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ON MARKOVIAN COCYCLE PERTURBATIONS IN CLASSICAL AND QUANTUM PROBABILITY

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We introduce Markovian cocycle perturbations of the groups of transformations associated with classical and quantum stochastic processes with stationary increments, which are characterized by a localization of the perturbation to the algebra of events of the past. The Markovian cocycle perturbations of the Kolmogorov flows associated with the classical and quantum noises result in the perturbed group of transformations which can be decomposed into the sum of two parts. One part in the decomposition is associated with a deterministic stochastic process lying in the past of the initial process, while another part is associated with the noise isomorphic to the initial one. This construction can be considered as some analog of the Wold decomposition for classical stationary processes excluding a nondeterministic part of the process in the case of the stationary quantum stochastic processes on the von Neumann factors which are the Markovian perturbations of the quantum noises. For the classical stochastic process with noncorrelated increments, the model of Markovian perturbations describing all Markovian cocycles up to a unitary equivalence of the perturbations has been constructed. Using this model, we construct Markovian cocycles transforming the Gaussian state ρ to the Gaussian states equivalent to ρ.

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1. Introduction. It is well known that every stochastic process with finite second moments and continuous in square mean can be considered as a continuous curve in the Hilbert space. In this framework, properties of the process, such as stationarity and noncorrelativity of the increments, appear as an invariance under the action of the group of unitaries in the Hilbert space and an orthogonality of the curve increments, respectively. Thus to investigate the stochastic process, one can use functional analysis techniques. This approach was introduced by Kolmogorov who considered in [26, 27] a classification problem for the equivalence classes of continuous curves $\xi = (\xi_t)_{t\in\mathbb{R}}$ in a Hilbert space H, which are invariant with respect to a strongly continuous one-parameter group of unitaries $U = (U_t)_{t\in\mathbb{R}}$ (i.e., $U_t(\xi_s - \xi_r) = \xi_{s+t} - \xi_{s+r}$) with the transformations of equivalence defined by the formula

$$\tilde{\xi}_t = W\xi_t + \eta, \tag{1.1}$$

where W and η are a unitary operator in H, and an element of H, correspondingly. Indeed, the continuous curve $\tilde{\xi} = (\tilde{\xi}_t)_{t\in\mathbb{R}}$ is invariant with respect to the group of unitaries $\tilde{U} = (W_tU_t)_{t\in\mathbb{R}}$, where the one-parameter family of unitaries

$$W_t = WU_tW^*U_{-t}, \quad t \in \mathbb{R}, \tag{1.2}$$

satisfies the condition of multiplicative U*-cocycle*,

$$W_{t+s} = W_tU_tW_sU_{-t}, \quad s,t \in \mathbb{R}. \tag{1.3}$$

The multiplicative cocycle generated by a unitary operator W as in (1.2) is said to be U*-coboundary*. Notice that not every cocycle (1.3) is coboundary (1.2). Here we use ordinary definitions of the cohomologies of group theory for 1-cocycle and 1-coboundary associated with the standard bar-resolvent of the group $\mathbb{R}$ with values in the multiplicative group $\mathcal{U}(H)$ of all unitaries in the Hilbert space H with the module structure defined by the group action $x \to U_txU_{-t}$, $x \in \mathcal{U}(H)$, $t \in \mathbb{R}$ (see, e.g., [12, 18]).

Suppose that there exists a continuous curve $\xi = (\xi_t)_{t\in\mathbb{R}}$ which is invariant with respect to the group U, and the increments of ξ are orthogonal such that $\xi_{t_1} - \xi_{s_1} \perp \xi_{t_2} - \xi_{s_2}$ for all disjoint intervals $(s_1,t_1)\cap(s_2,t_2) = \varnothing$. The curves of such type were called in [27] the Wiener spirals. Every multiplicative U-cocycle $(W_t)_{t\in\mathbb{R}}$, which is strongly continuous in t, defines a new strongly continuous one-parameter group of unitaries $\tilde{U} = (W_tU_t)_{t\in\mathbb{R}}$. We will call $\tilde{U}$ *a cocycle perturbation* of the group U. If $(W_t)_{t\in\mathbb{R}}$ is a coboundary, then, for the cocycle perturbation $\tilde{U}$, there also exists an invariant curve $\tilde{\xi}$ which is the Wiener spiral. Indeed, this curve can be constructed by formula (1.1), where one must substitute for W the unitary operator generating the coboundary by means of (1.2). For arbitrary cocycle $(W_t)_{t\in\mathbb{R}}$ which is not a coboundary, it is possible that there does not exist a Wiener spiral which is invariant with respect to the cocycle perturbation $\tilde{U}$. In this paper, we introduce a subset (2.2) of the set of all cocycles such that being given a cocycle from (2.2) generates the cocycle perturbation possessing the invariant Wiener spiral if the perturbing group satisfies this condition. The class of cocycle perturbations we propose is important because in applications (see [10, 21, 24, 29, 33]), the problem of constructing the dilation of a quantum positive evolution to a cocycle on the Fock space is often posed. The cocycles of this type are adapted with respect to a Fock filtration and can be constructed by the second quantization of cocycles from the subset (2.2) we introduce. Cocycles from (2.2) will be called Markovian. The usage of the term “Markovian cocycle” follows from [1, 2]. The Markovian cocycles of [1, 2] are connected with the classical Markov property in the sense that the perturbation of the Markovian stochastic process by the

Markovian cocycle is also the Markovian stochastic process with the same algebras of the present, future, and past. Our definition is different from the definition in [1, 2]. Nevertheless, under an appropriate interpretation, the perturbations we consider are Markovian in the sense of the definition in these cited papers, describing a wider class of perturbations. We consider this as a sufficient motivation for using this terminology.

The model example of the group of unitaries U possessing an invariant Wiener spiral is the group of shifts in the Hilbert space $H = L^2(\mathbb{R})$ defined by the formula $(U_t\eta)(x) = \eta(x+t)$, $\eta \in H$. In fact, the set of functions $\xi_t(x) = 1$, $x \in [-t,0]$, $\xi_t(x) = 0$, $x \notin [-t,0]$, is the Wiener spiral which is invariant with respect to U (see [27]). We will call "a past" and "a future" of the system the subspaces $H_{t]}$ and $H_{[t}$ containing functions with the support belonging to $[t,+\infty)$ and $(-\infty,t]$, respectively. Notice that $H_{t]} = U_tH_{0]}$ and $H_{[t} = U_tH_{[0}$, $t \in \mathbb{R}$. In the case we consider, we call the cocycle W *Markovian* if the restriction of the unitary operator W_t to the subspace of the future $H_{[t}$ is an identity transformation for every fixed $t \geq 0$. In particular, this definition guarantees that the subspace of the past $H_{0]}$ is invariant with respect to W_{-t} for all $t \geq 0$, which allows to consider the restriction of the cocycle perturbation $\tilde{U}_{-t}|_{H_{0]}}$, $t \geq 0$. Let $\tilde{U}$ be a cocycle perturbation of U by the cocycle which is Markovian in the sense of our definition. We will show that every such perturbation can be represented in the form $\tilde{U} = \tilde{U}^{(1)} \oplus \tilde{U}^{(2)}$, where $\tilde{U}^{(1)}$ is an arbitrary group of unitaries in the subspace of the past $H_{0]}$ and $\tilde{U}^{(2)}$ is unitarily equivalent to the initial group of shifts. This representation can be called the Wold decomposition of the cocycle perturbation of $\tilde{U}$. The groups $\tilde{U}^{(1)}$ and $\tilde{U}^{(2)}$ can be interpreted as associated with deterministic and nondeterministic parts of the process.

By a quantum stochastic process we mean (see [21, 24, 33]) a strongly continuous one-parameter family $x = (x_t)_{t\in\mathbb{R}}$ of linear (unbounded in general) operators in a Hilbert space H. Usually one needs to require that all operators x_t be closed and have the invariant common domain which is dense in H. Under this definition, stationary quantum stochastic process $x = (x_t)_{t\in\mathbb{R}}$ can be defined by the condition $x_t = \alpha_t(x^0) + x^1$, $t \in \mathbb{R}$, where α is a certain group of automorphisms and x^0, x^1 are two fixed linear operators. In this way, a quantum stochastic process with stationary increments is a continuous operator-valued curve which is invariant with respect to a certain group of automorphisms α. In quantum probability theory, the role of the σ-algebras of events associated with the stochastic process is played by the von Neumann algebras generated by increments of the process, that is, the ultra-weak closed algebras of bounded operators in H obtained as the second commutant of the set of the increments $x_t - x_s$ of the process x (see, e.g., [11, 21, 24]). Every classical stochastic process ξ consisting of the random variables $\xi_t \in L^\infty(\Omega)$ can be considered as a quantum, where the operators $x_t = M_{\xi_t}$ forming the process are the operators of multiplications by the functions ξ_t in the Hilbert space $L^2(\Omega)$.

Definition 1.1. The flow $\{T_t,\ t \in \mathbb{R}\}$ on the probability space $(\Omega, \mathcal{M}, \mu)$ is said to be a Kolmogorov flow (see [28]) if there exists a σ-algebra of events $\mathcal{M}_{0]} \subset \mathcal{M}$ such that $T_t \mathcal{M}_{0]} = \mathcal{M}_{t]}$,

$$\mathcal{M}_{s]} \subset \mathcal{M}_{t]}, \quad s < t, \tag{1.4}$$

$$\cup_t \mathcal{M}_{t]} = \mathcal{M}, \tag{1.5}$$

$$\cap_t \mathcal{M}_{t]} = \{\varnothing, \Omega\}. \tag{1.6}$$

In [15], a notion of the Kolmogorov flow was generalized into quantum probability. Here the σ-algebras of events $\mathcal{M}_{t]}$ are replaced by the corresponding von Neumann algebras. Notice that in the well-known monograph [25], where the conditions on the spectral function of process, which result in the Kolmogorov flow were investigated, a term “completely nondeterministic process” is used for the stochastic process generating the Kolmogorov flow, while the term “Kolmogorov flow” is not used anywhere. Every classical or quantum process with independent (in the classical sense) increments results in the Kolmogorov flow. We define Markovian cocycle perturbations of the classical and quantum Kolmogorov flows such that the perturbed flows contain the parts which are isomorphic to the initial Kolmogorov flow. Notice that in the model situation of the Hilbert space $L^2(\mathbb{R})$ we considered above, the Kolmogorov flow can be associated with the flow of shifts. Thus, the possibility to exclude the part being the Kolmogorov flow in the perturbed dynamics can be considered as some analogue of the Wold decomposition allowing to exclude a nondeterministic part of the process.

