\varepsilon} n^{1/2(1+\varepsilon)} \tag{3.3}$$

which is an even better inequality than inequality (2.4). We note that when k is fixed, then the fact that the equation $\gamma(n+1)-\gamma(n)=k$ has only finitely many positive integer solutions n satisfying $\max\{a,b\}\le 2|k|$ can be proved unconditionally as follows. Let $\mathcal{P}$ be the set of all prime numbers $p\le 2|k|$ and let $\mathcal{S}$ be the set of all positive integers whose prime factors belong to $\mathcal{P}$. In this case, both n and $n+1$ belong to $\mathcal{S}$, and therefore the pair $(x,y):=(n+1,n)$ is a solution of the equation $x-y=1$, with $x,y\in\mathcal{S}$, and it is known that such a diophantine equation has only finitely many solutions (x,y) which are effectively computable.

Thus, we may assume that $\max\{a,b\}>2|k|$. In this case, since $a-b=k$, it follows that $\max\{a,b\}<2\min\{a,b\}$. In particular, both inequalities $a<2b$ and $b<2a$ hold. Further, let $c:=(n+1)/\gamma(n+1)$ and $d:=n/\gamma(n)$. We may assume that $\max\{c,d\}>1$, for otherwise both n and $n+1$ are square-free, and this implies that $k=1$. We now have the system of equations

$$a-b=k,\qquad ca-db=1. \tag{3.4}$$

Applying the ABC-conjecture to the second equation of (3.4), we get

$$ca\ll_\varepsilon \gamma(abcd)^{1+\varepsilon}=(ab)^{1+\varepsilon}\ll_\varepsilon \left(2a^2\right)^{1+\varepsilon}\ll_\varepsilon a^{2+2\varepsilon}. \tag{3.5}$$

Inequality (3.5) implies that

$$c\ll_\varepsilon a^{1+2\varepsilon}\ll_\varepsilon (2b)^{1+2\varepsilon}\ll_\varepsilon b^{1+2\varepsilon}. \tag{3.6}$$

A similar argument shows that the inequality

$$d\ll_\varepsilon \left(\min\{a,b\}\right)^{1+2\varepsilon} \tag{3.7}$$

holds. We now multiply both sides of the first equation of (3.4) by c and subtract the second equation of (3.4) to get

$$kc-1=db-cb=b(d-c). \tag{3.8}$$

Note that d and c are coprimes and that at least one of them is larger than 1. Hence, $d-c\ne 0$. Thus, in view of (3.6) and (3.8),

$$|d-c|b=|kc-1|\le 2|k|c\ll_\varepsilon |k|b^{1+2\varepsilon} \tag{3.9}$$

so that

$$|d-c|\ll_\varepsilon |k|b^{2\varepsilon}. \tag{3.10}$$

In particular, since $b < 2a$, it follows that

$$|d-c| \ll |k|(\min\{a,b\})^{2\varepsilon}. \tag{3.11}$$

We now write $d-c=e$. Then, since $b=a-k$ and $d=c+e$, we have

$$1 = ca-db = ca-(c+e)(a-k) = ca-(ca+ea-kc-ke) = kc-ea+ke, \tag{3.12}$$

in which case

$$kc-ea = 1-ke. \tag{3.13}$$

Assume first that $1-ke \neq 0$. In this case, since $\gamma(c)|a$, we read from (3.13) that $\gamma(c)$ divides $|1-ke|$. In particular, it follows from (3.11) and (3.13) that

$$\gamma(c) \leq |1-ke| \leq 2|k||e| \ll_\varepsilon |k|^2 a^{2\varepsilon}. \tag{3.14}$$

Therefore,

$$1 = ca-db = (d-e)(b+k)-db = (db-eb+kd-ek)-db = -eb+kd-ek, \tag{3.15}$$

and hence,

$$kd-eb = 1+ek. \tag{3.16}$$

Assume now that $1+ek \neq 0$. In this case, since $\gamma(d)|b$, we get that $\gamma(d)$ divides $|1+ek|$ and therefore

$$\gamma(d) \leq |1+ek| \leq 2|k||e| \ll_\varepsilon |k|^2 a^{2\varepsilon}. \tag{3.17}$$

