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# SELF-SIMILAR RANDOM FRACTAL MEASURES USING CONTRACTION METHOD IN PROBABILISTIC METRIC SPACES

JÓZSEF KOLUMBÁN, ANNA SOÓS, and IBOLYA VARGA

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Self-similar random fractal measures were studied by Hutchinson and Rüschendorf. Working with probability metric in complete metric spaces, they need the first moment condition for the existence and uniqueness of these measures. In this paper, we use contraction method in probabilistic metric spaces to prove the existence and uniqueness of self-similar random fractal measures replacing the first moment condition.

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**1. Introduction.** Contraction methods for proving the existence and uniqueness of nonrandom self-similar fractal sets and measures were first applied by Hutchinson [7]. Further results and applications to image compression were obtained by Barnsley and Demko [3] and Barnsley [2]. At the same time, Falconer [5], Graf [6], and Mauldin and Williams [13] randomized each step in the approximation process to obtain self-similar random fractal sets. Arbeiter [1] and Olsen [15] studied self-similar random fractal measures applying nonrandom metrics. More recently, Hutchinson and Rüschendorf [8, 9, 10] introduced probability metrics defined by expectation for random measure and established existence, uniqueness, and approximation properties of self-similar random fractal measures. In these works a finite first moment condition is essential.

In this paper, we show that, using probabilistic metric spaces techniques, we can weaken the first moment condition for the existence and uniqueness of self-similar measures.

The theory of probabilistic metric spaces, introduced in 1942 by Menger [14], was developed by numerous authors, as it can be realized upon consulting [4, 18] and the references therein. The study of contraction mappings for probabilistic metric spaces was initiated by Sehgal [19] and Sherwood [20].

**2. Self-similar random fractal measures.** Based on contraction properties of random scaling operators with respect to $l_p^*$ and $l_p^{**}$, for $0 < p < \infty$, on a space of random measures and their distributions, respectively, defined below, Hutchinson and Rüschendorf [8, 9, 10] gave a simple proof for the existence and uniqueness of invariant random measures. The underlying probability space for the iteration procedure is also generated by selecting independent and identically distributed (i.i.d.) scaling laws for measures.

Let $(X,d)$ be a complete separable metric space.

**Definition 2.1.** A *scaling law with weights* is a $2N$-tuple

$$\mathbf{S} := (p_1, S_1, \dots, p_N, S_N), \quad N \geq 1, \tag{2.1}$$

of positive real numbers $p_i$ such that $\sum_{i=1}^{N} p_i = 1$ and of Lipschitz maps $S_i : X \to X$.

Let $r_i = \operatorname{Lip} S_i$, $i \in \{1,\dots,N\}$. Denote by $M = M(X)$ the set of finite-mass Radon measures on $X$ with weak topology. If $\mu \in M$, then the measure $\mathbf{S}\mu$ is defined by

$$\mathbf{S}\mu = \sum_{i=1}^{N} p_i S_i \mu, \tag{2.2}$$

where $S_i\mu$ is the usual push-forward measure, that is,

$$S_i\mu(A) = \mu\big(S_i^{-1}(A)\big) \quad \text{for } A \subseteq X. \tag{2.3}$$

**Definition 2.2.** The measure $\mu$ *satisfies the scaling law* $\mathbf{S}$ or *is a self-similar fractal measure* if $\mathbf{S}\mu = \mu$.

Let $M_q$ denote the set of unit mass Radon measures $\mu$ on $X$ with finite $q$th moment; that is,

$$M_q = \left\{ \mu \in M \mid \mu(X) = 1, \int_X d^q(x,a)\, d\mu(x) < \infty \right\}, \tag{2.4}$$

for some (and hence any) $a \in X$. Note that, if $p \geq q$, then $M_p \subset M_q$.

**Definition 2.3.** The *minimal metric* $l_q$ on $M_q$ is defined by

$$l_q(\mu,\nu) = \inf \left\{ \left( \int_X d^q(x,y)\, d\gamma(x,y) \right)^{1/q \wedge 1} \mid \pi_1\gamma = \mu,\ \pi_2\gamma = \nu \right\}, \tag{2.5}$$

where $\wedge$ denotes the minimum of the relevant numbers and $\pi_i\gamma$ denotes the $i$th marginal of $\gamma$, that is, projection of the measure $\gamma$ on $X \times X$ onto the $i$th component.

The $l_q$ metric has the following properties (see [16]).

(a) Suppose $\alpha$ is a positive real, $S : X \to X$ is Lipschitz, and $\vee$ denotes the maximum of the relevant numbers. Then, for $q > 0$ and for measures $\mu$ and $\nu$, we have the following properties:

$$l_q^{q\vee 1}(\alpha\mu,\alpha\nu) = \alpha l_q^{q\vee 1}(\mu,\nu), \tag{2.6}$$

$$l_q^{q\vee 1}(\mu_1+\mu_2,\nu_1+\nu_2) \le l_q^{q\vee 1}(\mu_1,\nu_1) + l_q^{q\vee 1}(\mu_2,\nu_2), \tag{2.7}$$

$$l_q(S\mu,S\nu) \le (\operatorname{Lip} S)^{q\wedge 1} l_q(\mu,\nu). \tag{2.8}$$

The first property follows from the definition by setting $\gamma = c\overline{\gamma}$, where $\overline{\gamma}$ is optimal for $(\mu,\nu)$, and the third follows by setting $\gamma = S\overline{\gamma}$. The second follows by setting $\gamma = \gamma_1+\gamma_2$, where $\gamma_i$ is optimal for $(\mu_i,\nu_i)$, and also by noting that $(a+b)^q \le a^q + b^q$ if $a,b \ge 0$ and $0 < q < 1$.

(b) The pair $(M_q,l_q)$ is a complete separable metric space and $l_q(\mu_n,\mu) \to 0$ if and only if

(i) $\mu_n \to \mu$ (weak convergence),

(ii) $\int_X d^q(x,a)d\mu_n(x) \to \int d^q(x,a)d\mu(x)$ (convergence of $q$th moments).