For every classical or quantum stochastic process, a great role is played by the set of all (not necessarily linear) functionals of the process. In particular, for the Wiener process, we have the Wiener-Itô decomposition of the space of all L^2-functionals into an orthogonal sum, which allows to solve effectively some stochastic differential equations (see [20]). Notice that from the viewpoint of the theory of the cohomologies of groups, the stochastic process with stationary increments determining the continuous curve being invariant with respect to the group of transformations U is an additive $(1-U)$-cocycle. It is natural to consider the ring of cohomologies of all degrees generated by the $(1-U)$-cocycle of such type which can be interpreted as the space of all (nonlinear) functionals of the initial stochastic process. We show that the Markovian cocycle perturbations we introduced define homomorphisms of this ring of cohomologies.

This paper is organized as follows. In Section 2, for the group of unitaries U and the continuous curve ξ which is invariant with respect to U, we define a class of the Markovian U-cocycles. The group of unitaries $\tilde{U}$ which is a perturbation of U by the cocycle of such type determines a continuous curve $\tilde{\xi}$ being invariant with respect to $\tilde{U}$ and connected with the curve ξ by formula (1.1), where W is (nonunitary in general) isometrical operator satisfying the additional property of the localization of action to "the past." We call such isometrical operators Markovian and we prove that every Markovian operator is associated with a certain perturbation by a Markovian cocycle. The investigation we give in Section 3 shows in detail that the Markovian cocycle we introduced determines the Wold decomposition for the cocycle perturbation. In Section 4 we construct a model of the Markovian cocycle for the stochastic process with independent increments. The model we give allows to construct the Markovian cocycles with the property $W_t - I \in s_2$ (the Hilbert-Schmidt class) which translate the fixed Gaussian measure to the equivalent Gaussian measures. In Section 5 we give the basic notion on the theory of Kolmogorov flows in classical and quantum probability. In Section 6, we define the rings of cohomologies generated by additive 1-cocycles and show in examples of the Wiener process and the quantum noise that the set of all functionals of the stochastic process with stationary increments can be considered as a ring of the cohomologies of the group which is composed by shifts of the increments in time. Moreover, we define a Markovian perturbation of the group resulting in a homomorphism of the ring of cohomologies of the group. Notice that for quantum stochastic processes $x = (\xi_t = x_t)_{t\in\mathbb{R}}$, it is also possible to define transformations of the form (1.1), where one must take a morphism for W and a linear operator for η. In Section 7, we introduce the Markovian cocycle perturbations of the quantum noises being a generalization of the classical processes with independent stationary increments for the quantum case, which result in transformations of the quantum stochastic processes of the form (1.1). The Markovian perturbations we introduce determine homomorphisms of the ring of cohomologies associated with the stochastic process in the sense of Section 6. The techniques involved in Section 6 allow to obtain in Section 7 some analogue of the Wold decomposition for the classical stochastic processes in the quantum case, which permits to exclude a nondeterministic part of the process.

2. Stochastic processes with stationary increments as curves in a Hilbert space. Let $\xi = (\xi_t)_{t\in\mathbb{R}}$ be continuous in square mean stochastic process with stationary increments on the probability space (Ω, Σ, μ). Without loss of generality, we can suppose that the condition $\xi(0) = 0$ holds. Then, in the Hilbert space H^ξ generated by the increments $\xi_t - \xi_s$, $s, t \in \mathbb{R}$, one can define a strongly continuous group of unitaries $U = (U_t)_{t\in\mathbb{R}}$ shifting the increments in time such that $U_t(\xi_s - \xi_r) = \xi_{s+t} - \xi_{r+t}$, $s, t, r \in \mathbb{R}$, where ξ satisfies the condition of (additive) $(1-U)$-cocycle, that is, $\xi_{t+s} = \xi_t + U_t\xi_s$, $s, t \in \mathbb{R}$. Denote by H the Hilbert space with the inner product $(\xi, \eta) = \mathbb{E}(\xi\overline{\eta})$ generated by the classes of equivalency of the random variables ξ, η in the space (Ω, μ), which possess finite second moments with respect to the Hilbert norm associated with the expectation $\mathbb{E}$. We will identify the random variables with the elements of the Hilbert space H. A stochastic process ξ can be considered as a curve in the

Hilbert space H^{ξ}, which is invariant with respect to the action of the group U (see [26, 27]). The space H^{ξ} is a subspace of H but does not coincide with it in general. Let $\tilde{U}$ be an arbitrary continuation of U to a strongly continuous group of unitaries in the Hilbert space H. Then ξ can also be considered as the curve in H, which is invariant with respect to the action of the group $\tilde{U}$. Denote by $\mathcal{I}(\xi)$ the set containing all possible strongly continuous groups of unitaries in H such that the stochastic process ξ is invariant with respect to them. One can define in the space H^{ξ} an increasing family of subspaces $H^{\xi}_{t]}$ generated by the increments $\xi_s - \xi_r$, $s, r \le t$, associated with "a past" before the moment t, and a decreasing family of subspaces $H^{\xi}_{[t}$ generated by the increments $\xi_s - \xi_r$, $s, r \ge t$, associated with "a future" after the moment t such that $H^{\xi} = \vee_t H^{\xi}_{t]} = \vee_t H^{\xi}_{[t}$. Notice that, for the processes with noncorrelated increments, the subspaces $H^{\xi}_{t]}$ and $H^{\xi}_{[t}$ are orthogonal. Fix the group $U \in \mathcal{I}(\xi)$. The strongly continuous one-parameter family of unitaries $W = (W_t)_{t\in\mathbb{R}}$ in H is said to be *a (multiplicative) U-cocycle* if the following condition holds:

$$W_{t+s} = W_t U_t W_s U_t^*, \quad s, t \in \mathbb{R},\ W_0 = I. \tag{2.1}$$

The cocycle W is called *Markovian* under the condition

$$W_t f = f, \quad f \in H^{\xi}_{[t},\ t > 0. \tag{2.2}$$

Property (2.1) exactly means that the strongly continuous one-parameter family of unitaries $\tilde{U} = (W_t U_t)_{t\in\mathbb{R}}$ forms a group. We consider Markovianity as a localization of the action of the cocycle W to the subspace of the past. Moreover, the Markovian property (2.2) preserves "a causality" such that "the future" of the system is not disturbed. Notice that our definition of a Markovian cocycle is based on the analogous definition introduced in [1, 2] in a considerably more general case. We defer the examples of Markovian cocycles to Section 4, where a model of the Markovian cocycle is given for the group of shifts on the line, which describes all cocycles up to the unitary equivalence of perturbations. Using (2.1), we get $I = W_{-t+t} = W_{-t} U_{-t} W_t U_{-t}^*$, $t > 0$, such that $W_{-t} = U_t^* W_t U_t$, $t > 0$. Thus one can rewrite (2.2) in the form

$$W_{-t} f = f, \quad f \in H^{\xi}_{[0},\ t > 0. \tag{2.3}$$

Consider the stochastic process $\xi^{(1)}$, $\xi^{(1)}(0) = 0$, being a continuous curve in the Hilbert space H, and take $U^{(1)} \in \mathcal{I}(\xi^{(1)})$. Suppose $W = (W_t)_{t\in\mathbb{R}}$ is a multiplicative $U^{(1)}$-cocycle in the space H.

Proposition 2.1. *Let W satisfy the Markovian property (2.2). Then the family $\xi_t^{(2)} = W_t \xi_t^{(1)}$, $t \le 0$, $\xi_t^{(2)} = \xi_t^{(1)}$, $t > 0$, is continuous in t and is a stochastic process with stationary increments. Furthermore, it is a curve in H invariant with respect to the group of unitaries $U_t^{(2)} = W_t U_t^{(1)}$, $t \in \mathbb{R}$.*

Proof. Check that $\xi^{(2)}_{t+s} = \xi^{(2)}_t + U^{(2)}_t \xi^{(2)}_s$, $s,t \in \mathbb{R}$. In fact, for $s,t \le 0$, we obtain

$$\begin{aligned}\xi^{(2)}_{t+s} &= W_{t+s}\xi^{(1)}_{t+s} = W_{t+s}\xi^{(1)}_t + W_{t+s}U^{(1)}_t\xi^{(1)}_s\\ &= W_tU^{(1)}_tW_sU^{(1)}_{-t}\xi^{(1)}_t + W_tU^{(1)}_tW_s\xi^{(1)}_s\\ &= -W_tU^{(1)}_tW_s\xi^{(1)}_{-t} + U^{(2)}_t\xi^{(2)}_s\\ &= -W_tU^{(1)}_t\xi^{(1)}_{-t} + U^{(2)}_t\xi^{(2)}_t\\ &= \xi^{(2)}_t + U^{(2)}_t\xi^{(2)}_s,\end{aligned} \tag{2.4}$$

where we used the identity $W_s\xi^{(1)}_{-t} = \xi^{(1)}_{-t}$, $s,t \le 0$, which is correct due to (2.3). For $s,t > 0$, the property we prove is true because $\xi^{(2)}_t = \xi^{(1)}_t$, $t \ge 0$, by definition. Notice that

$$\begin{aligned}\xi^{(2)}_t = \xi^{(1)}_t &= -U^{(1)}_t\xi^{(1)}_{-t} = -W_tU^{(1)}_tW_{-t}U^{(1)}_{-t}U^{(1)}_t\xi^{(1)}_{-t}\\ &= -W_tU^{(1)}_tW_{-t}\xi^{(1)}_{-t} = -U^{(2)}_t\xi^{(2)}_{-t}, \quad t > 0,\end{aligned} \tag{2.5}$$

by means of the cocycle condition (2.1) for W. □

We will call an isometrical operator $R : H^{\xi^{(1)}} \to H^{\xi^{(2)}}$ Markovian if $Rf = f$, $f \in H^{\xi^{(1)}}_{[0}$.

Proposition 2.2. *Given a Markovian cocycle W, there exists a limit*

$$\lim_{t\to+\infty} W_{-t}\eta = W_{-\infty}\eta, \quad \eta \in H^{\xi^{(1)}}, \tag{2.6}$$

such that $W_{-\infty}$ is a Markovian isometrical operator with the property $W_{-\infty}f = W_{-s}f$, $f \in H^{\xi^{(1)}}_{[-s}$, $s \ge 0$.

Proof. Notice that $W_{-t-s}f = W_{-s}U^{(1)}_{-s}W_{-t}U^{(1)}_sf = W_{-s}f$, $f \in H^{\xi^{(1)}}_{[-s}$, $s,t \ge 0$, due to the Markovian property in the form (2.3). Hence, the limit exists for the set of elements $f \in H^{\xi^{(1)}}_{[-s}$, $s \ge 0$, which is dense in $H^{\xi^{(1)}}$. Thus, the strong limit exists by the Banach-Steinhaus theorem. The limiting operator $W_{-\infty}$ is Markovian because all operators W_{-t}, $t \ge 0$, satisfy this condition. □

Proposition 2.3. *In the space H, there exists a Markovian isometrical operator R with the property $\xi^{(2)}_t = R\xi^{(1)}_t$, $t \in \mathbb{R}$, if and only if the stochastic processes $\xi^{(1)}$ and $\xi^{(2)}$ are connected by a Markovian cocycle $W = (W_t)_{t\in\mathbb{R}}$ by the formula $\xi^{(2)}_t = W_t\xi^{(1)}_t$, $t \le 0$.*

Proof

Necessity. Suppose that there exists a Markovian isometrical operator R such that $\xi^{(2)}_t = R\xi^{(1)}_t$, $t \in \mathbb{R}$. Check condition (2.2) for W:

$$\begin{aligned}W_t\left(\xi^{(1)}_r - \xi^{(1)}_s\right) &= U^{(2)}_tU^{(1)}_{-t}\left(\xi^{(1)}_r - \xi^{(1)}_s\right) = U^{(2)}_t\left(\xi^{(1)}_{r-t} - \xi^{(1)}_{s-t}\right)\\ &= U^{(2)}_t\left(\xi^{(2)}_{r-t} - \xi^{(2)}_{s-t}\right) = \xi^{(2)}_r - \xi^{(2)}_s, \quad r,s \ge t > 0.\end{aligned} \tag{2.7}$$

Here we used the Markovian property for the operator R, which implies that $\xi_t^{(2)} = R\xi_t^{(1)} = \xi_t^{(1)}$, $t \geq 0$.