Applying the ABC-conjecture to the equation

$$d-c=e, \tag{3.18}$$

we get, using (3.11), (3.14), and (3.17), that

$$\begin{aligned} \max\{d,c\} &\ll_\varepsilon \gamma(dc|e|)^{1+\varepsilon} \ll_\varepsilon (\gamma(d)\gamma(c)|e|)^{1+\varepsilon} \\ &\ll_\varepsilon (|k|^4 a^{4\varepsilon}|e|)^{1+\varepsilon} \ll |k|^{4(1+\varepsilon)} a^{5\varepsilon}|e|^{1+\varepsilon}, \end{aligned} \tag{3.19}$$

provided that $\varepsilon < 1/4$. On the one hand, inequality (3.19) combined with (3.11) gives

$$\max\{d,c\} \ll_\varepsilon |k|^{4(1+\varepsilon)} a^{5\varepsilon}|e|^{1+\varepsilon} \ll_\varepsilon |k|^{5(1+\varepsilon)} a^{\varepsilon(5+2(1+\varepsilon))} \ll_\varepsilon |k|^{5(1+\varepsilon)} a^{8\varepsilon}, \tag{3.20}$$

provided that $\varepsilon < 1/2$, while on the other hand, returning to (3.8), it follows from (3.19) and (3.11) that

$$b|e| = b|d-c| = |kc-1| \le 2|k|c \ll_\varepsilon |k|^{5+4\varepsilon} a^{5\varepsilon}|e|^{1+\varepsilon}, \tag{3.21}$$

and therefore

$$b \ll_\varepsilon |k|^{5+4\varepsilon} a^{5\varepsilon}|e|^{\varepsilon} \ll_\varepsilon |k|^{5(1+\varepsilon)} a^{7\varepsilon}. \tag{3.22}$$

Since $a \le 2b$, it follows from (3.22) that

$$b \ll_\varepsilon |k|^{5(1+\varepsilon)/(1-7\varepsilon)} \ll_\varepsilon |k|^{5(1+10\varepsilon)}, \tag{3.23}$$

provided that $\varepsilon < 1/35$. Substituting (3.23) into (3.20) and using again the fact that $a \le 2b$, we get

$$\max\{d,c\} \ll_\varepsilon |k|^{5((1+\varepsilon)+8\varepsilon(1+10\varepsilon))} \ll_\varepsilon |k|^{5(1+10\varepsilon)}, \tag{3.24}$$

provided that $\varepsilon < 1/80$. From (3.23) and (3.24), we immediately get that

$$n = bd \ll_\varepsilon |k|^{10(1+10\varepsilon)}, \tag{3.25}$$

leading to

$$|k| \gg_\varepsilon n^{1/10(1+10\varepsilon)} \gg_\varepsilon n^{1/10-\varepsilon}, \tag{3.26}$$

which is precisely inequality (2.4).