(c) If $\delta_a$ is the Dirac measure at $a \in X$, then

$$\begin{aligned} l_q(\mu,\mu(X)\delta_a) &= \left(\int_X d^q(x,a)d\mu(x)\right)^{1/q\wedge 1}, \\ l_q(\delta_a,\delta_b) &= d^{1\wedge q}(a,b). \end{aligned} \tag{2.9}$$

Let $\mathbf{M}$ denote the set of all random measures $\mu$ with value in $M$, that is, random variables $\mu : \Omega \to M$. Let $\mathbf{M}_q$ denote the space of random measures $\mu : \Omega \to M_q$ with finite expected $q$th moment. That is,

$$\mathbf{M}_q := \left\{\mu \in \mathbf{M} \mid \mu^\omega(X) = 1 \ a.s.,\ E_\omega \int_X d^q(x,a)d\mu^\omega(x) < \infty\right\}. \tag{2.10}$$

The notation $E_\omega$ indicates that the expectation is with respect to the variable $\omega$. It follows from (2.10) that $\mu^\omega \in M_q$ a.s. Note that $\mathbf{M}_p \subset \mathbf{M}_q$ if $q \le p$. Moreover, since $E^{1/q}|f|^q \to \exp(E\log|f|)$ as $q \to 0$,

$$\mathbf{M}_0 := \cup_{q>0}\mathbf{M}_q = \left\{\mu \in \mathbf{M} \mid \mu_\omega(X) = 1 \ a.s.,\ E_\omega \int_X \log d(x,a)d\mu^\omega(x) < \infty\right\}. \tag{2.11}$$

For random measures $\mu,\nu \in \mathbf{M}_q$, define

$$l_q^*(\mu,\nu) := \begin{cases} E_\omega^{1/q} l_q^q(\mu^\omega,\nu^\omega), & q \ge 1, \\ E_\omega l_q(\mu^\omega,\nu^\omega), & 0 < q < 1. \end{cases} \tag{2.12}$$

One can check, as in [16], that $(\mathbf{M}_q,\mathbf{l}_q^*)$ is a complete separable metric space. Note that $l_q^*(\mu,\nu) = l_q(\mu,\nu)$ if $\mu$ and $\nu$ are constant random measures.

Let $\mathcal{M}$ denote the class of probability distributions on $\mathbf{M}$, that is,

$$\mathcal{M} = \{\mathcal{D} = \operatorname{dist}\mu \mid \mu \in \mathbf{M}\}. \tag{2.13}$$

Let $\mathcal{M}_q$ be the set of probability distributions of random measures $\mu \in \mathbf{M}_q$. For $q \le p$, it is to be noticed that $\mathcal{M}_p \subset \mathcal{M}_q$. Let

$$\mathcal{M}_0 := \cup_{q>0}\mathcal{M}_q. \tag{2.14}$$

The minimal metric on $\mathcal{M}_q$ is defined by

$$l_q^{**}(\mathcal{D}_1,\mathcal{D}_2) = \inf\left\{l_q^*(\mu,\nu) \mid \mu \overset{d}{=} D_1,\ \nu \overset{d}{=} D_2\right\}. \tag{2.15}$$

It follows that $(\mathcal{M}_q, l_q^{**})$ is a complete separable metric space with the next properties (see [16]):

(a) $l_q^{**}(\alpha\mathcal{D}_1,\alpha\mathcal{D}_2) = \alpha l_q^{**}(\mathcal{D}_1,\mathcal{D}_2)$,

(b) $l_q^{**}(\mathcal{D}_1+\mathcal{D}_2,\mathcal{D}_3+\mathcal{D}_4) \le l_q^{**q}(\mathcal{D}_1,\mathcal{D}_3)+l_q^{**q}(\mathcal{D}_2,\mathcal{D}_4)$,

for $\mathcal{D}_i \in \mathcal{M}_q$, $i = 1,2,3,4$.

**Definition 2.4.** *A random scaling law with weights* or a *random scaling law for measure* $\mathbb{S} = (p_1,S_1,p_2,S_2,\dots,p_N,S_N)$ is a random variable whose values are scaling laws, with $\sum_{i=1}^N p_i = 1$ a.s.

We write $\mathcal{S} = \operatorname{dist}\mathbb{S}$ for the probability distribution determined by $\mathbb{S}$.

If $\mu$ is a random measure, then the random measure $\mathbb{S}\mu$ is defined (up to probability distribution) by

$$\mathbb{S}\mu := \sum_{i=1}^{N} p_i S_i \mu^{(i)}, \tag{2.16}$$

where $\mathbb{S}$, $\mu^{(1)},\dots,\mu^{(N)}$ are independent of one another, and $\mu^{(i)} \overset{d}{=} \mu$. If $\mathcal{D} = \operatorname{dist}\mu$, we define $\mathcal{S}\mathcal{D} = \operatorname{dist}\mathbb{S}\mu$.

**Definition 2.5.** The measure $\mu$ *satisfies the scaling law* $\mathbb{S}$ or is a *self-similar random fractal measure* if $\mathbb{S}\mu \overset{d}{=} \mu$, or equivalently $\mathcal{S}\mathcal{D} = \mathcal{D}$, where $\mathcal{D}$ is called a self-similar random fractal distribution.

To generate a random self-similar fractal measure, we use the *iterative procedure* described as follows. Fix $q > 0$. Beginning with a nonrandom measure $\mu_0 \in M_q$ (or, more generally, a random measure $\mu_0 \in \mathbf{M}_q$), one iteratively applies i.i.d. scaling laws with distribution $\mathcal{S}$ to obtain a sequence $\mu_n$ of random measures in $\mathbf{M}_q$ and a corresponding sequence $\mathcal{D}_n$ of distributions in $\mathcal{M}_q$ as follows.