Sufficiency. Suppose that $\xi_t^{(2)} = W_t\xi_t^{(1)}$, $t \leq 0$, and a cocycle W is Markovian. Then, due to Proposition 2.2, one can define a Markovian isometrical operator $W_{-\infty} = s - \lim_{t\to+\infty} W_{-t}$. Consider the process $\tilde{\xi}_t^{(2)} = W_{-\infty}\xi_t^{(1)}$. For $t \geq 0$, we obtain $\tilde{\xi}_t^{(2)} = \xi_t^{(1)}$ by means of the Markovian property for $W_{-\infty}$. For $t \leq 0$, the representation $W_{-\infty}f = W_{-s}f$, $f \in H_{[-s}^{\xi^{(1)}}$, $s \geq 0$, gives $\tilde{\xi}_t^{(2)} = W_t\xi_t^{(1)} = \xi_t^{(2)}$, $t \leq 0$.

3. Processes with noncorrelated increments: the Wold decomposition.

Given a stationary stochastic process ξ, there exists a group of unitaries $U = (U_t)_{t\in\mathbb{R}}$ in the Hilbert space with the inner product defined by the formula $(\cdot,\cdot) = \mathbb{E}(\cdot\bar{\cdot})$ such that $\xi_t = U_t\xi_0$, $t \in \mathbb{R}$. Recall that the process ξ is said to be nondeterministic if $\wedge_{t\in\mathbb{R}} H_{t]}^{\xi} = 0$ and deterministic if $\wedge_{t\in\mathbb{R}} H_{t]}^{\xi} = H^{\xi}$. It is evident that there exist processes which are neither nondeterministic nor deterministic. A stationary process ξ has the unique decomposition $\xi = \xi^{(1)} \oplus \xi^{(2)}$, where $\xi^{(1)}$ and $\xi^{(2)}$ are deterministic and nondeterministic processes, respectively, such that $\xi^{(1)}$ and $\xi^{(2)}$ have noncorrelated increments. In its turn, a nondetermnistic process $\xi^{(2)}$ is uniquely defined by the Wold decomposition

$$\xi_t^{(2)} = \int_{-\infty}^{t} c(t-s)\zeta(ds), \tag{3.1}$$

where $\zeta(ds)$ is a noncorrelated measure such that $\mathbb{E}|\zeta(ds)|^2 = ds$ and $\mathbb{E}(\zeta(\Delta)\overline{\zeta}(\Delta')) = 0$ for all measurable disjoint sets Δ and Δ' (see [35]). Thus, every stationary process $\xi_t = U_t\xi_0$ uniquely defines the process with noncorrelated stationary increments ζ_t, which is an invariant curve with respect to the group U. This process can be called "the nondeterministic part" of ξ_t. In the following, we will call the Wold decomposition the possibility to associate with the fixed stationary process the process with stationary noncorrelated increments, which is its nondeterministic part in the sense given above. Let $V = (V_t)_{t\in\mathbb{R}_+}$ be a strongly continuous semigroup of nonunitary isometrical operators in a Hilbert space H. In functional analysis, the Wold decomposition is a decomposition of the form $H = H^{(1)} \oplus H^{(2)}$, where subspaces $H^{(1)}$ and $H^{(2)}$ reduce the semigroup V to a semigroup of unitary operators and a semigroup of completely nonunitary isometrical operators, respectively. A completely nonunitary isometrical operator is characterized by the property that there is no subspace reducing it to a unitary operator. Every strongly continuous semigroup consisting of completely nonunitary isometrical operators is unitary equivalent to its model, which is the semigroup of right shifts $S = (S_t)_{t\in\mathbb{R}_+}$ in the Hilbert space $L^2(\mathbb{R}_+, \mathcal{K})$ defined by $(S_tf)(x) = f(x-t)$, $x > t$, and $(S_tf)(x) = 0$, $0 \leq x \leq t$. Recall that a deficiency index of the generator $d = s - \lim_{t\to 0}((V_t - I)/t)$ of the strongly continuous semigroup V

is a number of linear independent solutions to the equation $d^*f = -f$. The Hilbert space of values $\mathcal{K}$ has the dimension equal to the deficiency index of the generator of V (see [32]). In the following, we will call the deficiency index of the generator an index of the semigroup. Notice that every semigroup of completely nonunitary isometrical operators V with the index $n > 0$ determines n noncorrelated processes $\xi^{(i)}$, $1 \le i \le n$, with noncorrelated increments such that $\xi^{(i)}_{t+s} = \xi^{(i)}_t + V_t\xi^{(i)}_s$, $s,t \ge 0$. In the model case of the semigroup of right shifts S in $L^2(\mathbb{R}_+,\mathcal{K})$, the processes of such type can be constructed in the following way. Choose the orthonormal basis of the space $\mathcal{K}$ consisting of the elements e_i, $1 \le i \le n \le +\infty$, and put $\xi^{(i)} = e_i \otimes \chi_{[0,t]}$, where $\chi_{[0,t]}$ is an indicator function of the interval $[0,t]$. We will investigate a behavior of the processes with noncorrelated increments with respect to perturbations by the Markovian cocycles we introduced in Section 2.

Proposition 3.1. *Let ξ and W be the process with noncorrelated increments and the Markovian cocycle, respectively. Then, $\xi'_t = W_t\xi_t$, $t \le 0$ and $\xi'_t = \xi_t$, $t > 0$, is a process with noncorrelated increments.*

Proof. It follows from Proposition 2.3 that the Markovian isometrical operator $W_{-\infty} = s-\lim_{t\to+\infty} W_{-t}$ connects the perturbed process with the initial process by the formula $\xi'_t = W_{-\infty}\xi_t$, $t \in \mathbb{R}$. □

Let the process with noncorrelated increments $\xi = (\xi)_{t\in\mathbb{R}}$ be invariant with respect to the group of unitaries $U = (U_t)_{t\in\mathbb{R}}$.

Proposition 3.2. *The restriction $V_t = U_{-t}|_{H^{\xi}_{0]}}$, $t \ge 0$, determines a semigroup of completely nonunitary isometrical operators with unit index in the Hilbert space $H^{\xi}_{0]}$.*

Proof. Every semigroup of completely nonunitary isometrical operators with unit index is unitarily equivalent to its model which is the semigroup of right shifts $S = (S_t)_{t\ge0}$ acting in the Hilbert space $L^2(\mathbb{R}_+)$ by $(S_tf)(x) = f(x-t)$, $x > t$, and $(S_tf)(x) = 0$, $0 \le x \le t$ (see [32]). Define a continuous curve $\eta = (\eta_t)_{t\ge0}$ in $L^2(\mathbb{R}_+)$ such that $\eta_t(x) = 1$, $0 \le x \le t$, and $\eta_t(x) = 0$, $x > t$. Linear combinations of the elements of the curve η form a dense set in the space $L^2(\mathbb{R}_+)$, and $\eta_{t+s} = \eta_t + S_t\eta_s$, $s,t \ge 0$. Notice that the stationarity and the orthogonality of the curve increments imply $\|\xi_t\|^2 = \|\sum_{i=1}^n(\xi_{i(t/n)} - \xi_{(i-1)(t/n)})\|^2 = n\|\xi_{t/n}\|^2$ and $\|\xi_t\|^2/t = \sigma^2 = \text{const}$. Thus, the curve ξ can be represented in the form $\xi_t = \sigma^2\mu_t$, where the measure $d\mu_t$ with values in H satisfies the condition $\mu(\Delta_1) \perp \mu(\Delta_2)$ for disjoint measurable sets $\Delta_1,\Delta_2 \subset \mathbb{R}$, where $\|d\mu_t\|^2 = dt$. Define a unitary operator $W : H^{\xi}_{0]} \to L^2(\mathbb{R}_+)$ by the formula $W\mu_t = \eta_t$, then $V_t = W^*S_tW$, $t \ge 0$. □

Let $V = (V_t)_{t\ge0}$ be the semigroup of completely nonunitary isometrical operators in $H^{\xi}_{0]}$ defined in Proposition 3.2.

Proposition 3.3. *The Markovian cocycle W determines a semigroup of isometrical operators $\tilde V = (W_{-t}V_t)_{t\in\mathbb{R}_+}$ in $H^{\xi}_{0]}$ with unit index. The Wold decomposition $H^{\xi}_{0]} = H^{(1)} \oplus H^{(2)}$ associated with the semigroup $\tilde V$ can be done by the condition $H^{(2)} = W_{-\infty}H^{\xi}_{0]}$, where $W_{-\infty} = s-\lim_{t\to+\infty} W_{-t}$.*

Proof. It follows from Propositions 2.2 and 2.3 that there exists a limit $W_{-\infty} = s-\lim_{t\to+\infty} W_{-t}$ such that the Markovian isometrical operator $W_{-\infty}$ satisfies the condition $W_t\xi_t = W_{-\infty}\xi_t$, $t \le 0$. Let $H^{(1)}$ be a subspace of H^{ξ} defined by the condition of orthogonality to all elements $W_{-\infty}\xi_t$, $t\in\mathbb{R}$. This subspace is invariant with respect to the action of the group of unitaries W_tU_t, $t\in\mathbb{R}$. On the other hand, by means of the Markovian property for $W_{-\infty}$, the subspace $H^{(1)}$ is orthogonal to all elements $W_{-\infty}\xi_t = \xi_t$, $t \ge 0$, and, therefore, $H^{(1)} \subset H^{\xi}_{0]}$. Thus, we have proved that the subspace $H^{(1)} \subset H^{\xi}_{0]}$ is invariant with respect to the group of unitaries $(W_tU_t)_{t\in\mathbb{R}}$. Hence, the restriction $V_t|_{H^{(1)}} = W_{-t}U_{-t}|_{H^{(1)}}$, $t \ge 0$, consists of unitary operators. The subspace $H^{(2)} \subset H^{\xi}$ is determined by the condition of orthogonality to $H^{(1)}$, which is $H^{(2)} = W_{-\infty}H^{\xi}$ by the definition of $H^{(1)}$. In this way, the restriction $V|_{H^{(2)}}$ is a semigroup of completely nonunitary isometrical operators by means of Proposition 3.2. □