Our reasoning was based on the fact that we assumed, aside from the ABC-conjecture, that $1-ke \neq 0$ and $1+ke \neq 0$. Hence, we now assume that $(1-ke)(1+ke) = 0$. Note that this is possible only when $|k| = 1$, which, together with the previous arguments, justifies Theorem 2.1(i). Now assume that $1-ke = 0$. In this case, $ke = 1$ and therefore $1+ke = 2$. Equation (3.16) now tells us that $\gamma(d)|2$ and therefore $d = 2^r$ for some integer $r \ge 0$. Since $ke = 1$, we either have $k = e = 1$ or $k = e = -1$. When $k = e = 1$, we have $d-c = 1$, in which case $c = d-1 = 2^r-1$ and $a-b = 1$. From (3.13), we also have $c-ea = 0$ so that $a = ea = c = 2^r-1$, and $b = a-1 = 2^r-2 = 2(2^{r-1}-1)$. The condition that a and b are positive and square-free forces $r > 1$ and both 2^r-1 and $2^{r-1}-1$ to be square-free. Hence $n+1 = ac = (2^r-1)^2$, while $n = (2^r-1)^2-1 = 2^{r+1}(2^{r-1}-1)$, which is exactly the parametric family mentioned in Section 2. Now assume that $k = e = -1$. In this case, we have $d-c = -1$, which implies that $c = d+1 = 2^r+1$, and $a-b = -1$. From (3.13), we also have $c = a$ so that $a = 2^r+1$, and $b = a+1 = 2^r+2 = 2(2^{r-1}+1)$. The condition that a and b are square-free forces $r > 1$ and $2^{r-1}+1$ and 2^r+1 to be square-free. Thus $n+1 = (2^r+1)^2$ and $n = (2^r+1)^2-1 = 2^{r+1}(2^{r-1}+1)$. Now assume that $1+ke = 0$, in which case (3.16) shows that $kd = eb$. Since $ke = -1$, we get that $k = -e \in \{\pm 1\}$, and therefore $d = -b$. This is impossible because both d and b are positive. The proof of Theorem 2.1 is thus complete.

4. The proof of Theorem 2.3. The proof of this result can be achieved by following the same procedure as in the proof of Theorem 2.1, and we will only sketch it. Let $\varepsilon > 0$ be a very small number. Put $j := m-n$, $k := \gamma(m)-\gamma(n)$, and $K := \max\{j, |k|\}$. We may assume that $j > |k|$, for otherwise we already have that $|\gamma(m)-\gamma(n)| = |k| \geq j = |m-n|$, which implies inequality (2.6).

We write $a := \gamma(m)$ and $b := \gamma(n)$. If $\max\{a,b\} \leq 2K$, then the ABC-conjecture applied to the equation $m-n = j$ shows that

$$j \ll_\varepsilon (abK)^{1+\varepsilon} \ll_\varepsilon K^{3(1+\varepsilon)}, \tag{4.1}$$

which gives

$$K \gg_\varepsilon j^{1/3(1+\varepsilon)}, \tag{4.2}$$

which is a better inequality than the one asserted at (2.6). Thus, we may assume that $\max\{a,b\} > 2K$. As in the proof of Theorem 2.1, we set $c := m/\gamma(m)$ and $d := n/\gamma(n)$ and we have the system of equations

$$a-b = k, \qquad ca-db = j. \tag{4.3}$$

Applying the ABC-conjecture to the second equation of (4.3), we get

$$ca \ll_\varepsilon \left(\gamma(abcd)j\right)^{1+\varepsilon} \ll_\varepsilon K^{1+\varepsilon}a^{2+2\varepsilon}, \tag{4.4}$$

which, together with the fact that $a \leq 2b$, leads easily to the fact that

$$c \ll_\varepsilon K^{1+\varepsilon}\left(\min\{a,b\}\right)^{1+2\varepsilon}. \tag{4.5}$$

In the same way, one shows that

$$d \ll_\varepsilon K^{1+\varepsilon}\left(\min\{a,b\}\right)^{1+2\varepsilon}. \tag{4.6}$$

We now multiply both sides of the first equation of (4.3) by c and subtract the second equation of (4.3) to get

$$kc-j = b(d-c). \tag{4.7}$$

Note that d and c are coprimes, thus $d-c = 0$ only when $d = c = 1$. This, in turn, is possible only when both m and n are square-free, therefore $|k| = j$, which is a contradiction. Hence, $d-c \neq 0$. Thus, in view of (4.5) and of (4.7),

$$b|d-c| = |kc-j| \leq Kc \ll_\varepsilon K^{2+\varepsilon} b^{1+2\varepsilon} \tag{4.8}$$

so that

$$|d-c| \ll_\varepsilon K^{2+\varepsilon} b^{2\varepsilon}, \tag{4.9}$$

and since $b \leq 2a$, we get that

$$|d-c| \ll_\varepsilon K^{2+\varepsilon} (\min\{a,b\})^{2\varepsilon}. \tag{4.10}$$