(i) Select a scaling law $\mathbb{S}$ via the distribution $\mathcal{S}$ and define

$$\mu_1 = \mathbb{S}\mu_0 = \sum_{i=1}^{n} p_i S_i \mu_0, \tag{2.17}$$

that is,

$$\mu_1(\omega) = \mathbb{S}\mu_0 = \sum_{i=1}^{n} p_i(\omega) S_i(\omega)\mu_0, \qquad \mathcal{D}_1 \stackrel{d}{=} \mu_1. \tag{2.18}$$

(ii) Select $\mathbb{S}^1,\dots,\mathbb{S}^N$ via $\mathcal{S}$ with $\mathbb{S}^i = (p_1^i, S_1^i, \dots, p_N^i, S_N^i)$, $i \in \{1,2,\dots,N\}$, independent of each other and of $\mathbb{S}$, and define

$$\mu_2 := \mathbb{S}^2\mu_0 = \sum_{i,j} p_i p_j^i S_i \circ S_j^i \mu_0, \qquad \mathcal{D}_2 \stackrel{d}{=} \mu_0. \tag{2.19}$$

(iii) Select $\mathbb{S}^{ij} = (p_1^i, S_1^{ij}, \dots, p_N^i, S_N^{i,j})$ via $\mathcal{S}$, independent of one another and of $\mathbb{S}^1,\dots,\mathbb{S}^N, S$, and define

$$\mu_3 = \mathbb{S}^3\mu_0 = \sum_{i,j,k} p_i p_j^i p_k^{ij} S_i \circ S_j^i \circ S_k^{ij} \mu_0, \qquad \mathcal{D}_3 \stackrel{d}{=} \mu_3, \tag{2.20}$$

and so forth.

Thus $\mu_{n+1} = \sum_{i=1}^N p_i S_i \mu_n^{(i)}$, where $\mu_n^{(i)} \stackrel{d}{=} \mu_n \stackrel{d}{=} \mathcal{D}_n$, $\mathbb{S} \stackrel{d}{=} \mathcal{S}$, and $\mu_n^{(i)}$ and $\mathbf{S}$ are independent. It follows that $\mathcal{D}_n = \mathcal{S}\mathcal{D}_{N-1} = \mathcal{S}^n\mathcal{D}_0$, where $\mathcal{D}_0$ is the distribution of $\mu_0$. In the case $\mu_0 \in M_q$, $\mathcal{D}_0$ is constant.

In the following, we define the underlying probability space for a.s. convergence (see [10]).

A *construction tree* (or a construction process) is a map $\omega : \{1,\dots,N\}^* \to \Gamma$, where $\Gamma$ is the set of (nonrandom) scaling laws. A construction tree specifies, at each node of the scaling law used for constructive definition, a recursive sequence of random measures. Denote the scaling law of $\omega$ at the node $\sigma$ by the $2N$-tuple

$$\mathbf{S}^\sigma(\omega) = \omega(\sigma) = \big(p_1^\sigma(\omega), S_1^\sigma(\omega), \dots, p_N^\sigma(\omega), S_N^\sigma(\omega)\big), \tag{2.21}$$

where $p_i^\sigma$ are weights and $S_i^\sigma$ Lipschitz maps. The sample space of all construction trees is denoted by $\tilde{\Omega}$. The underlying probability space $(\tilde{\Omega}, \tilde{\mathcal{K}}, \tilde{P})$ for the iteration procedure is generated by selecting i.i.d. scaling laws $\omega(\sigma) \stackrel{d}{=} \mathbf{S}$ for each $\sigma \in \{1,\dots,N\}^*$. We use the notation

$$\begin{aligned} \overline{p}^\sigma &= p_{\sigma_1} p_{\sigma_2}^{\sigma_1} p_{\sigma_3}^{\sigma_1\sigma_2} \cdots p_{\sigma_n}^{\sigma_1\cdots\sigma_{n-1}}, \\ \overline{S}^\sigma &= S_{\sigma_1} S_{\sigma_2}^{\sigma_1} p_{\sigma_3}^{\sigma_1\sigma_2} \cdots S_{\sigma_n}^{\sigma_1\cdots\sigma_{n-1}}, \end{aligned} \tag{2.22}$$

where $|\sigma| = n$ and where $p_i^\sigma$ and $S_i^\sigma$ denote the $i$th components of scaling law. For a fixed measure $\mu_0 \in M_q$, define

$$\mu_n = \mu_n(\omega) = \sum_{|\sigma|=n} \overline{p}^\sigma(\omega)\overline{S}^\sigma(\omega)\mu_0 \tag{2.23}$$

for $n \ge 1$. This is identical to the sequence defined in an iterative procedure with an underlying space $\Omega = \overline{\Omega}$. To see this, for $\omega \in \overline{\Omega}$ and $1 \le i \le N$, let $\omega^{(i)} \in \overline{\Omega}$ be defined by

$$\omega^{(i)}(\sigma) = \omega(i * \sigma) \tag{2.24}$$

for $\sigma \in \{1,\dots,N\}^*$. Then

$$\begin{aligned} \overline{p}^{i*\sigma} &= p_i(\omega)\overline{p}^{\sigma}(\omega^{(i)}), \\ \overline{S}^{i*\sigma} &= S_i(\omega) \circ \overline{p}^{\sigma}(\omega^{(i)}). \end{aligned} \tag{2.25}$$

By construction, $\omega^{(i)}$ are i.i.d. with the same distribution as $\omega$, and are independent of $(p_1(\omega),S_1(\omega),\dots,p_N(\omega),S_N(\omega))$. More precisely, for any $\overline{P}$ measurable sets $E,F \subset \overline{\Omega}$ and $B \subset \Gamma$,

$$\overline{P}(\{\omega \mid \omega \in E\}) = \overline{P}(\{\omega \mid \omega^{(i)} \in E\}), \tag{2.26}$$

where $\{\omega \mid \omega^{(i)} \in E\}$ and $\{\omega \mid \omega^{(j)} \in E\}$ are independent if $i \ne j$, and $\{\omega \mid (p_1\omega,S_1(\omega),\dots,p_N(\omega),S_N(\omega)) \in B\}$ and $\{\omega \mid \omega^{(i)} \in E\}$ are independent. It follows that

$$\mu_{n+1}(\omega) = \sum_{i=1}^{N}\sum_{|\sigma|=n} \overline{p}^{i*\sigma}\overline{S}^{i*\sigma}(\omega)\mu_0 = \sum_{i=1}^{N} p_i(\omega)S_i(\omega)\mu_n(\omega^{(i)}) = \mathbf{S}\mu_n(\omega). \tag{2.27}$$

In [8], Hutchinson and Rüschendorf proved the following theorem.