4. A model of the Markovian cocycle for the process with noncorrelated increments. Let $S = (S_t)_{t\in\mathbb{R}}$ be a flow of shifts on the line acting by the formula $(S_t\eta)(x) = \eta(x+t)$, $x,t\in\mathbb{R}$, $\eta \in H = L^2(\mathbb{R})$. The group S is naturally associated with the stochastic process $\xi_t = \chi_{[-t,0]}$ with noncorrelated increments such that $\xi_{t+s} = \xi_t + S_t\xi_s$, $s,t\in\mathbb{R}$. Let the subspace $H_{t]}$ consist of functions f with the support $\operatorname{supp} f \subset [-t,+\infty)$. Then $H_{t]}$ is generated by the increments $\xi_s - \xi_r$, $s,r \le t$. The restrictions $T_t = S_{-t}|_{H_{0]}}$, $t \ge 0$, form the semigroup of right shifts T. Every invariant subspace $T_t\mathcal{V} \subset \mathcal{V}$, $t \ge 0$, can be described as an image of the isometrical operator M_Θ, $\mathcal{V} = M_\Theta H_{0]}$, where $M_\Theta = \mathcal{L}^{-1}\Theta\mathcal{L}$. Here $\mathcal{L}$ is the Laplace transformation and Θ is a multiplication operator by Θ which is an inner function in the semiplane $\operatorname{Re}\lambda \ge 0$. Recall that a function $\Theta(\lambda)$ is said to be inner if it is analytical in the halfplane $\operatorname{Re}\lambda \ge 0$ and its modulus equals one on the imaginary axis (see [32]). Denote by $P_{[0,t]}$, $P_{[t,+\infty)}$, $P_{\mathcal{V}}$, and $P_{\mathcal{V}^\perp}$ orthogonal projections on the subspaces of functions with the support belonging to the segment $[0,t]$ and the interval $[t,+\infty)$, and on $\mathcal{V}$ and $\mathcal{V}^\perp$, respectively.

Proposition 3.3 shows that arbitrary perturbation of the group of shifts associated with the process with noncorrelated increments by a Markovian cocycle W is completely described by a unitary part $R = \tilde V|_{H^{(1)}}$ of the semigroup of isometries $\tilde V = (W_{-t}V_t)_{t\in\mathbb{R}_+}$. In the following theorem, we construct the Markovian cocycles resulting in the semigroup R which is unitarily equivalent to the one we set. In this way, we introduce a model describing all Markovian cocycles up to unitary equivalence of perturbations.

Theorem 4.1. *Let $R = (R_t)_{t\in\mathbb{R}_+}$ be a strongly continuous semigroup of unitaries in the space $\mathcal{V}^\perp$, where $\mathcal{V}$ is invariant with respect to the semigroup of* *right shifts T. Then the family of unitary operators $(W_t)_{t\in\mathbb{R}}$, defined for negative t by the formula*

$$\begin{gathered} W_{-t}\eta = (R_tP_{\mathcal{V}^\perp}S_t - P_{\mathcal{V}})P_{[t,+\infty)}\eta + M_\Theta P_{[0,t]}\eta, \quad \eta \in H_{0]}, \\ W_{-t}\eta = \eta, \quad \eta \in H_{[0}, \ t \geq 0, \end{gathered} \tag{4.1}$$

and extended for positive t by the formula $W_t = S_tW^*_{-t}S_{-t}$, $t \geq 0$, is a Markovian cocycle such that $\lim_{t\to+\infty} W_{-t}\eta = M_\Theta\eta$, $\eta \in H_{0]}$. The semigroup R determines a unitary part of the semigroup of isometries $(W_{-t}T_t)_{t\in\mathbb{R}_+}$ in the space $H_{0]}$ associated with it according to the Wold decomposition.

Proof. Notice that the projection $P_{\mathcal{V}}$ and the isometrical operator M_Θ are commuting with the right shifts $T_t = S_{-t}$, $t \geq 0$, and the projections $P_{[t,+\infty)}$. Hence, the family of unitary operators $W_{-t}S_{-t} = (R_tP_{\mathcal{V}^\perp} - S_{-t}P_{\mathcal{V}})P_{H_{0]}}\eta + (M_\Theta P_{[0,t]} + P_{H_{[0}})S_{-t}$, $t \geq 0$, forms a semigroup in H. The restriction $V_t = W_{-t}S_{-t}|_{H_{0]}}$, $t \geq 0$, is a semigroup of nonunitary isometrical operators in $H_{0]}$ with the Wold decomposition $H = \mathcal{V}^\perp \oplus \mathcal{V}$. Really, the restriction $V_t|_{\mathcal{V}}$ is intertwined with the semigroup of right shifts by the isometrical operator M_Θ implementing a unitary map of $H_{0]}$ to $\mathcal{V}$ such that $V_t|_{\mathcal{V}}M_\Theta = M_\Theta T_t$, $t \geq 0$. Thus, $V|_{\mathcal{V}}$ and T are unitarily equivalent. The restriction $V|_{\mathcal{V}^\perp} = R$. Therefore, $V = (W_{-t}T_t)_{t\geq 0}$ is the semigroup of nonunitary isometrical operators with unit index and unitary part R. □

Below, using the model of Markovian cocycle introduced in Theorem 4.1, we construct the Markovian cocycle satisfying the property $W_t - I \in s_2$, $t \in \mathbb{R}$. Cocycles of such type can be named inner for further applications in quantum probability (see [3, 5, 6]). In fact, this condition appears, particularly, as the condition of innerness for the quasifree automorphism of the Fermion algebra, which is generated by the unitary operator W (see [3, 5, 31]). It is possible to explain why this condition appears in the following way. Attempts to define a measure in a Hilbert space H result in constructing the measure of white noise on the space E^* involved in the triple $E \subset H \subset E^*$, where we denote by E^* the adjoint space of linear functionals on the space E which is dense in H (see [20]). This situation is realized, particularly, if $\xi \in H$ are included in the parameter set of the generalized stochastic process. Given $\xi \in E$, $x \in E^*$, denote by $\langle x,\xi\rangle$ the corresponding dual product. Notice that, if $x \in H$, then $\langle x,\xi\rangle = (x,\xi)$ coincides with the inner product in H. Fix a positive bounded operator R in the space H. Then there exists a space (Ω,μ) with the Gaussian measure μ such that $\int_{E^*} e^{i\langle x,\xi\rangle}d\mu(x) = e^{-(\xi,R\xi)}$, $\xi \in E \subset H$. Suppose that a unitary operator W in H does not map elements of H but the measure μ to certain other Gaussian measure $\tilde{\mu}$ such that $\int_{E^*} e^{i<x,\xi>}d\tilde{\mu}(x) = e^{-(\xi,W^*RW\xi)}$, $\xi \in E \subset H$. It is natural to ask: when are the Gaussian measures determined by the operators R and W^*RW equivalent? The Feldman criterion (see [16, 17]) gives the condition $(\xi,R\xi) - (\xi,W^*RW\xi) = (\xi,\Delta\xi)$, where Δ is a Hermitian operator of the Hilbert-Schmidt class. Thus, $R - W^*RW \in s_2$, which can be rewritten as $WR - RW \in s_2$.

The condition of the type given above is satisfied for all positive operators R if $W-I\in s_2$. Notice that the condition of the Feldman type appears in [7] as the condition of quasiequivalence for Gaussian states on the Boson algebra.

Let the inner function Θ involved in the condition of Theorem 4.1 be the Blaschke product (see [32]) constructed from the complex numbers λ_k, $1\le k\le N\le +\infty$, in the following way. By means of [3, 4, 5], take real parts such that $\operatorname{Re}\lambda_k<0$, $\sum_{k=1}^N|\operatorname{Re}\lambda_k|<+\infty$; imaginary parts can be chosen arbitrary, then $\Theta(\lambda)=\prod_{k=1}^N((\lambda+\overline{\lambda}_k)/(\lambda-\lambda_k))$. The Blaschke product $\Theta(\lambda)$ is a regular analytical function in the semiplane $\operatorname{Re}\lambda>0$ and its module equals one on the imaginary axis. The corresponding subspace $\mathcal{V}$ of the Hilbert space $H_{0]}=L^2(\mathbb{R}_+)$, which is invariant with respect to the semigroup of right shifts, is determined by the condition of orthogonality to all exponents $e^{\lambda_k x}$, $1\le k\le N$. Let the functions g_k, $1\le k\le N$, be obtained by the successive orthogonalization of the system $e^{\lambda_k x}$. Then $(g_k,g_l)=\delta_{kl}$ and g_k, $1\le k\le N$, form an orthonormal basis of the space $\mathcal{V}^\perp$. Define a C_0-semigroup of unitaries $R=(R_t)_{t\in\mathbb{R}}$ by the formula $R_tg_k=e^{i\operatorname{Im}\lambda_k t}g_k$, $1\le k\le N$, $t\in\mathbb{R}_+$.

Theorem 4.2. *The Markovian cocycle $W=(W_t)_{t\in\mathbb{R}}$ associated with the inner function Θ and with the semigroup of unitaries R constructed above as in Theorem 4.1 is inner, that is, it satisfies the condition $W_t-I\in s_2$, $t\in\mathbb{R}$.*

Corollary 4.3. *Perturbing the semigroup of right shifts T by the inner Markovian cocycle, it is possible to obtain the semigroup of isometrical operators with the unitary part possessing the pure point spectrum we introduced.*

Proof. The semigroup R is a unitary part of the semigroup $(W_{-t}T_t)_{t\in\mathbb{R}_+}$ in the space $H_{0]}$ by means of Theorem 4.1. The point spectrum of R consists of imaginary numbers $i\operatorname{Im}\lambda_k$, $1\le k\le N\le+\infty$, which can be chosen arbitrary. □

Proof of Theorem 4.2. Notice that $W_{-t}\eta-\eta=0$, $\eta\in\mathcal{H}_{0]}^\perp$, $t>0$. Thus, we need to prove a convergence of the series $\sum_{i=1}^{+\infty}\|W_{-t}\eta_i-\eta_i\|^2$ for an orthonormal basis $(\eta_i)_{i=1}^{+\infty}$ of the space $\mathcal{H}_{0]}$. Represent $W_{-t}-I|_{\mathcal{H}_{0]}}$ as a sum of two parts such that $W_{-t}\eta-\eta=(R_tP_{\mathcal{V}^\perp}S_t-P_{\mathcal{V}^\perp})P_{[t,+\infty)}\eta+(M_\Theta-I)P_{[0,t]}\eta$, $\eta\in H_{0]}$, $t>0$, and prove a convergence of the series associated with these parts, that is,

$$\sum_{k=1}^{+\infty}\left|\left|(R_tP_{\mathcal{V}^\perp}S_t-P_{\mathcal{V}^\perp})P_{[t,+\infty)}\eta_k\right|\right|^2<+\infty, \tag{4.2}$$

$$\sum_{k=1}^{+\infty}\left|\left|(M_\Theta-I)P_{[0,t]}\eta_k\right|\right|^2<+\infty. \tag{4.3}$$