As before, we let $e = d-c$ and using the fact that $b = a-k$ and $d = c+e$, we rewrite the second equation of (4.3) as

$$kc - ea = j - ke. \tag{4.11}$$

Assume that $j - ke \neq 0$. Since $\gamma(c)|a$, we then get from (4.11) and (4.10) that

$$\gamma(c) \leq |j-ke| \ll K|e| \ll_\varepsilon K^{3+\varepsilon} (\min\{a,b\})^{2\varepsilon}. \tag{4.12}$$

An inequality similar to (4.12) holds with c replaced by d provided that $j+ke \neq 0$, and now the ABC-conjecture applied to the equation $d-c=e$ gives

$$\begin{aligned} \max\{c,d\} &\ll_\varepsilon (\gamma(c)\gamma(d)|e|)^{1+\varepsilon} \ll_\varepsilon K^{(6+2\varepsilon)(1+\varepsilon)} (\min\{a,b\})^{4\varepsilon(1+\varepsilon)} |e|^{1+\varepsilon} \\ &\ll_\varepsilon K^{6(1+2\varepsilon)} (\min\{a,b\})^{5\varepsilon} |e|^{1+\varepsilon}, \end{aligned} \tag{4.13}$$

provided that $\varepsilon < 1/4$. On the one hand, inequality (4.13), together with (4.10), gives

$$\begin{aligned} \max\{c,d\} &\ll_\varepsilon K^{6(1+2\varepsilon)+(2+\varepsilon)(1+\varepsilon)} (\min\{a,b\})^{5\varepsilon+2\varepsilon(1+\varepsilon)} \\ &\ll_\varepsilon K^{8(1+2\varepsilon)} (\min\{a,b\})^{8\varepsilon}, \end{aligned} \tag{4.14}$$

provided that $\varepsilon < 1/2$, while, on the other hand, inequality (4.13) and (4.7) show that

$$b|e| = |kc-j| \leq Kc \ll_\varepsilon K^{7+12\varepsilon} (\min\{a,b\})^{5\varepsilon} |e|^{1+\varepsilon} \tag{4.15}$$

and therefore

$$\begin{aligned} b &\ll_\varepsilon K^{7+12\varepsilon} (\min\{a,b\})^{5\varepsilon} |e|^{\varepsilon} \\ &\ll_\varepsilon K^{7+12\varepsilon+\varepsilon(2+\varepsilon)} (\min\{a,b\})^{5\varepsilon+2\varepsilon^2} \ll_\varepsilon K^{7+15\varepsilon} (\min\{a,b\})^{7\varepsilon}. \end{aligned} \tag{4.16}$$

Multiplying (4.14) and (4.16) and using the fact that $a \leq m$, we get

$$m = bd \ll_\varepsilon K^{15+31\varepsilon} m^{15\varepsilon}, \tag{4.17}$$

leading to the conclusion that

$$K \gg_{\varepsilon} m^{(1-15\varepsilon)/(15+31\varepsilon)} \gg_{\varepsilon} m^{1/15-2\varepsilon} \gg_{\varepsilon} j^{1/15-2\varepsilon}, \tag{4.18}$$

which implies inequality (2.6) in light of the fact that ε can be taken arbitrarily small.