**Theorem 2.6.** *Let $\mathbb{S} = (p_1,S_1,p_2S_2,\dots,p_N,S_N)$ be a random scaling law with $\sum_{i=1}^N p_i = 1$ a.s. Assume $\lambda_q := E_\omega(\sum_{i=1}^N p_i r_i^q) < 1$ and*

$$E_\omega\left(\sum_{i=1}^{N} p_i d^q(S_i a,a)\right) < \infty \quad \textit{for some } q > 0, \textit{ and for } a \in Y. \tag{2.28}$$

*Then the following facts hold.*

(a) *The operator $\mathbb{S} : M_q \to M_q$ is a contraction map with respect to $l_q^*$.*

(b) *There exists a self-similar random measure $\mu^*$.*

(c) *If $\mu_0 \in M_p$ (or, more generally, $\mathbf{M}_q$), then*

$$\begin{aligned} E_\omega^{1/q} l_q^q(\mu_k,\mu^*) &\le \frac{\lambda_q^{k/q}}{1-\lambda_q^{1/q}} E_\omega^{1/q} l_q^q(\mu_1,\mathbf{S}\mu_0) \longrightarrow 0, \quad q \ge 1, \\ E_\omega l_q(\mu_k,\mu^*) &\le \frac{\lambda_q^{k}}{1-\lambda_q} E_\omega l_q(\mu_1,\mathbf{S}\mu_0) \longrightarrow 0, \quad 0 < q < 1, \end{aligned} \tag{2.29}$$

*as $k \to \infty$. In particular $\mu_n \to \mu^*$ a.s. in the sense of weak convergence of measures.*

*Moreover, up to probability distribution, $\mu^*$ is the unique unit mass random measure with $E_\omega \int \ln d(x,a)\,d\mu^\omega < \infty$, which satisfies $\mathbb{S}$.*

Using contraction method in probabilistic metric spaces, instead of condition (2.28), we can give a weaker condition for the existence and uniqueness of invariant measure. More precisely, we prove the following theorem.

**Theorem 2.7.** *Let $\mathbb{S} = (p_1, S_1, p_2, S_2, \dots, p_N, S_N)$ be a random scaling law which satisfies $\sum_{i=1}^N p_i = 1$ a.s., and suppose $\lambda_q := \operatorname{ess\,sup}(\sum_{i=1}^N p_i r_i^q) < 1$ for some $q > 0$. If there exist $\alpha \in M_q$ and a positive number $\gamma$ such that*

$$P(\{\omega \in \Omega \mid l_q(\alpha(\omega), \mathbb{S}\alpha(\omega)) \geq t\}) \leq \frac{\gamma}{t} \quad \forall t > 0, \tag{2.30}$$

*then there exists $\mu^*$ such that $\mathbb{S}\mu^* = \mu^*$ a.s.*

*Moreover, up to probability distribution, $\mu^*$ is the unique unit mass random measure which satisfies $\mathbb{S}$.*

**Remark 2.8.** If condition (2.28) is satisfied, then condition (2.30) also holds. To see this, let $a \in X$ and $\alpha(\omega) := \delta_a$ for all $\omega \in \Omega$. We have

$$\begin{aligned}
&P(\{\omega \in \Omega \mid l_q(\delta_a(\omega), \mathbb{S}\delta_a(\omega)) \geq t\}) \\
&\quad = P\left(\left\{\omega \in \Omega \mid l_q\left(\sum_{i=1}^N p_i\delta_a(\omega), \sum_{i=1}^N p_i S_i \delta_a(\omega)\right) \geq t\right\}\right) \\
&\quad \leq P\left(\left\{\omega \in \Omega \mid \sum_{i=1}^N p_i l_q(\delta_a(\omega), S_i\delta_a(\omega)) \geq t\right\}\right) \\
&\quad = P\left(\left\{\omega \in \Omega \mid \sum_{i=1}^N p_i d^q(S_i a, a) \geq t\right\}\right) \\
&\quad \leq \frac{1}{t} E_\omega\left(\sum_{i=1}^N p_i d^q(S_i a, a)\right) = \frac{\gamma}{t}.
\end{aligned} \tag{2.31}$$

However, condition (2.30) can also be satisfied if

$$E_\omega\left(\sum_{i=1}^N p_i d^q(S_i a, a)\right) = \infty \quad \forall q > 0. \tag{2.32}$$

Let $\Omega = ]0,1]$ with the Lebesque measure, let $X$ be the interval $[0, \infty[$, and let $N = 1$. Define $S : X \to X$ by $S^\omega(x) = x/2 + e^{1/\omega}$. This map is a contraction with ratio $1/2$. For $q > 0$, the expectation $E_\omega d^q(S0, 0) = \infty$, however

$$P(\{\omega \in \Omega \mid l_q(S0, 0) \geq t\}) = \frac{1}{t} \tag{2.33}$$

for all $t > 0$.

## 3. Invariant sets in $E$-spaces

**3.1. Menger spaces.** Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$. A mapping $F : \mathbb{R} \to [0,1]$ is called a *distribution function* if it is nondecreasing, left continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$ (see [4]). By $\Delta$ we will denote the set of all distribution functions $F$. Let $\Delta$ be ordered by the relation "$\le$", that is, $F \le G$ if and only if $F(t) \le G(t)$ for all real $t$. Also $F < G$ if and only if $F \le G$ but $F \ne G$. We set $\Delta^+ := \{F \in \Delta : F(0) = 0\}$.