To check (4.2), it is sufficient to prove a convergence of the series $\sum_{k=1}^N\|(R_t-P_{\mathcal{V}^\perp}S_t^*P_{\mathcal{V}^\perp})g_k\|^2$, where the functions $(g_k)_{k=1}^N$ forming the orthonormal basis of the space $\mathcal{V}^\perp$ are obtained by a successive orthogonalization of the exponents

$e^{\lambda_k x}$. One can represent R_t as a sum of $R_t^{(1)}$ and $R_t^{(2)}$, where $R_t^{(1)}g_k = e^{\lambda_k t}g_k$, $R_t^{(2)}g_k = (e^{i\operatorname{Im}\lambda_k t} - e^{\lambda_k t})g_k$, $1 \le k \le N$, $t > 0$. Then the series $\sum_{k=1}^{N}\|(R_t^{(1)} - P_{\mathcal{V}^\perp}S_t^* P_{\mathcal{V}^\perp})g_k\|^2$ converges by the theorem on the triangulation of the truncated shift (see [32]). Notice that $\sum_{k=1}^{N}|e^{i\operatorname{Im}\lambda_k t} - e^{\lambda_k t}| = \sum_{k=1}^{N}|1 - e^{\operatorname{Re}\lambda_k t}|$ and the last series converges because $\sum_{k=1}^{N}|\operatorname{Re}\lambda_k| < +\infty$. Therefore, $R_t^{(2)} \in s_1$ (the first Schatten class) such that

$$\begin{aligned}\sum_{k=1}^{N}&||(R_t - P_{\mathcal{V}^\perp}S_t^* P_{\mathcal{V}^\perp})g_k||^2\\ &\le \sum_{k=1}^{N}||(R_t^{(1)} - P_{\mathcal{V}^\perp}S_t^* P_{\mathcal{V}^\perp})g_k||^2 + \sum_{k=1}^{N}||R_t^{(2)}g_k||^2 < +\infty.\end{aligned}\tag{4.4}$$

To prove (4.3), it is sufficient to find the set of functions $f_k(x)$, $k \in \mathbb{Z}$, which is dense in the space $H_{[0,t]} = P_{[0,t]}H$ with the property that there exists a bounded operator V with the bounded inverse V^{-1} such that the set of functions $(Vf_k)_{k=-\infty}^{+\infty}$ forms an orthonormal basis in $H_{[0,t]}$ and the series $\sum_{k=-\infty}^{+\infty}\|(M_\Theta - I)f_k\|^2 < +\infty$ converges. The set of functions $(f_k)_{k=-\infty}^{+\infty}$ satisfying the property given above is called a Riesz basis of the space $H_{[0,t]}$. A canonical example of the Riesz basis is the set of exponents $f_k = e^{\mu_k x}$, $x \in [0,t]$, with the indicators $\mu_k = -1/2|k| + i(2\pi k/t)$, $k \in \mathbb{Z}$. Put $f_k(x) = 0$ for $x \notin [0,t]$. Then $f_k = f_k^{(1)} - f_k^{(2)}$, where $f_k^{(1)} = e^{\mu_k x}$, $x \in \mathbb{R}_+$, $f_k^{(2)} = e^{\mu_k x}$, $x \ge t$, $f_k^{(1)}(x) = 0$, $x \in \mathbb{R}_-$, and $f_k^{(2)}(x) = 0$, $x < t$. Using the Parseval equality for the Laplace transformation and taking into account that the Blaschke product Θ is an isometrical operator, we obtain

$$\begin{aligned}\sum_{k\in\mathbb{Z}}&||(M_\Theta - I)f_k||^2\\ &\le 2\sum_{k\in\mathbb{Z}}\left(||(M_\Theta - I)f_k^{(1)}||^2 + ||(M_\Theta - I)f_k^{(2)}||^2\right)\\ &= \frac{1}{\pi}\sum_{k\in\mathbb{Z}}\left(||(\Theta - I)\tilde{f}_k^{(1)}||^2 + ||(\Theta - I)\tilde{f}_k^{(2)}||^2\right)\\ &= \frac{2}{\pi}\sum_{k\in\mathbb{Z}}\Big\{\left(||\tilde{f}_k^{(1)}||^2 - \operatorname{Re}\left(\Theta\tilde{f}_k^{(1)}, \tilde{f}_k^{(1)}\right)\right)\\ &\qquad + \left(||\tilde{f}_k^{(2)}||^2 - \operatorname{Re}\left(\Theta\tilde{f}_k^{(2)}, \tilde{f}_k^{(2)}\right)\right)\Big\},\end{aligned}\tag{4.5}$$

where $\tilde{f}_k^{(1)}(\lambda) = 1/\lambda\mu_k$ and $f_k^{(2)} = e^{\mu_k t}(e^{-t\lambda}/(\lambda - \mu_k))$ are the Laplace transformations of the functions $f_k^{(1)}$ and $f_k^{(2)}$. Applying the analytical functions techniques, we get

$$\left(\Theta\tilde{f}_k^{(i)}, \tilde{f}_k^{(i)}\right) = \Theta(-\overline{\mu_k})||f_k^{(i)}||^2, \quad i = 1, 2.\tag{4.6}$$

Notice that

$$\ln\Theta(\lambda) = \sum_{k=1}^{N}\left(\ln\left(1+\frac{\overline{\lambda}_k}{\lambda}\right) - \ln\left(1-\frac{\lambda_k}{\lambda}\right)\right) = \frac{\sum_{k=1}^{N}\operatorname{Re}\lambda_k}{\lambda} + o\left(\frac{1}{\lambda}\right). \tag{4.7}$$

Hence,

$$\Theta(\lambda) = 1 + \frac{\sum_{k=1}^{+\infty}\operatorname{Re}\lambda_k}{\lambda} + o\left(\frac{1}{\lambda}\right). \tag{4.8}$$

Substituting in (4.5) the formulas (4.6) and (4.8), we obtain

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \|(M_\Theta - I)f_k\|^2 &\le C_1 \sum_{k\in\mathbb{Z}} \left(1 - \operatorname{Re}\Theta(-\overline{\mu_k})\right) \\ &\le C_2 \sum_{k\in\mathbb{Z}} \frac{1/2|k|}{1/4k^2 + 4\pi^2k^2/t^2} \le C_3 \frac{1}{|k|^3} < +\infty,\end{aligned} \tag{4.9}$$

where C_1, C_2, and C_3 are some positive constants. □

5. Processes with independent increments in classical and quantum probability: Kolmogorov flows. Denote by $\mathcal{L}_s(H)$ and $\sigma_1(H)$ the sets of linear Hermitian and positive unit-trace operators in a Hilbert space H. In the quantum probability theory, the elements $x \in \mathcal{L}_s(H)$ and $\rho \in \sigma_1(H)$ are called random variables or observables and states of the system, respectively. Consider the spectral decomposition $x = \int \lambda\, dE_\lambda$ of the random variable $x \in \mathcal{L}_s(H)$, where E_λ is a resolution of the identity in H. Then a probability distribution of x in the state $\rho \in \sigma_1(H)$ is defined by the formula $P(x < \lambda) = \operatorname{Tr}\rho E_\lambda$. Thus, the expectation of x in the state ρ can be calculated as $\mathbb{E}(x) = \operatorname{Tr}\rho x$ (see [21, 24]). Notice that the classical random variables from $L^\infty(\Omega)$ can be considered as linear operators of multiplication by the function in the Hilbert space $H = L^2(\Omega)$, where Ω is some probability space. A quantum stochastic process (in the narrow sense of the word) is a strongly continuous family of operators (unbounded in general) $x_t \in \mathcal{L}_s(H)$, $t \in \mathbb{R}$—"quantum observables." In applications (see [8, 9, 33]), one needs to require that the operators x_t be closed and have the invariant common domain which is dense in H. We will call the quantum stochastic process a process with stationary increments if there exists an ultraweak continuous one-parameter group of $*$-automorphisms α_t, $t \in \mathbb{R}$, of the algebra of all bounded operators in H such that $x_{t+s} = x_t + \alpha_t(x_s)$, $s, t \in \mathbb{R}$. We do not suppose that the operators x_t are bounded and assume that the action of $\alpha = (\alpha_t)_{t\in\mathbb{R}}$ is correctly defined on $x = (x_t)_{t\in\mathbb{R}}$. The quantum stochastic process with stationary increments x_t is called a stationary process if there exists $x \in \mathcal{L}_s(H)$ such that $x_t = \alpha_t(x) - x$, $t \in \mathbb{R}$. As in the case of classical stochastic processes, the quantum stochastic process which is continuous in the square mean determines a continuous curve in the Hilbert space with the inner product defined by the expectation. This curve will be denoted by $[x] = ([x_t])_{t\in\mathbb{R}}$. As we have identified continuous curves in a

Hilbert space with classical stochastic processes, we obtain that a quantum stochastic process x can be associated with the classical stochastic process $\xi = [x]$. If x is a process with stationary increments or a stationary process and the state $\rho \in \sigma_1(H)$ is invariant with respect to the action of the group α, that is, $\mathrm{Tr}(\rho\alpha_t(a)) = \mathrm{Tr}(\rho a)$, $t \in \mathbb{R}$, where a is an arbitrary linear combination of the random variables x_t, $t \in \mathbb{R}$, then the classical stochastic process $\xi = [x]$ is also stationary. The group of unitaries U shifting the increments ξ in time is defined by the formula $(U_t\xi_s, \xi_r) = \mathrm{Tr}(\rho\alpha_t(x_s)x_r)$, $s,t,r \in \mathbb{R}$. Because the notion of expectation in the quantum probability theory plays a major role, we will use it to define an independence in the classical probability theory and also, for convenience, to pass to quantum probability in the following. The classical stochastic process $x = (x_t)_{t\in\mathbb{R}}$ is said to be a process with independent increments or a Lévy process if the identity

$$\mathbb{E}(\phi_1(x_{t_1} - x_{s_1})\phi_2(x_{t_2} - x_{s_2}) \cdots \phi_n(x_{t_n} - x_{s_n})) = \prod_{i=1}^{n} \mathbb{E}(\phi_i(x_{t_i} - x_{s_i})) \quad (5.1)$$

holds for arbitrary choice of functions $\phi_i \in L^\infty$ and disjoint intervals (s_i, t_i). There are different approaches to a definition of the Lévy processes in quantum probability (see [21, 24]). Anyway, besides that the condition (5.1) must be satisfied, they involve certain additional conditions concerning the algebraic structure of the process x. One of them is commutativity for increments, that is,