It remains to consider the degenerate cases in which $j \pm ke = 0$. Assume first that $j - ke = 0$. In this case, we have that $k|j$. Put $j = kj_0$. Then $e = j_0$. Note that k and j_0 have the same sign. Equation (4.11) then shows that $kc = ea = j_0 a$. Write $D := \gcd(k, j_0) > 0$ and write $k = Dk_1$, $j_0 = Dj_1$. Note that k_1 and j_1 have the same sign. We then get that $c = j_1\rho$ and $a = k_1\rho$, and since $\gamma(c)|a$ and a is square-free, we get that $j_1|\rho$, and therefore $\rho = j_1\rho_1$, which implies that $c = j_1^2\rho_1$ and $a = k_1 j_1 \rho_1$. The analogue of relation (3.16) is now $kd - eb = j + ek = 2j$, which can be rewritten as $kd - j_0 b = 2kj_0$. Simplifying D, we get $k_1 d - j_1 b = 2k_1 j_0$. Reducing this modulo k_1, we get that k_1 divides $j_1 b$, and since k_1 is coprime to j_1, we get that k_1 divides b. But k_1 also divides a, therefore k_1 divides both m and n, which shows that $k_1 = \pm 1$. Since j_0 is a multiple of j_1, we may reduce the above equation modulo j_1 and read that j_1 divides $k_1 d$, which implies that j_1 divides d. Since it also divides a, it follows that $j_1 = \pm 1$. Thus, we have showed that $k = \eta j_0$, $j = j_0^2$, $a = c$, and $d = b + 2\eta j_0$, where $\eta \in \{\pm 1\}$. The relation $a - b = k$ gives $a = b + \eta j_0$, therefore $m = ac = (b + \eta j_0)^2$ and $n = bd = b(b + 2\eta j_0)$. It is clear that $m - n = j$, and the only restriction now is that both b and $b + \eta j_0 = a$ are square-free, coprimes, and that every prime number p dividing $b + 2\eta j_0 = c$ divides b. Every prime dividing $b + 2\eta j_0$ and b must divide $2j_0$, but if it divides j_0, then it will divide both b and $a = b + \eta j_0$, which is impossible. Therefore, the only possibility is that either $c = b + 2\eta j_0 = 1$ or j_0 is odd, and that $b + 2\eta j_0$ is a power of 2, say, $b + 2\eta j_0 = 2^r$ for some nonnegative integer r. This gives that $b = 2^r - 2\eta j_0$ and that both $2^r - 2\eta j_0$ and $a = 2^r - \eta j_0$ are positive and square-free. At any rate, note that in this case we have that $K = j_0^2$, therefore $|\gamma(m) - \gamma(n)| = |m - n|^{1/2}$, which confirms inequality (2.6) in this case as well. The remaining case, that is, the one for which $k - ej = 0$, also does not lead to any solution by sign considerations.

The above arguments take care of part (i) of Theorem 2.3. Part (ii) can be proved in an identical manner as Theorem 2.1 (simply treat the number j as a constant); note that since $|k| > j$, the degenerate instances considered above do not occur here. Finally, part (iii) of Theorem 2.3 follows from the above discussion of the degenerate instances. Theorem 2.3 is therefore proved.

5. Computational results. There exist 106 integers r, $1 < r < 200$, such that $2^r - 1$ and $2^{r-1} - 1$ are both square-free. These are

2, 3, 4, 5, 8, 9, 10, 11, 14, 15, 16, 17, 23, 26, 27, 28, 29, 32, 33, 34,
35, 38, 39, 44, 45, 46, 47, 50, 51, 52, 53, 56, 57, 58, 59, 62, 65, 68,
69, 70, 71, 74, 75, 76, 77, 82, 83, 86, 87, 88, 89, 92, 93, 94, 95, 98,