Throughout this paper, $H$ will denote the heaviside distribution function defined by

$$H(x) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0. \end{cases} \tag{3.1}$$

Let $X$ be a nonempty set. For a mapping $\mathcal{F} : X \times X \to \Delta^+$ and $x, y \in X$, we will denote $\mathcal{F}(x,y)$ by $F_{x,y}$, and the value of $F_{x,y}$ at $t \in \mathbb{R}$ by $F_{x,y}(t)$, respectively. The pair $(X, \mathcal{F})$ is a *probabilistic metric space* (briefly *PM space*) if $X$ is a nonempty set and $\mathcal{F} : X \times X \to \Delta^+$ is a mapping satisfying the following conditions:

(1) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t \in \mathbb{R}$;
(2) $F_{x,y}(t) = 1$, for every $t > 0$, if and only if $x = y$;
(3) if $F_{x,y}(s) = 1$ and $F_{y,z}(t) = 1$, then $F_{x,z}(s+t) = 1$.

A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a $t$-*norm* if the following conditions are satisfied:

(4) $T(a,1) = a$ for every $a \in [0,1]$;
(5) $T(a,b) = T(b,a)$ for every $a, b \in [0,1]$;
(6) if $a \ge c$ and $b \ge d$, then $T(a,b) \ge T(c,d)$;
(7) $T(a, T(b,c)) = T(T(a,b), c)$ for every $a, b, c \in [0,1]$.

A *Menger space* is a triplet $(X, \mathcal{F}, T)$, where $(X, \mathcal{F})$ is a PM space, $T$ is a $t$-norm, and instead of condition (3), we have the stronger condition

(8) $F_{x,y}(s+t) \ge T(F_{x,z}(s), F_{z,y}(t))$ for all $x, y, z \in X$ and $s, t \in \mathbb{R}_+$.

The $(t,\epsilon)$-topology in a Menger space was introduced in 1960 by Schweizer and Sklar [17]. The base for the neighbourhoods of an element $x \in X$ is given by

$$\{U_x(t,\epsilon) \subseteq X : t > 0,\ \epsilon \in ]0,1[\}, \tag{3.2}$$

where

$$U_x(t,\epsilon) := \{y \in X : F_{x,y}(t) > 1-\epsilon\}. \tag{3.3}$$

In 1969, Sehgal [19] introduced the notion of a contraction mapping in PM spaces. The mapping $f : X \to X$ is said to be a *contraction* if there exists $r \in ]0,1[$ such that

$$F_{f(x),f(y)}(rt) \ge F_{x,y}(t) \tag{3.4}$$

for every $x, y \in X$ and $t \in \mathbb{R}_+$.

A sequence $(x_n)_{n\in\mathbb{N}}$ from $X$ is said to be *fundamental* if

$$\lim_{n,m\to\infty} F_{x_m,x_n}(t) = 1 \tag{3.5}$$

for all $t > 0$. The element $x \in X$ is called *limit* of the sequence $(x_n)_{n\in\mathbb{N}}$, and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ if $\lim_{n\to\infty} F_{x,x_n}(t) = 1$ for all $t > 0$. A PM (Menger) space is said to be *complete* if every fundamental sequence in that space is convergent.

Let $A$ and $B$ be nonempty subsets of $X$. The *probabilistic Hausdorff-Pompeiu distance* between $A$ and $B$ is the function $F_{A,B} : \mathbb{R} \to [0,1]$ defined by

$$F_{A,B}(t) := \sup_{s<t} T\left(\inf_{x\in A}\sup_{y\in B} F_{x,y}(s), \inf_{y\in B}\sup_{x\in A} F_{x,y}(s)\right). \tag{3.6}$$

In the following, we recall some properties proved in [11, 12].

**Proposition 3.1.** *If $\mathscr{C}$ is a nonempty collection of nonempty closed bounded sets in a Menger space $(X,\mathscr{F},T)$ with $T$ continuous, then $(\mathscr{C},\mathscr{F}_{\mathscr{C}},T)$ is also Menger space, where $\mathscr{F}_{\mathscr{C}}$ is defined by $\mathscr{F}_{\mathscr{C}}(A,B) := F_{A,B}$ for all $A,B \in \mathscr{C}$.*

**Proposition 3.2.** *Let $T_m(a,b) := \max\{a+b-1,0\}$. If $(X,\mathscr{F},T_m)$ is a complete Menger space and $\mathscr{C}$ is the collection of all nonempty closed bounded subsets of $X$ in $(t,\epsilon)$-topology, then $(\mathscr{C},\mathscr{F}_{\mathscr{C}},T_m)$ is also a complete Menger space.*

**3.2. *E*-spaces.** The notion of $E$-space was introduced by Sherwood [20] in 1969. Next we recall this definition. Let $(\Omega,\mathscr{K},P)$ be a probability space and let $(Y,\rho)$ be a metric space. The ordered pair $(\mathscr{E},\mathscr{F})$ is an *$E$-space over the metric space* $(Y,\rho)$ (briefly, an $E$-space) if the elements of $\mathscr{E}$ are random variables from $\Omega$ into $Y$ and $\mathscr{F}$ is the mapping from $\mathscr{E}\times\mathscr{E}$ into $\Delta^+$ defined via $\mathscr{F}(x,y) = F_{x,y}$, where

$$F_{x,y}(t) = P(\{\omega \in \Omega \mid d(x(\omega),y(\omega)) < t\}) \tag{3.7}$$

for every $t \in \mathbb{R}$. Usually $(\Omega,\mathscr{K},P)$ is called the base and $(Y,\rho)$ the target space of the $E$-space. If $\mathscr{F}$ satisfies the condition

$$\mathscr{F}(x,y) \neq H \quad \text{for } x \neq y, \tag{3.8}$$

with $H$ defined in Section 3.1, then $(\mathscr{E},\mathscr{F})$ is said to be a *canonical $E$-space*. Sherwood [20] proved that every canonical $E$-space is a Menger space under $T = T_m$, where $T_m(a,b) = \max\{a+b-1,0\}$. In the following, we suppose that $\mathscr{E}$ is a canonical $E$-space.