$$[x_{t_1} - x_{s_1}, x_{t_2} - x_{s_2}] = 0 \quad (5.2)$$

for disjoint intervals (s_i, t_i). Condition (5.2) is associated with the bosonic independency and it is not a unique possibility (for the fermionic case, see [9]). Notice that, for the quantum processes with independent increments, it is possible to define the representation of the Lévy-Hinchin type (see [22, 23]). For convenience, we recall Definition 1.1. In formula (1.6), we mean that the intersection of σ-algebras contains only two events which are the empty set and the whole space Ω. Let $\mathcal{M}_{t]}$ be generated by the events associated with the stationary stochastic process ξ_s, $s < t$. The investigation of the conditions which lead to the Kolmogorov flow generated by ξ is given in [25]. In particular, the flow of the Wiener process is a Kolmogorov flow (see [20]). Notice that to obtain the Kolmogorov flow from the stochastic process, it is not necessary to require the independence of the increments. Consider the quantum stochastic process with stationary increments $x = (x_t)_{t\in\mathbb{R}}$. In the quantum probability theory, a role of the σ-algebras of events is played by the von Neumann algebras generated by the quantum random variables. In this way, the conditional expectation is a completely positive projection on the von Neumann algebra (see [21, 24]). Let $\mathcal{M}_{t]} = \{x_s, s < t\}''$ and $\mathcal{M}_{[t} = \{x_s, s > t\}$ be the von Neumann algebras generated by the past before the moment t and the future after the moment t of the quantum stochastic process with stationary increments x. Here A' denotes the set of all bounded operators in H which are commuting with the operators (not bounded in general) from the set A. Notice that the operators x_s, $s \le t$, and x_s, $s > t$, are affiliated to the von Neumann algebras $\mathcal{M}_{t]}$ and $\mathcal{M}_{[t}$, respectively. It follows from the stationarity of the increments of the process x that $\mathcal{M}_{t]} = \alpha_t(\mathcal{M}_{0]})$, $t \in \mathbb{R}$, for the group α shifting the increments. It is natural to call the group of automorphisms α a Kolmogorov flow on the von Neumann algebra $\mathcal{M}$ if the conditions (1.4), (1.5), and (1.6) are satisfied, where, in condition (1.6), the trivial σ-algebra $\{\emptyset, \Omega\}$ is replaced by the trivial von Neumann algebra $\{\mathbb{C}\mathbf{1}\}$ containing only operators which are multiple of the identity. Thus we obtain the condition of the algebraic Kolmogorov flow α (see [15]).

For classical random variables ξ with $\mathbb{E}\xi = 0$, the condition $D\xi = \mathbb{E}\xi^2 = 0$ means that $\xi = 0$ almost surely. The situation is different in the quantum case. For a quantum random variable x with $\mathbb{E}x = 0$, the equality $Dx = \mathbb{E}x^2 = \operatorname{Tr}\rho x^2 = 0$ does not imply that $x = 0$. The property of this type characterizes the expectation $\mathbb{E} = \mathbb{E}_\rho$. If the situation appears, that is, $\operatorname{Tr}\rho x = 0$ implies $x = 0$ for all positive operators x belonging to a certain algebra $\mathcal{M}$, then ρ is said to determine a faithful state $\mathbb{E}(\cdot) = \operatorname{Tr}\rho\,\cdot$ on $\mathcal{M}$.

Proposition 5.1. *Let the expectation $\mathbb{E}(\cdot) = \operatorname{Tr}\rho\,\cdot$ determine a faithful state on the von Neumann algebra $\mathcal{M}$ generated by the stationary increments of the quantum stochastic process satisfying the condition (5.1). Then the group of automorphisms α shifting the increments in time is a Kolmogorov flow.*

Proof. Let $x \in \cap_t \mathcal{M}_{t]}$, then given $y \in \cup_t \mathcal{M}_{[t} = \mathcal{M}$, condition (5.1) implies that $\mathbb{E}((x - \mathbb{E}(x))y) = \mathbb{E}(x - \mathbb{E}(x)\mathbf{1})\mathbb{E}(y) = 0 = \operatorname{Tr}\rho(x - \mathbb{E}(x))y$. Put $y = x - \mathbb{E}(x)$. Because the state ρ is faithful, it follows that the identity $\operatorname{Tr}\rho(x - \mathbb{E}(x))^2 = 0$ implies $x = \mathbb{E}(x) \in \{\mathbb{C}\mathbf{1}\}$. □

In the following section, we will give an important example of the algebraic Kolmogorov flow, which complements the example of Proposition 5.1 in some sense.

6. Cohomology of groups as a language describing the perturbations of the space of all functionals from stochastic process. We recall some notion of the cohomology of groups (see, e.g., [12, 18]). Define a certain action α of the real line $\mathbb{R}$ as an additive group on the algebra $\mathcal{M}$. The element $I \in \operatorname{Hom}(\mathbb{R}^k, \mathcal{M})$ is said to be (additive) $(k - \alpha)$-cocycle if the following identity is satisfied:

$$\begin{aligned}\alpha_{t_1} I(t_2, t_3, \ldots, t_{k+1}) &- I(t_1 + t_2, t_3, \ldots, t_{k+1}) + \cdots \\ &+ (-1)^i I(t_1, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{k+1}) + \cdots \\ &+ (-1)^{k+1} I(t_1, \ldots, t_k) = 0,\end{aligned} \tag{6.1}$$

$t_i \in \mathbb{R}$, $1 \le i \le k+1$. The $(k-\alpha)$-cocycle I is said to be a coboundary if there exists an element $J \in \operatorname{Hom}(\mathbb{R}^{k-1}, \mathcal{M})$ such that

$$\begin{aligned} I(t_1,\dots,t_k) = \alpha_{t_1}(J(t_2,\dots,t_k)) - J(t_1+t_2,\dots,t_k) + \cdots \\ + (-1)^i J(t_1,\dots,t_{i-1},t_i+t_{i+1},t_{i+2},\dots,t_k) + \cdots \\ + (-1)^{k+1} J(t_1,\dots,t_{k-1}), \end{aligned} \tag{6.2}$$

$t_i \in \mathbb{R}$, $1 \le i \le k$. Denote by C^k and B^k the sets of all additive $(k-\alpha)$-cocycles and $(k-\alpha)$-coboundaries, correspondingly. Then $H^k = H^k(\alpha) = C^k/B^k$ is called a kth group of cohomologies of α with values in $\mathcal{M}$. Define a cohomological multiplication $\cup : H^k \times H^l \to H^{k+l}$ by the formula

$$(I_k \cup I_l)(t_1,\dots,t_{k+l}) = I_k(t_1,\dots,t_k)\alpha_{t_1+\cdots+t_k}(I_l(t_{k+1},\dots,t_{k+l})), \tag{6.3}$$

$I_k \in C^k$, $I_l \in C^l$, $t_i \in \mathbb{R}$, $1 \le i \le k+l$. The group $H = H(\alpha) = \oplus_{i=1}^{+\infty} H^i$ is a graded ring with respect to the multiplication $\cup$. Notice that a group of 0-cohomologies was omitted because we do not need it in the following. Now let α and α' be two actions on the algebra $\mathcal{M}$. We will say that the rings $H(\alpha)$ and $H(\alpha')$ are isomorphic if there is a one-to-one correspondence $w : H(\alpha) \to H(\alpha')$ mapping each $H^k(\alpha)$ to $H^k(\alpha')$ and the action of α to the action of α'.

The one-parameter family of automorphisms $w = (w_t)_{t\in\mathbb{R}}$ of the algebra $\mathcal{M}$ is said to be *a (multiplicative) α-cocycle* if the following condition holds: $w_{t+s} = w_t \circ \alpha_t \circ w_s \circ \alpha_{-t}$, $s,t \in \mathbb{R}$, $w_0 = \mathrm{Id}$. We will call a multiplicative α-cocycle *Markovian* with respect to a certain set $\mathcal{I}$ of additive $(1-\alpha)$-cocycles $I(t) \in \mathcal{I}$ if the condition $w_t(I(t+s)-I(t)) = I(t+s)-I(t)$, $t,s \ge 0$, is satisfied. The following proposition can be considered as some “abstract generalization” of the properties of the Markovian perturbations we have described in the previous sections.

Proposition 6.1. *Let the ring $A \subset H(\alpha)$ be generated by a set of additive $(1-\alpha)$-cocycles $I(t) \in \mathcal{I}$ and the multiplication $\cup$. Then, a multiplicative α-cocycle w which is Markovian with respect to $\mathcal{I}$ determines the homomorphism of A into $H(\alpha')$, where the group α' is defined by the formula $\alpha'_t = w_t \circ \alpha_t$, $t \in \mathbb{R}$. The image A' of this homomorphism is a ring generated by the set of additive $(1-\alpha')$-cocycles $I'(t) = w_t(I(t))$, $t \le 0$, $I'(t) = I(t)$, $t > 0$, $I(t) \in \mathcal{I}$.*

Proof. Check that $I'(t) = w_t(I(t))$ satisfies the condition for a $(1-\alpha')$-cocycle. Fix $s,t > 0$ and notice that

$$\begin{aligned} w_{-s}(I(-s)) &= w_{-s-t+t}(I(-s)) \\ &= w_{-s-t} \circ \alpha_{-s-t} \circ w_t \circ \alpha_{s+t}(I(-s)) \\ &= w_{-s-t} \circ \alpha_{-s-t} \circ w_t(I(t) - I(s+t)) \\ &= w_{-s-t} \circ \alpha_{-s-t}(I(t) - I(s+t)) \\ &= w_{-s-t}(I(-s)) \end{aligned} \tag{6.4}$$

by means of the cocycle condition and the Markovian property for w. Therefore,

$$\begin{aligned} w_{-t-s}(I(-t-s)) &= w_{-t-s}(I(-t)+\alpha_{-t}(I(-s))) \\ &= w_{-t-s}(I(-t)) + w_{-t-s}\circ\alpha_{-t}(I(-s)) \\ &= w_{-t}(I(-t)) + w_{-t}\circ\alpha_{-t}\circ w_{-s}(I(-s)) \\ &= I'(-t)+\alpha'_{-t}(I'(-s)). \end{aligned} \tag{6.5}$$

Using the Markovian property of w, we obtain $I'(t) = I(t) = -\alpha_t(I(-t)) = -w_t\circ\alpha_t\circ w_{-t}(I(-t)) = -\alpha'_t(I'(-t))$, $t>0$, because $w_t\circ\alpha_t\circ w_{-t}\circ\alpha_{-t} = \mathrm{Id}$ due to the cocycle condition for w. □

We give examples showing how the language of the theory of cohomology of groups can describe the set of all functionals from the classical and quantum stochastic processes with the stationary independent increments.

