IJMMS 2003:62, 3957–3962
PII. S0161171203301425
http://ijmms.hindawi.com

PARRONDO'S PARADOX

GEOFFREY C. BERRESFORD and ANDREW M. ROCKETT

Received 3 January 2003

Since coming to the attention of the general news media several years ago, the paradoxical combination of two losing games into a winning game by J. M. R. Parrondo has been the subject of numerous numerical investigations and simulations. This note provides a clear statement of the nature of the paradox together with a straightforward analysis and resolution that is accessible to a wide audience.

2000 Mathematics Subject Classification: 91A60, 60J10.

1. Introduction. A recently discovered paradox by J. M. R. Parrondo involves combining two losing games into a winning game [1, 3, 4, 5, 6, 7]. In this note, we set Parrondo's games in a continuum of such games, find an explicit expression for the difference between the losing and winning games, and maximize this difference, thereby eliminating the probabilistic components and explaining the paradox simply in deterministic terms.

2. Parrondo's paradox. Parrondo's paradox involves two games, A and B, each based on tossing biased coins, with the player winning \$1 on heads and losing \$1 on tails. In game A, the coin (which we call coin 1) has probability $1/2-c$ of heads, and so probability $1/2+c$ of tails, where c is a small positive number such as 0.005. For $c=0$, game A is fair, so c represents the bias against the player. Game B uses two coins: one (coin 2) has probability $1/10-c$ of heads and the other (coin 3) has probability $3/4-c$ of heads. The choice of which coin to use is determined by the following rule: use coin 2 if the current winnings are a multiple of 3, and otherwise use coin 3. For $c=0$, game B is also fair. To see this, we represent the game as a Markov chain (see Figure 2.1), with the states representing the winnings being 0,1, or $2 \pmod 3$ and the transition probabilities on the arrows giving the probabilities of winning or losing \$1 with the appropriate coin, so that the transition matrix is

$$T=\begin{pmatrix} 0 & \frac{1}{10} & \frac{9}{10} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{pmatrix}. \tag{2.1}$$

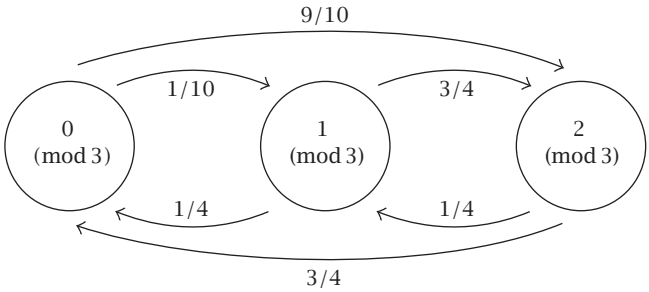

FIGURE 2.1. Game B as a Markov chain.

Standard methods (see, e.g., [8, pages 479–480]) allow one to calculate the "stationary distribution" d with $dT = d$, giving the long-run probabilities of the various states, obtaining $d = (5/13,\ 2/13,\ 6/13)$. Multiplying these probabilities by the expected winnings for those states, $(-4/5,\ 1/2,\ 1/2)$, gives 0, showing that for $c = 0$, the game is fair. For $c > 0$, the game is *unfavorable* to the player since the probabilities for heads are reduced.

Parrondo's paradox is the observation that if games A and B (with $c > 0$ to make them unfavorable) are played in certain combinations, the result is favorable in that the long-run expected winnings for each play, or the "value" of the game, are positive. The games can be played in strict alternation ($ABABAB\ldots$), by repeating blocks like $AAABB$, or by randomly choosing game A with a probability p and B with probability $1-p$. For example, if games A and B are played with $c = 0.005$, their long-run expected winnings per play are, respectively, -1 cent and about -0.87 cent, while if they are combined by choosing each randomly with probability $1/2$, the value is about 1.57 cents (found by representing the combined game as another Markov chain and multiplying the new long-run probabilities by the expected gains). Taking instead $c = 0$ raises the long-run value of the combined game to about 2.54 cents. Other combinations are discussed (along with simulations) in [2] and elsewhere on the world wide web.