The convergence in an $E$-space is exactly the probability convergence. The $E$-space $(\mathscr{E},\mathscr{F})$ is said to be complete if the Menger space $(\mathscr{E},\mathscr{F},T_m)$ is complete.

**Proposition 3.3.** *If $(Y,\rho)$ is a complete metric space, then the $E$-space $(\mathscr{E},\mathscr{F})$ is also complete.*

**Proof.** This property is well known for $Y = R$ (see, e.g., [21, Theorem VII.4.2]). In the general case, the proof is analogous.

Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of elements of $\mathscr{E}$, that is,

$$\lim_{n,m\to\infty} F_{x_n,x_{n+m}}(t) = 1 \quad \forall t > 0. \tag{3.9}$$

First we show that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of the given sequence which is convergent almost everywhere to a random variable $x$. We set positive numbers $\epsilon_i$ so that $\sum_{i=1}^{\infty}\epsilon_i < \infty$ and put $\delta_p = \sum_{i=p}^{\infty}\epsilon_i$, $p = 1,2,\dots$. For each $i$, there is a natural number $k_i$ such that

$$P(\{\omega\in\Omega \mid \rho(x_k(\omega),x_l(\omega)) \geq \epsilon_i\}) < \epsilon_i \quad \text{for } k,l \geq k_i. \tag{3.10}$$

We can assume that $k_1 < k_2 < \cdots < k_i < \cdots$. Then

$$P(\{\omega\in\Omega \mid \rho(x_{k_{i+1}}(\omega),x_{k_i}(\omega)) \geq \epsilon_i\}) < \epsilon_i \quad \text{for } k,l \geq k_i. \tag{3.11}$$

We put

$$D_p = \cup_{i=p}^{\infty}\{\omega\in\Omega \mid \rho(x_{k_{i+1}},x_{k_i}) \geq \epsilon_i\}. \tag{3.12}$$

Then $P(D_p) < \delta_p$. Finally, for the intersection $D' = \cap_{p=1}^{\infty}D_p$, we obviously have $P(D') = 0$ since $\delta_p \to 0$. We will show that the sequence $(x_{k_i}(\omega))$ has a finite limit $x(\omega)$ at every point $\omega \in \{\omega\in\Omega \mid \rho(x_k(\omega),x_m(\omega)) > t\}\setminus D'$. For some $p$ we have $x \notin D_p$. Consequently, $\rho(x_{k_{i+1}}(\omega),x_{k_i}(\omega)) < \epsilon_i$, for all $i \geq p$. It follows that for any two indices $i$ and $j$ such that $j > i \geq p$, we have

$$\rho(x_{k_j}(\omega),x_{k_i}(\omega)) \leq \sum_{m=i}^{j-1}\rho(x_{k_{m+1}}(\omega),x_{k_m}(\omega)) < \sum_{m=i}^{j-1}\epsilon_m < \sum_{m=i}^{\infty}\epsilon_m = \delta_i. \tag{3.13}$$

Thus $\lim_{i,j\to\infty}\rho(x_{k_j}(\omega),x_{k_i}(\omega)) = 0$. This means that $(x_k(\omega))_{k\in\mathbb{N}}$ is a Cauchy sequence for every $\omega$ which implies the pointwise convergence of $(x_{k_i})_{i\in\mathbb{N}}$ to a finite-limit function. Now remains only to put

$$x(\omega) = \begin{cases} \lim x_{k_i}(\omega) & \text{for } \omega \notin D', \\ 0 & \text{for } \omega \in D' \end{cases} \tag{3.14}$$

to obtain the desired limit random variable. By Lebesque theorem (see, e.g., [21, Theorem VI.5.2]), $x_{k_i} \to x$ with respect to $d$. Thus, every Cauchy sequence in $\mathscr{E}$ has a limit, which means that the space $\mathscr{E}$ is complete. □

The next result was proved in [12].

**Theorem 3.4.** *Let $(\mathscr{E},\mathscr{F})$ be a complete $E$-space, $N \in \mathbb{N}^*$, and let $f_1,\dots,f_N : \mathscr{E} \to \mathscr{E}$ be contractions with ratios $r_1,\dots,r_N$, respectively. Suppose that there exist an element $z \in \mathscr{E}$ and a real number $\gamma$ such that*

$$P(\{\omega \in \Omega \mid \rho(z(\omega), f_i(z(\omega))) \geq t\}) \leq \frac{\gamma}{t} \tag{3.15}$$

*for all $i \in \{1,\dots,N\}$ and for all $t > 0$. Then there exists a unique nonempty closed bounded and compact subset $K$ of $\mathscr{E}$ such that*

$$f_1(K) \cup \cdots \cup f_N(K) = K. \tag{3.16}$$

**Corollary 3.5.** *Let $(\mathscr{E},\mathscr{F})$ be a complete $E$-space and let $f : \mathscr{E} \to \mathscr{E}$ be a contraction with ratio $r$. Suppose there exist $z \in \mathscr{E}$ and a real number $\gamma$ such that*

$$P(\{\omega \in \Omega \mid \rho(z(\omega), f(z)(\omega)) \geq t\}) \leq \frac{\gamma}{t} \quad \forall t > 0. \tag{3.17}$$

*Then there exists a unique $x_0 \in \mathscr{E}$ such that $f(x_0) = x_0$.*

## 4. Proof of Theorem 2.7.

Before the proof of the theorem, we give two lemmas.