TABLE 5.1

k	$n < 10^9$ such that $\gamma(n+1) - \gamma(n) = k$
3	4, 49
7	9, 12
11	20, 27, 288, 675, 71199
13	18, 152, 3024
15	16, 28
17	1681, 59535, 139239, 505925
19	98, 135, 11375
21	25, 2299, 18490
23	75, 1215, 1647, 2624
27	52, 39325
29	171, 847, 1616, 4374
31	32, 36, 40, 45, 60, 1375
39	76, 775
41	50, 63000
43	56, 84
45	22747, 182182
47	92, 1444, 250624
49	54, 584, 21375, 23762, 71874, 177182720
53	147, 315, 9152, 52479
55	512, 9408, 12167, 129311
59	324, 4239
67	72, 88, 132, 5576255
69	82075, 656914
71	140, 3509, 114375
73	872, 1274, 3249
75	148, 105412, 843637
79	81, 104, 117, 156, 343, 375, 7100, 47375, 76895
83	164, 275, 5967, 33124, 89375, 7870625, 38850559
85	126, 1016, 16128, 471968, 10028976
89	531, 11736
91	96, 100, 1050624
93	832, 201019, 1608574
97	3807, 4067, 12716, 73304
99	112, 1975, 8575

TABLE 5.2

k	$n < 10^9$ such that $\gamma(n+1) - \gamma(n) = -k$
5	7, 11
7	44, 80
9	19
11	17, 360, 31212
13	15, 175, 944, 69375
17	23, 351, 1183, 5750, 240064
19	63, 116, 120, 242, 29645
21	43, 424
23	26, 99, 279, 2400, 110079
25	51, 1808, 2808
27	1519
29	31, 35, 39, 59, 168, 2375, 6655, 167112000
31	350
33	67
35	423, 1376
37	9800
41	47, 55, 62, 83, 296, 824, 3699, 3968, 100499
43	207, 260, 528, 5687
45	91
47	53, 539
49	1475, 3536, 317600, 834272
51	9250
55	332
57	115, 124
59	74, 89, 711, 735
61	123, 62000, 945624
65	71, 87, 131
67	1224, 11583, 362556
69	79, 139, 18784
71	855, 2988
73	188, 549, 624, 783, 975, 2645, 28593
77	103, 155, 1368, 129032
79	476, 725, 2600, 2783
81	163, 10624
83	97, 4655, 26568, 334719
85	6128
87	244
89	95, 119, 440, 58080, 1292400
91	548, 1025, 2208, 50255
93	187
95	1143
97	111, 6992, 44375, 68607

99, 104, 107, 112, 113, 116, 117, 118, 119, 122, 123, 124, 125, 128, 129, 130, 131, 134, 135, 142, 143, 146, 149, 152, 153, 154, 158, 159, 164, 165, 166, 167, 170, 171, 172, 173, 176, 177, 178, 179, 182, 183, 184, 185, 188, 191, 194, 195, 196, 197.

There exist 113 integers r, $1 < r < 200$, such that $2^r + 1$ and $2^{r-1} + 1$ are both square-free. These are

5, 6, 7, 8, 12, 13, 14, 17, 18, 19, 20, 23, 24, 25, 26, 29, 32, 35, 36, 37, 38, 41, 42, 43, 44, 47, 48, 49, 53, 54, 59, 60, 61, 62, 65, 66, 67, 72, 73, 74, 77, 80, 83, 84, 85, 86, 89, 92, 95, 96, 97, 98, 101, 102, 103, 104, 107, 108, 109, 113, 114, 115, 116, 119, 120, 121, 122, 125, 126, 127, 128, 132, 133, 134, 137, 138, 139, 140, 143, 144, 145, 146, 149, 152, 155, 156, 157, 158, 161, 162, 163, 164, 167, 168, 169, 173, 174, 175, 176, 179, 180, 181, 185, 186, 187, 188, 192, 193, 194, 197, 198, 199, 200.

Moreover, here Table 5.1 presents all the solutions $n < 10^9$ to the equation $\gamma(n+1) - \gamma(n) = k$, for $1 < k < 100$ (note that this equation has no solution $n < 10^9$ for $k = 5, 9, 25, 33, 35, 37, 51, 57, 61, 63, 65, 77, 81, 87$, and 95.)

Finally, Table 5.2 presents all the solutions $n < 10^9$ to the equation $\gamma(n+1) - \gamma(n) = -k$, for $1 < k < 100$ (note that this equation has no solution $n < 10^9$ for $k = 3, 15, 39, 53, 63, 75$, and 99).

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