3. Analyzing the paradox by coin selection. Since two unfavorable games combining into a favorable game is no more paradoxical than two fair games combining into an "even more" favorable game, we take $c = 0$, both for simplicity and because it increases the difference between the value of the favorable game and the larger of the two values of games A and B. A paradox is a *seemingly* contradictory occurrence, and Parrondo's paradox may be analyzed as follows. There are three coins: coin 1 is fair (probability of heads is $1/2$), coin 2 is unfavorable (probability of heads is $1/10$), and coin 3 is favorable to the player (probability of heads is $3/4$). Game B uses the unfavorable and favorable coins in proportions that make the game fair, using the coins 0, $5/13 \approx 38\%$,

and $8/13 \approx 62\%$ of the time, respectively. In the combined game, however, they are used $1/2 = 50\%$, $245/1418 \approx 17\%$, and $232/709 \approx 33\%$ of the time, and the favorable outcome should not be unexpected since the unfavorable coin is used only about half as often.

4. Improving Parrondo. How can we modify game B to maximize the value of the combined game? The probabilities $1/10$ and $3/4$ of heads for coins 2 and 3 that make game B fair are not unique—they can be replaced by any other probabilities x and $(1-x-\sqrt{x-x^2})/(1-2x)$, such as $1/5$ and $2/3$, and game B will remain fair. We could also vary the proportions with which games A and B are played, from 50:50 to any probability p for game A and $1-p$ for game B. The graph in Figure 4.1 shows the values $V(x,p)$ of the combined games for different values of x and p:

$$V(x,p) = \frac{3p(p^2-3p+2)(1-2x)(4(x-x^2)-2-w(x))}{(8-4(x-x^2)(9+w(x)))+(2p-p^2)(1+4(x-x^2)w(x))}, \quad (4.1)$$

where $w(x) = -3+4\sqrt{x-x^2}$.

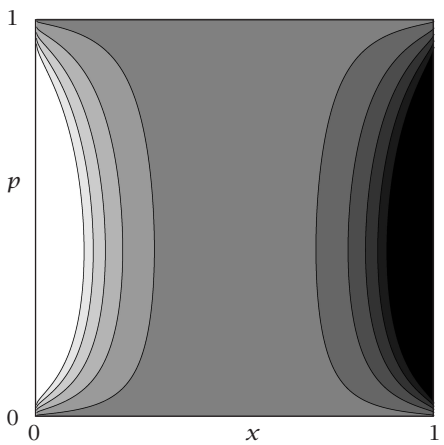

FIGURE 4.1. Values of combined games with probability x of winning with coin 2 and probability p of playing game A (a darker shade means a smaller value). The maximum value occurs at $x = 0$, $p = 1-\sqrt{13-4\sqrt{10}}$.

The value of the combined game is maximized at $x = 0$, meaning that in game B coins 2 and 3 have probabilities 0 and 1, respectively, for heads. This makes game B deterministic: if your winnings are a multiple of 3 you lose \$1, and otherwise you win \$1. Game B is still fair since ultimately it oscillates between states of winning or losing \$1, as shown in Figure 4.2.

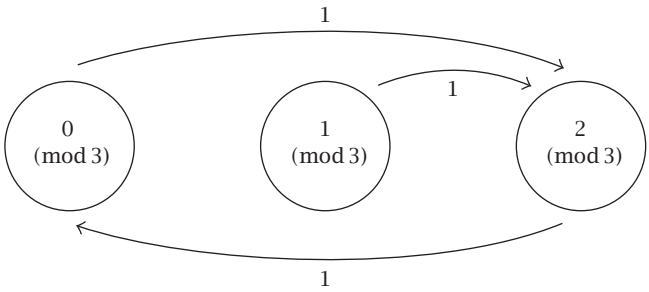

FIGURE 4.2. Extreme game *B*.

Combining this extreme version of game *B* with game *A* leads to a positive expected value because game *A*, with equal chances of winning and losing \$1, will occasionally lead to the middle state above, resulting in the gain of \$1 before resuming the oscillation.

5. Extreme Parrondo. It should now be clear that we may push matters still further by replacing oscillations with absorbing states and then switching between games so as to *always* win. Such deterministic games can be disguised behind a screen of probabilistic rules for choosing which game is played next to achieve Parrondo-like effects.

The "ultimate" Parrondo paradox would be two fair games that when played in some combination result in winning \$1 on each play. We describe such a situation below, beginning with one where the combined game wins \$0.50 on each play, and then a deterministic version with winnings of \$1 on each play.

Game *A*: If your winnings are even, you win \$1; if not, you win nothing.

Game *B*: If your winnings are odd, you win \$1; if not, you win nothing.