Let $\mathscr{E}_q$ be the set of random variables with values in $M_q$ and let $\mathscr{E}_q(\alpha)$ be the set

$$\mathscr{E}_q(\alpha) := \left\{\beta \in \mathscr{E}_q \mid \exists \gamma > 0,\ P(\{\omega \in \Omega \mid l_q(\alpha(\omega), \beta(\omega)) \geq t\}) \leq \frac{\gamma}{t}\ \forall t > 0\right\}. \tag{4.1}$$

**Lemma 4.1.** *For all $\alpha \in M_q$, $\mathbf{M}_q \subset \mathscr{E}_q(\alpha)$.*

**Proof.** For $\beta \in \mathbf{M}_q$, we have

$$\begin{aligned} &P(\{\omega \in \Omega \mid l_q(\alpha(\omega), \beta(\omega)) \geq t\}) \\ &\quad = \int_{l_q(\alpha(\omega),\beta(\omega)) \geq t} dP \leq \frac{1}{t} \int_{\Omega} l_q(\alpha(\omega), \beta(\omega))\, dP = \frac{1}{t} E_\omega l_q(\alpha(\omega), \beta(\omega)). \end{aligned} \tag{4.2}$$

Since $\beta \in \mathcal{M}_q$, we have $\gamma = E_\omega l_q(\alpha(\omega), \beta(\omega)) < \infty$ for all $t > 0$. □

**Lemma 4.2.** *The pair $(\mathscr{E}_q,\mathscr{F})$ is a complete $E$-space.*

**Proof.** The lemma follows by choosing $Y := \mathscr{E}_q$ and $\mathscr{F}_{\mu,\nu}(t) := P(\{\omega \in \Omega \mid l_q(\mu(\omega), \nu(\omega)) < t\})$ in Proposition 3.3. □

**Proof of Theorem 2.7.** Let $\mathbb{S}$ be a random scaling law. Define $f : \mathscr{E}_q \to \mathscr{E}_q$ by $f(\mu) = \mathbf{S}\mu$, that is,

$$\mathbf{S}\mu(\omega) = \sum_i p_i^\omega S_i^\omega \mu(\omega^{(i)}). \tag{4.3}$$

We first claim that if $\mu \in \mathscr{E}_q$, then $\mathbb{S}\mu \in \mathscr{E}_q$. For this, choose i.i.d. $\mu(\omega^{(i)}) \overset{d}{=} \mu(\omega)$ and $(p_1^\omega, S_1^\omega, \dots, p_N^\omega, S_N^\omega) \overset{d}{=} \mathbb{S}$ independent of $\mu(\omega)$. For $q \geq 1$ and $b_i = S_i^{-1}(a)$, using (2.8), we compute

$$\begin{aligned} \int_X d^q(x,a)\, d(\mathbb{S}\mu^\omega(x)) &= l_q^q\left(\sum_{i=1}^N p_i^\omega S_i^\omega \mu(\omega^{(i)}), \delta_a\right) \\ &= l_q^q\left(\sum_{i=1}^N p_i^\omega S_i^\omega \mu(\omega^{(i)}), \sum_{i=1}^N p_i^\omega S_i^\omega \delta_{b_i}\right) \\ &\leq \sum_{i=1}^N p_i^\omega r_i^q l_q^q\left(\mu(\omega^{(i)}), \delta_{b_i}\right). \end{aligned} \tag{4.4}$$

Since $\mu(\omega^{(i)}) \in M_q$, we have

$$\int_X d^q(x,a)\, d(\mathbb{S}\mu(x)) < \infty. \tag{4.5}$$

We can deal with the case $0 < q < 1$ similarly by replacing $l_q^q$ with $l_q$:

$$\begin{aligned} \int_X d^q(x,a)\, d(\mathbb{S}\mu^\omega(x)) &= l_q\left(\sum_{i=1}^N p_i^\omega S_i^\omega \mu(\omega^{(i)}), \delta_a\right) \\ &= l_q\left(\sum_{i=1}^N p_i^\omega S_i^\omega \mu(\omega^{(i)}), \sum_{i=1}^N p_i^\omega S_i^\omega \delta_{b_i}\right) \\ &\leq \sum_{i=1}^N p_i^\omega r_i^q l_q\left(\mu(\omega^{(i)}), \delta_{b_i}\right) < \infty. \end{aligned} \tag{4.6}$$

To establish the contraction property, we consider $\mu, \nu \in \mathscr{E}_q$,

$$\mu(\omega^{(i)}) \overset{d}{=} \mu(\omega), \quad \nu(\omega^{(i)}) \overset{d}{=} \nu(\omega), \quad i \in \{1, 2, \dots, N\}, \tag{4.7}$$

and $q \geq 1$. We have

$$\begin{aligned} F_{f(\mu),f(\nu)}(t) &= P\left(\{\omega \in \overline{\Omega} \mid l_q(f(\mu(\omega)), f(\nu(\omega))) < t\}\right) \\ &= P\left(\left\{\omega \in \overline{\Omega} \mid l_q\left(\sum_{i=1}^N p_i^\omega S_i^\omega \mu(\omega^{(i)}), \sum_{i=1}^N p_i^\omega S_i^\omega \nu(\omega^{(i)})\right) < t\right\}\right) \\ &\geq P\left(\left\{\omega \in \overline{\Omega} \mid \left[\sum_{i=1}^N p_i^\omega (r_i)^q l_q^q\left(\mu(\omega^{(i)}), \nu(\omega^{(i)})\right)\right]^{1/q} < t\right\}\right) \\ &\geq P\left(\left\{\omega \in \overline{\Omega} \mid \left[\lambda_q l_q^q(\mu(\omega), \nu(\omega))\right]^{1/q} < t\right\}\right) = F_{\mu,\nu}\left(\frac{t}{\lambda_q^{1/q}}\right) \end{aligned} \tag{4.8}$$

for all $t > 0$.