6.1. The Wiener process. Consider the Wiener process $\{B(t), t\in\mathbb{R}\}$ implemented on the probability space (Ω,μ). Let $S=(S_t)_{t\in\mathbb{R}}$ be the group of transformations shifting the increments of the process in time. Then the Wiener process satisfies the condition of the additive $(1-S)$-cocycle, that is, $B(t+s) = B(t)+S_t(B(s))$, $s,t\in\mathbb{R}$. Consider the ring of cohomologies $H(S)$ for the group S with values in $L^\infty(\Omega)$. Then the Wiener process $B(t)$ generates the subring $A\subset H(S)$. Denote by $\mathcal{H}$ the Hilbert space of all L^2-functionals from the Wiener process. As it is known, one can define for $\mathcal{H}$ the Wiener-Itô decomposition $\mathcal{H} = \oplus_{i=0}^{+\infty}\mathcal{H}_i$ in the orthogonal sum of spaces formed by polynomials of increasing degrees (see, e.g., [20]). Notice that the representation of the graded ring A as the sum of cohomologies of all degrees $A=\oplus_{i=1}^{+\infty}A_i$ is a cohomological analog of the Wiener-Itô decomposition. Take a function $a(x)\in L^2_{\mathrm{loc}}(\mathbb{R})$ and determine a one-parameter family of linear maps $w=(w_t)_{t\in\mathbb{R}}$ acting on A by

$$\begin{gathered} w_t(B(t+s)-B(t)) = B(t+s)-B(t), \quad s\ge 0, \\ w_t(B(s)) = \exp\left\{-\frac{1}{2}\int_0^s a(x)\,dB(x) - \frac{1}{4}\int_0^t |a(x)|^2dx\right\} \\ \cdot\left(B(s)+\int_0^t a(x)\,dB(x)\right), \quad s\le t. \end{gathered} \tag{6.6}$$

Every w_t defines a unitary transformation in the space of L^2-functionals of the white noise (see [20]) and satisfies the property for the Markovian cocycle by the definition. Thus the following proposition holds.

Proposition 6.2. *The family w is a Markovian cocycle.*

6.2. Quantum noises. Let G be a certain Lie algebra with the involution. Then (see [36]) there exists a one-parameter family of $*$-homomorphisms

$j : x \to j_t(x) \in \mathcal{L}(H)$ mapping each element $x \in G$ to a strongly continuous one-parameter family of operators $j_t(x)$, $t \in \mathbb{R}$, in the symmetric Fock space $\mathcal{H} = \mathcal{F}(L^2(\mathbb{R},\mathcal{K}))$ over the one-particle Hilbert space $L^2(\mathbb{R},\mathcal{K})$ consisting of functions on the real line with values in the Hilbert space $\mathcal{K}$. Every homomorphism j_t preserves the commutator such that $[j_t(x), j_t(y)] = j_t([x,y])$, $x, y \in G$, and satisfies the condition of additive $(1-\alpha)$-cocycle with respect to the group of automorphisms α generated by the group of shifts in the space $L^2(\mathbb{R},\mathcal{K})$. Then j can be extended to a $*$-homomorphism of the universal enveloping algebra $\mathcal{U}(G)$ determining the quantum Lévy process. Notice that the classification of possible Lévy processes over Lie algebras is given in [30, 37]. The extension to a universal enveloping algebra is constructed in [19]. In the case if $j_t(x) \in \mathcal{L}_s(\mathcal{H})$, $t \in \mathbb{R}$, under fixed $x \in G$, the one-parameter family $x_t = j_t(x)$, $t \in \mathbb{R}$, is the quantum stochastic process in the sense of the definition given in Section 5. Moreover, $x_t = j_t(x)$ is the process with independent increments, where the independence means that the conditions (5.1) and (5.2) of Section 5 are satisfied. Nevertheless, we will consider the curves $x_t = j_t(x)$, $t \in \mathbb{R}$, consisting of non-Hermitian operators as well. The scope of all curves $x_t = j_t(x)$, $x \in G$, will be named *a quantum noise* because the increments of all curves x_t are independent. Notice that the quantum stochastic processes $j_t(x)$ can play the roles of the Wiener and Poisson processes in the quantum case (see [21, 24, 33] for references and comments). Consider the ring of cohomologies A generated by additive $(1-\alpha)$-cocycles $j_t(x)$, $x \in G$. Notice that the Fock space can be factorized such that $\mathcal{F}(L^2(\mathbb{R},\mathcal{K})) = \mathcal{F}(L^2(\mathbb{R}_+,\mathcal{K})) \otimes \mathcal{F}(L^2(\mathbb{R}_-,\mathcal{K}))$. In the following, we construct the example of a multiplicative α-cocycle generating the homomorphism of A which uses this factorization. Put $w_{-t}(\cdot) = W_t \otimes I \cdot W_t^* \otimes I$, $t \geq 0$, where the family of unitary operators W_t, $t \geq 0$, in the Fock space $\mathcal{F}(L^2(\mathbb{R}_+,\mathcal{K}))$ satisfies the quantum stochastic differential equation constructed in [33, Example 25.17, page 198], that is,

$$dW = W\left(dA_m^+ + d\Lambda_{U^{-1}} - dA_{U^{-1}m} - \frac{1}{2}\langle\langle m, m\rangle\rangle\right), \tag{6.7}$$

where A_m^+ and A_m are the basic processes of creation and annihilation, respectively, generated by a function $m \in \mathcal{K}$, and $\Lambda_{U^{-1}}$ is the number of particles process generated by a unitary operator U (see [33]).

Let $\mathcal{M}_{t]}$ and $\mathcal{M}_{[t}$ be the von Neumann algebras generated by the increments of the quantum noise $j_t(x)$, $x \in G$, before the moment of time t and after the moment of time t, respectively. Then, for the von Neumann algebra $\mathcal{M}$ generated by all operators $j_t(x)$, $x \in G$, $t \in \mathbb{R}$, we obtain $\mathcal{M} = \vee_t \mathcal{M}_{t]} = \vee_t \mathcal{M}_{[t}$, where $\mathcal{M}_{t]} = \alpha_t(\mathcal{M}_{0]})$, $t \in \mathbb{R}$.

Proposition 6.3. *Let the von Neumann algebra $\mathcal{M}$ generated by increments of the quantum noise $j_t(x)$, $x \in G$, be a factor. Then the group of automorphisms α is a Kolmogorov flow on $\mathcal{M}$.*

Proof. Take $x \in \cap_t \mathcal{M}_{t]} \subset \mathcal{M}$. The increments of the quantum noise are independent quantum random variables by the condition of the proposition. Hence, the algebra $\mathcal{M}_{[t}$ belongs to the commutant of the algebra $\mathcal{M}_{t]}$. Therefore, the operator x is commuting with all operators $y \in \cup_t \mathcal{M}_{[t} = \mathcal{M}$. As the von Neumann algebra $\mathcal{M}$ is a factor, that is, $\mathcal{M} \cap \mathcal{M}' = \{\mathbb{C}\mathbf{1}\}$, we get $x = \mathrm{const}\,\mathbf{1}$. □

Notice that in the applications the algebra $\mathcal{M}$ is often the algebra of all bounded operators, that is, the factor of type I. In this case, the condition of Proposition 6.3 is satisfied. Take into account that the expectation in the quantum case is often determined by a pure state ρ such that $\mathbb{E}(x) = (\Omega, x\Omega)$. Here Ω is some vector in the Hilbert space $\mathcal{H}$, where the operators x act. The pure state on the algebra of all bounded operators in $\mathcal{H}$ cannot be faithful (see [11]). Hence, Proposition 6.3 complements Proposition 5.1.

7. Markovian perturbations of stationary quantum stochastic processes. At first, we will generalize the definition of the quantum stochastic process given in Section 5. Following [2], *a quantum stochastic process* will refer to the one-parameter family of $*$-homomorphisms $j = (j_t)_{t\in\mathbb{R}}$ from certain algebra with the involution $\mathcal{A}$ to the algebra of linear (unbounded in general) operators $\mathcal{L}(H)$ in some Hilbert space $\mathcal{H}$. We will suppose that $j_0(x) = 0$, $x \in \mathcal{A}$. For $\mathcal{A}$, one can, in particular, take a certain Lie algebra. Consider the minimal von Neumann algebra $\mathcal{M}$ generated by all operators $j_t(x)$, $x \in \mathcal{A}$, such that $\mathcal{M}=\{j_t(x), x \in \mathcal{A}\}''$, where $'$ denotes the commutant in the algebra $\mathcal{B}(H)$ consisting of all bounded operators in $\mathcal{H}$. For fixed $s, t \in \mathbb{R}$, we denote by $\mathcal{M}_{t]}$, $\mathcal{M}_{[t}$, and $\mathcal{M}_{[s,t]}$ the von Neumann algebras generated by the operators $\{j_t(x) - j_r(x),\ r \le t\}$, $\{j_r(x) - j_t(x),\ r \ge t\}$, and $\{j_r(x) - j_t(x),\ s \le r \le t\}$, respectively. Further, we will assume that the state $\rho > 0$, $\mathrm{Tr}\,\rho = 1$, determining the expectation $\mathbb{E}(y) = \mathrm{Tr}\,\rho y$ for quantum random variables $y \in \mathcal{M}$ is fixed. Moreover, let $\mathcal{M}$ be in "the standard form" with respect to the state ρ, that is, the Hilbert space $\mathcal{H}$, where $\mathcal{M}$ acts, is defined by the map $x \to [x]$ from $\mathcal{M}$ to the dense set in $\mathcal{H}$ such that the inner product is given by the formula $([x],[y]) = \mathbb{E}(xy^*)$, $x, y \in \mathcal{M}$. The quantum stochastic process j is said to be *stationary* if there exists a group of automorphisms $\alpha_t \in \mathrm{Aut}(\mathcal{M})$, $t \in \mathbb{R}$, whose actions are correctly defined on the operators $j_s(x)$, $\mathcal{A}$ such that $\alpha_t(j_s(x)) = j_{s+t}(x)$, $x \in \mathcal{A}$, and the expectation $\mathbb{E}(\cdot)$ is invariant with respect to the group $\alpha = (\alpha_t)_{t\in\mathbb{R}}$, that is, $\mathbb{E}(\alpha_t(y)) = \mathbb{E}(y)$, $y \in \mathcal{M}$. The stationary quantum stochastic process is a particular case of the quantum stochastic process with the stationary increments generated by $*$-homomorphisms j_t which determine the curves $j_t(x)$ being additive $(1-\alpha)$-cocycles for every fixed $x \in \mathcal{A}$. For the example of the quantum stochastic process with stationary increments, the quantum noise can be chosen. We will suppose that α has ultraweak continuous orbits; that is, all functions $\eta(\alpha_t(y))$ are continuous in t for arbitrary $\eta \in \mathcal{M}_*$, $y \in \mathcal{M}$. The one-parameter family of $*$-automorphisms w_t, $t \in \mathbb{R}$, of the algebra $\mathcal{M}$ is said to be a *(multiplicative) Markovian α-cocycle* if the following two conditions are satisfied:

(i) $w_{s+t} = w_s \circ \alpha_s \circ w_t \circ \alpha_{-s}$, $s,t \in \mathbb{R}$,

(ii) $w_t(x) = x$, $x \in \mathcal{M}_{[t}$, $t \geq 0$.