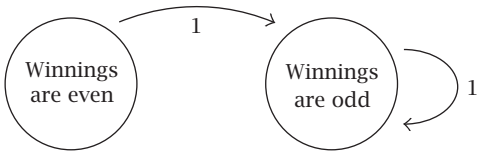

FIGURE 5.1. Extreme game *A*.

The diagram for game *B* is similar to Figure 5.1 but "odd" leads to "even" with probability 1 and "even" is the absorbing state. Clearly, both games are fair in that after at most one step you are in the absorbing state with winnings of 0. If we now combine games *A* and *B* by playing each randomly, with probability 1/2 each, we have the situation shown in Figure 5.2.

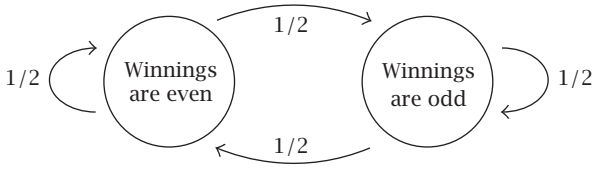

FIGURE 5.2. A random choice between extreme games.

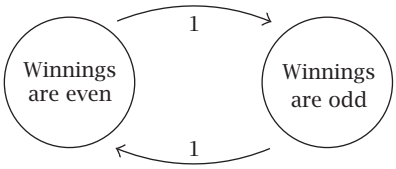

FIGURE 5.3. The ultimate, nonprobabilistic Parrondo game.

By symmetry, the long-term invariant distribution for this chain is $(1/2, 1/2)$, and multiplying by the expected payoff vector $(1/2, 1/2)$ gives expected winnings of $1/2$.

Even more favorable, simply alternate $ABABAB\ldots$, with no probabilities in the game at all. Ultimately, the chain is as shown in Figure 5.3, in which you win \$1 each play.

This can be made into a more Parrondo-like game by introducing probabilities, but at a cost of decreasing the value of the game. For example, consider two games, C and D, with each play of each based on a toss of a coin that comes up heads with probability $9/10$. In Game C, if your current winnings are *even*, you win \$1 on heads and lose \$1 on tails; while if your current winnings are *odd*, you win nothing on heads and lose \$1 on tails. Modeling this as a Markov chain with states *even* and *odd*, the invariant vector is $(1/11, 10/11)$, and multiplying by the payoffs $(4/5, -1/10)$ gives a negative value of $-1/55$. Game D is the same as game C but reversing even and odd, and therefore has the same negative value. However, playing these in strict alternation $CDCDCD\ldots$ and beginning with 0 winnings, each play wins \$1 with probability $9/10$ and loses \$1 with probability $1/10$, so the combined game has a value of $+4/5$.

REFERENCES

[1] *CHANCE News 9.01 (November 17, 1999 to January 2, 2000)*, http://www.dartmouth.edu/~chance/chance_news/recent_news/chance_news_9.01.html.

[2] S. B. Ekhad and D. Zeilberger, *Remarks on the Parrondo paradox*, Personal Journal of Ekhad and Zeilberger, http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/parrondo.html.

[3] G. P. Harmer and D. Abbott, *Game theory: losing strategies can win by Parrondo's paradox*, Nature **402** (1999), no. 6764, 864.

[4] G. P. Harmer, D. Abbott, and P. G. Taylor, *The paradox of Parrondo's games*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **456** (2000), no. 1994, 247–259.
[5] E. Klarreich, *Playing both sides*, Sciences **41** (2001), no. 1, 25–29.
[6] J. M. R. Parrondo, *Parrondo's paradoxical games*, http://seneca.fis.ucm.es/parr/GAMES/.
[7] O. E. Percus and J. K. Percus, *Can two wrongs make a right? Coin-tossing games and Parrondo's paradox*, Math. Intelligencer **24** (2002), no. 3, 68–72.
[8] A. Rényi, *Probability Theory*, North-Holland Series in Applied Mathematics and Mechanics, vol. 10, American Elsevier Publishing, New York, 1970.

Geoffrey C. Berresford: Department of Mathematics, C.W. Post Campus, Long Island University, 720 Northern Boulevard, Brookville, NY 11548, USA
E-mail address: geoffrey.berresford@liu.edu

Andrew M. Rockett: Department of Mathematics, C.W. Post Campus, Long Island University, 720 Northern Boulevard, Brookville, NY 11548, USA
E-mail address: andrew.rockett@liu.edu