In case $0 < q < 1$, one replaces $l_q^q$ everywhere by $l_q$:

$$\begin{aligned}
F_{f(\mu),f(\nu)}(t) &= P(\{\omega \in \overline{\Omega} \mid l_q(f(\mu(\omega)), f(\nu(\omega))) < t\}) \\
&= P\left(\left\{\omega \in \overline{\Omega} \mid l_q\left(\sum_{i=1}^{N} p_i^{\omega} S_i^{\omega} \mu(\omega^{(i)}), \sum_{i=1}^{N} p_i^{\omega} S_i^{\omega} \nu(\omega^{(i)})\right) < t\right\}\right) \\
&\geq P\left(\left\{\omega \in \overline{\Omega} \mid \left[\sum_{i=1}^{N} p_i^{\omega} (r_i)^q l_q(\mu(\omega^{(i)}), \nu(\omega^{(i)}))\right]^{1/q} < t\right\}\right) \\
&\geq P(\{\omega \in \overline{\Omega} \mid [\lambda_q l_q(\mu(\omega), \nu(\omega))] < t\}) = F_{\mu,\nu}\left(\frac{t}{\lambda_q}\right)
\end{aligned} \tag{4.9}$$

for all $t > 0$. Thus $\mathbb{S}$ is a contraction map with ratio $\lambda_q^{1/q \wedge 1}$. We can apply Corollary 3.5 for $r = \lambda_q^{1/q \wedge 1}$. If $\mu^*$ is the unique fixed point of $\mathbb{S}$ and $\mu_0 \in M_q$, then

$$\begin{aligned}
F_{\mathbb{S}^n \mu_0, \mu^*}(t) &= P(\{\omega \in \overline{\Omega} \mid l_q(\mathbb{S}^n \mu_0, \mu^*) < t\}) \\
&\geq P\left(\left\{\omega \in \overline{\Omega} \mid \frac{\lambda_q^{n/q}}{1-\lambda_q^{1/q}} l_q(\mu_0, \mathbb{S}\mu_0) < t\right\}\right) \\
&= F_{\mu_0, \mathbb{S}\mu_0}\left(\frac{t(1-\lambda_q^{1/q})}{\lambda_q^{n/q}}\right), \\
&\lim_{n \to \infty} F_{\mathbb{S}^n \mu_0, \mu^*}(t) = 1 \quad \forall t > 0.
\end{aligned} \tag{4.10}$$

From $\mu_{n+1}(\omega) = \mathbb{S}\mu_n(\omega)$, it follows that $\mu_n \to \mu^*$ exponentially fast. Moreover, for $q \geq 1$,

$$\sum_{i=1}^{\infty} \overline{P}(l_q^q(\mathbf{S}^n \nu_0, \mu^*) \geq \epsilon) \leq \sum_{i=1}^{\infty} \frac{e l_q^q(\mathbf{S}^n \mu_0, \mu^*)}{\epsilon} \leq c \sum_{i=1}^{\infty} \frac{\lambda_q^n}{\epsilon} < \infty. \tag{4.11}$$

This implies by Borel-Cantelli lemma that $l_q(\mu_n, \mu^*) \to 0$ a.s.

For the uniqueness, let $\mathcal{D}$ be the set of probability distribution of members of $\mathcal{E}_q$. We define the probability metric on $\mathcal{D}$ by

$$F_{\mathcal{A},\mathcal{B}}(t) = \sup_{s<t} \sup \left\{F_{\mu,\nu}(s) \mid \mu \overset{d}{=} \mathcal{A},\ \nu \overset{d}{=} \mathcal{B}\right\}. \tag{4.12}$$

To establish the contraction property of $\mathcal{S}$, we consider $\mathcal{A}, \mathcal{B} \in \mathcal{D}$. For $q \geq 1$, we get

$$\begin{aligned}
F_{\mathcal{S}\mathcal{A},\mathcal{S}\mathcal{B}}(t) &= \sup_{s<t} \sup \left\{F_{\mathbf{S}\mu,\mathbf{S}\nu}(s) \mid \mu \overset{d}{=} \mathcal{A},\ \nu \overset{d}{=} \mathcal{B}\right\} \\
&\geq \sup_{s<t} \sup \left\{F_{\mu,\nu}\left(\frac{s}{\lambda_q^{1/q}}\right) \mid \mu \overset{d}{=} \mathcal{A},\ \nu \overset{d}{=} \mathcal{B}\right\} = F_{\mathcal{A},\mathcal{B}}\left(\frac{t}{\lambda_q^{1/q}}\right)
\end{aligned} \tag{4.13}$$

for all $t > 0$. For $0 < q < 1$, the demonstration is similar.

Consider $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $\mathcal{S}\mathcal{D}_1 = \mathcal{D}_1$ and $\mathcal{S}\mathcal{D}_2 = \mathcal{D}_2$. Since $\mathcal{D}_1 = \mathcal{S}^n(\mathcal{D}_1)$ and $\mathcal{D}_2 = \mathcal{S}^n(\mathcal{D}_2)$, we have

$$F_{\mathcal{D}_1,\mathcal{D}_2}(t) \geq F_{\mathcal{D}_1,\mathcal{D}_2}\left(\frac{t}{r^n}\right) \tag{4.14}$$

for all $t > 0$. Using $\lim_{n\to\infty} r^n = 0$, it follows that

$$F_{\mathcal{D}_1,\mathcal{D}_2}(t) = 1 \tag{4.15}$$

for all $t > 0$. □

**Remark 4.3.** Since $\lambda_q^{1/q} \to \max_i r_i$ as $q \to \infty$, we can regard [12, Theorem 4.2] as a limit case of Theorem 2.7. More precisely, if $\max_i r_i < 1$, then $\operatorname{sprt}\mu^*$ is the unique compact set satisfying the random scaling law for sets $(S_1,\dots,S_N)$.

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József Kolumbán: Faculty of Mathematics and Computer Science, Babes-Bolyai University, 3400 Cluj-Napoca, Romania

*E-mail address*: kolumban@math.ubbcluj.ro

Anna Soós: Faculty of Mathematics and Computer Science, Babes-Bolyai University, 3400 Cluj-Napoca, Romania

*E-mail address*: asoos@math.ubbcluj.ro

Ibolya Varga: Faculty of Mathematics and Computer Science, Babes-Bolyai University, 3400 Cluj-Napoca, Romania

*E-mail address*: ivarga@cs.ubbcluj.ro