Notice that if the von Neumann algebra $\mathcal{M}_{[t}$ is generated by the increments of the quantum noise $\{j_t(x), x \in \mathcal{A}\}$ the Markovian cocycle w in the sense of the definition given here is a cocycle, being Markovian with respect to the set $\{j_t(x), x \in \mathcal{A}\}$ in the sense of the definition of Section 6.

Proposition 7.1. *Let $j_t(x)$, $x \in \mathcal{A}$, and w be the quantum noise and the Markovian cocycle, respectively. Suppose that w does not change values of the expectation $\mathbb{E}$ determining the probability distribution of j. Then the one-parameter family of homomorphisms $\tilde{j}_t(x) = w_t \circ j_t(x)$, $t \leq 0$, and $\tilde{j}_t(x) = j_t(x)$, $t > 0$, $x \in \mathcal{A}$, is the quantum noise isomorphic to the initial one. In particular, the processes $\tilde{j}_t(x)$, $x \in \mathcal{A}$, have independent increments.*

Proof. The quantum noise j is a quantum stochastic process with independent increments. Denote by α the corresponding group of automorphisms shifting the increments in time. Then, due to Proposition 6.1, $\tilde{j}_t(x)$ is an additive $(1-\tilde{\alpha})$-cocycle for each fixed $x \in \mathcal{A}$, where the group $\tilde{\alpha}$ consists of automorphisms $\tilde{\alpha}_t = w_t \circ \alpha_t$, $t \in \mathbb{R}$. By means of property (5.2) guaranteeing the independence of increments of the quantum noise j, we obtain the commutator

$$[j_{t_1}(x_1) - j_{s_1}(x_1), j_{t_2}(x_2) - j_{s_2}(x_2)] = 0 \tag{7.1}$$

for all $x_1, x_2 \in \mathcal{A}$ and disjoint intervals (s_1, t_1) and (s_2, t_2). We will prove that this property holds for the process $\tilde{j}$ also. For s_1, t_1, s_2, and t_2, (7.1) takes place because $\tilde{j}_r(x) = j_r(x)$, $x \in \mathcal{A}$, $r \geq 0$. The identity for the case where at least one of s_1, t_1, s_2, and t_2 is less than zero can be obtained by applying the automorphisms $\tilde{\alpha}_{-r} = w_{-r} \circ \alpha_{-r}$, $r > 0$, to formula (7.1). In fact, the automorphism $\tilde{\alpha}_{-r}$ shifts the increments for r units backward, and formula (7.1), which is true for the process $\tilde{j}$ with positive values s_1, t_1, s_2, and t_2, automatically appears to be true for negative values. Moreover, arguing in the way given above, it is easy to obtain that all algebraic properties satisfied for the operators $j_t(x)$, $x \in \mathcal{A}$, $t \in \mathbb{R}$, are also satisfied for the operators $\tilde{j}_t(x)$, $x \in \mathcal{A}$, $t \in \mathbb{R}$. It remains to check the condition of independence (5.1) for increments of the process $\tilde{j}$. But it holds because it is true for the process j and the expectation is invariant with respect to the action of the cocycle w by the condition. Thus, the processes j and $\tilde{j}$ are isomorphic. □

Consider the restriction $\beta_t = \alpha_{-t}$, $t \in \mathbb{R}_+$, to the subalgebra $\mathcal{M}_{0]}$ which is invariant with respect to the action of α_{-t}, $t \in \mathbb{R}_+$. The unital semigroup $\beta = (\beta_t)_{t \in \mathbb{R}_+}$ consists of the endomorphisms of $\mathcal{M}_{0]}$ possessing the property $\beta(\mathcal{M}_{0]}) \neq \mathcal{M}_{0]}$ and has the orbits continuous in the sense that $\eta(\beta_t(x))$ is a continuous function for all $\eta \in \mathcal{M}_{0]*}$, $x \in \mathcal{M}_{0]}$. Hence, β is an E_0-semigroup by the definition introduced in [34]. If α is a Kolmogorov flow, the E_0-semigroup β is a semiflow of Powers shifts, that is, each β_t, $t > 0$, is a Powers shift [34] such that $\cap_{n\in\mathbb{N}}\beta_{tn}(\mathcal{M}_{0]}) = \{\mathbb{C}\mathbf{1}\}$, $t > 0$ (see [14]). Notice that the Markovian cocycle w generates a new group of automorphisms $\tilde{\alpha}_t = w_t \circ \alpha_t$, $t \in \mathbb{R}$, on the von Neumann algebra $\mathcal{M}$ and a new E_0-semigroup $\tilde{\beta}_t = w_{-t} \circ \beta_t$, $t \in \mathbb{R}_+$, on the von Neumann algebra $\mathcal{M}_{0]}$. Recall that the conditional expectation of the unital algebra $\mathcal{M}$ onto the unital algebra $\mathcal{N} \subset \mathcal{M}$ is a completely positive projection of $\mathcal{M}$ onto $\mathcal{N}$ (see, e.g., [21, 24]). Using the techniques in [13], one can exclude the maximum subalgebra of the algebra $\mathcal{M}_{0]}$ such that the restriction of the semigroup $\tilde{\beta}$ to it is a semigroup of automorphisms and there exists a conditional expectation onto this algebra. The quantum random variables $\tilde{j}_t(x) = w_t \circ j_t(x)$, $x \in \mathcal{A}$, generate the von Neumann subalgebras $\tilde{\mathcal{M}} = \{\tilde{j}_t(x),\ x \in \mathcal{A},\ t \in \mathbb{R}\}''$ and $\tilde{\mathcal{M}}_+ = \{\tilde{j}_{-t}(x),\ x \in \mathcal{A},\ t \in \mathbb{R}_+\}''$ of the algebras $\mathcal{M}$ and $\mathcal{M}_{0]}$, respectively. Suppose that $\mathcal{M}$ is a factor. Then, due to Proposition 6.3, α is a Kolmogorov flow on $\mathcal{M}$, and by means of Proposition 7.1, the restriction $\tilde{\alpha}|_{\tilde{\mathcal{M}}}$ is also a Kolmogorov flow. Thus, $\tilde{\beta}|_{\tilde{\mathcal{M}}_+}$ is a semiflow of Powers shifts. The Wold decomposition (3.1) allows to uniquely determine the stochastic process with noncorrelated increments associated with the stationary process. In the following theorem we establish the possibility to exclude a restriction of the group of automorphisms obtained by a perturbation of the Kolmogorov flow generated by the quantum noise, which is isomorphic to the initial Kolmogorov flow. In this way, our conjecture can be considered as some analogue of the Wold decomposition for the quantum case.

Theorem 7.2. *Let the group of automorphisms $\tilde{\alpha}$ on the von Neumann factor $\mathcal{M}$ be obtained through a Markovian cocycle perturbation of the Kolmogorov flow generated by the quantum noise j with the expectation defining the probability distribution, which is invariant with respect to the Markovian cocycle w. Then there exists a subfactor $\tilde{\mathcal{M}} \subset \mathcal{M}$ such that the restriction $\tilde{\alpha}|_{\tilde{\mathcal{M}}}$ is the Kolmogorov flow generated by the quantum noise $\tilde{j}$ which is isomorphic to the initial one. The limit $\lim_{t\to+\infty} w_{-t} = w_{-\infty}$ correctly defines a normal $*$-endomorphism $w_{-\infty}$ with the property $\tilde{\mathcal{M}} = w_{-\infty}(\mathcal{M})$, $\tilde{j}_t = w_{-\infty} \circ j_t$, $t \in \mathbb{R}$.*

Proof. The first part of the theorem follows from Proposition 7.1. We will prove that there exists the limit $\lim_{t\to+\infty} w_{-t} = w_{-\infty}$ defining the normal $*$-endomorphism on $\mathcal{M}$ with the properties we claimed. Notice that $w_{-t-s}(y) = w_{-t}(y)$, $y \in \mathcal{M}_{t]}$. It follows that the limit exists on the dense set of elements. In the following, we will prove that it exists. The formulas $(U_t[x],[y]) = \mathbb{E}(\alpha_t(x)y^*)$ and $(W_t[x],[y]) = \mathbb{E}(w_t(x)y^*)$, $x, y \in \mathcal{M}$, define a strongly continuous group $U = (U_t)_{t\in\mathbb{R}}$ of unitary operators in $\mathcal{H}$ and a unitary U-cocycle $W = (W_t)_{t\in\mathbb{R}}$, respectively. Let $\mathcal{H}_{[t}$ be a subspace of $\mathcal{H}$ generated by the elements $[x]$, $x \in \mathcal{M}_{[t}$. Then $W_t\xi = \xi$, $\xi \in \mathcal{H}_{[t}$, $t \geq 0$, by means of the Markovian property for w. Therefore, W is a Markovian U-cocycle in the sense of the definition of Section 2, and the limit $s-\lim_{t\to+\infty} W_{-t} = W_{-\infty}$ exists due to Proposition 2.2. Notice that $w_t(x) = W_t x W_t^*$, where $x \in \mathcal{M}$ and $t \in \mathbb{R}$. It implies that $w_{-t-s}(x)\xi = W_{-t-s}xW_{-t-s}\xi = W_{-t-s}xW_{-t}\xi \to W_{-\infty}xW_{-t}\xi$, where $x \in \mathcal{M}$, $\xi \in \mathcal{H}_{[-t}$, and $s \to +\infty$. Hence, the limit $\lim_{t\to+\infty} w_{-t}(x)\xi$ exists for a dense set of the elements $\xi \in \mathcal{H}$. Because w_{-t} is an automorphism, we get $\|w_{-t}(x)\| = \|x\|$. Therefore, the strong limit $s-\lim_{t\to+\infty} w_{-t}(x)$ is defined by the Banach-Steinhaus theorem for all $x \in \mathcal{M}$. Moreover, the limiting map $w_{-\infty}$ preserves the identity because all w_{-t} satisfy this property and $\|w_{-\infty}\| = 1$. In this way, the map $w_{-\infty}$ is positive (see [11]). On the other hand, $w_{-\infty}$ is a normal $*$-endomorphism for it is a limit of the series of normal $*$-automorphisms w_{-t}. Thus, $w_{-\infty}$ is completely positive. Notice that $w_{-\infty}(\mathcal{M}_{0]}) = \tilde{\mathcal{M}}_{0]}$. The Markovian property gives $w_{-t-s}(y) = w_{-t}(y)$, where $y \in \mathcal{M}_{[-t}$ and $s,t \geq 0$. Due to Proposition 7.1, $\tilde{j}_{-t} = w_{-t} \circ j_{-t}$, $t \geq 0$. Hence, $\tilde{j}_{-t} = w_{-t-s} \circ j_{-t} = w_{-\infty} \circ j_{-t}$, $t \geq 0$. For positive values of time, $\tilde{j}_t = j_t = w_{-\infty} \circ j_t$, $t \geq 0$, by means of $w_{-t}(y) = w_{-\infty}(y) = y$, where $t \geq 0$ and $y \in \mathcal{M}_{[0}$. $\square$

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