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KKM THEOREM WITH APPLICATIONS TO LOWER AND UPPER BOUNDS EQUILIBRIUM PROBLEM IN G-CONVEX SPACES

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We give some new versions of KKM theorem for generalized convex spaces. As an application, we answer a question posed by Isac et al. (1999) for the lower and upper bounds equilibrium problem.

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1. Introduction. In [5], Isac et al. raised the following open problem which is closely related to the equilibrium problem. Given a closed nonempty subset K in a locally convex semireflexive topological space, a mapping $f: K\times K \to \mathbb{R}$, and two real numbers α,β, where $\alpha \le \beta$, it is interesting to know under what conditions there exists an $\bar{x}\in K$ such that

$$\alpha \le f(\bar{x},y) \le \beta, \quad \forall y \in K. \tag{1.1}$$

First, Li [8] gave some answers to the open problem (1.1) by introducing and using the concept of extremal subsets. Then Chadli et al. [1] gave some answers to this open problem by a method different from that Li used. Our goal in this paper is to derive some more results in answering this problem in G-convex spaces. In fact, we will derive some results of problem (1.1) for bifunctions that are defined on $X\times X$, for which X is a G-convex space.

Let X be nonempty set. We denote by 2^X the family of all subsets of X, by $\mathcal{F}(X)$ the family of all nonempty finite subsets of X, and by $|A|$ the cardinality of $A\in\mathcal{F}(X)$.

Let Y be a nonempty set and let X be a topological space. If $F: Y \to 2^X$ is a multivalued map, then we say that F is transfer closed-valued if, for any $(y,x)\in Y\times X$ with $x\notin F(y)$, there exists $y'\in Y$ such that $x\notin \operatorname{cl}F(y')$; see Tian [14]. It is clear that this definition is equivalent to saying that $\bigcap_{y\in Y}F(y) = \bigcap_{y\in Y}\operatorname{cl}F(y)$. If $B\subseteq Y$ and $A\subseteq X$, then we say that $F: B\to 2^A$ is transfer closed-valued if the multivalued map $y\to F(y)\cap A$ is transfer closed-valued. In the case when $X=Y$ and $A=B$, we say that F is transfer closed-valued on A.

Let f be a bifunction on $X\times Y$, then f is called λ-transfer lower semicontinuous (l.s.c.) on the first variable on X if, for each $(x,y)\in X\times Y$ with $f(x,y)>\lambda$, there exist $y'\in Y$ and a neighborhood $U(x)$ of x in X such that $f(z,y')>\lambda$ for all $z \in U(x)$. The bifunction f is said to be λ-transfer upper semicontinuous (u.s.c.) on the first variable on X if $-f$ is λ-transfer l.s.c. on the first variable. If f is defined on $Y \times X$, then λ-transfer l.s.c. (u.s.c.) bifunction on second variable on X is defined by a similar method. It is easily seen that an l.s.c (u.s.c.) bifunction is λ-transfer l.s.c (u.s.c.) bifunction for each λ.

A generalized convex space or G-convex space was first introduced by Park and Kim [12], and more recently, it has been generalized by Park [10]. A G-convex space $(X,D;\Gamma)$ consists of a topological space X, a nonempty set D, and a multivalued map $\Gamma : \mathcal{F}(D) \to 2^X \setminus \{\varnothing\}$ such that, for each $A \in \mathcal{F}(D)$ with the cardinality $|A| = n+1$, there exists a continuous function $\Phi_A : \Delta_n \to \Gamma(A)$ such that each $J \in \mathcal{F}(A)$ implies $\Phi_A(\Delta_J) \subset \Gamma(J)$, for which if $A = \{a_0, a_1, \dots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \dots, a_{i_j}\}$, then $\Delta_J = \operatorname{co}\{e_{i_0}, \dots, e_{i_j}\}$. When $D = X$, we will write $(X;\Gamma)$ in place of $(X,X;\Gamma)$. If $(X,D;\Gamma)$ is a G-convex space, $D \subseteq X$, and $K \subset X$, then K is G-convex if for each $A \in \mathcal{F}(D)$, $A \subset K$ implies $\Gamma(A) \subset K$. The G-convex hull of K denoted by G-$\operatorname{co}K$ is the set $\bigcap\{B \subset X : B$ is a G-convex subset of X containing $K\}$.

Notice that G-convex spaces contain most of the well-know spaces such as topological vector spaces, convex spaces, generalized H-spaces, L-spaces, C-spaces, and hyperconvex metric spaces (see [10, 11, 12, 13] and the references therein).

Let $(X,D;\Gamma)$ be a G-convex space, then the multivalued mapping $F : D \to 2^X$ is called a KKM map if, for each finite subset A of D, we have $\Gamma(A) \subseteq F(A)$; see Park and Lee [13]. If $x \mapsto \operatorname{cl}F(x)$ is a KKM map, then we say that $\operatorname{cl}F$ is a KKM map.

2. Main results. The KKM theorem is a very important tool in the study of the equilibrium problem. To solve problem (1.1) on G-convex spaces, we first give some refined versions of the KKM theorem. The following KKM theorem, due to Park and Lee [13, Theorem 1], is essential for obtaining our main results.

Theorem 2.1. *Let $(X,D;\Gamma)$ be a G-convex space and let $F : D \to 2^X$ be a multimap such that*

(1) *F has closed (resp., open) values,*

(2) *F is a KKM map.*

Then $\{F(z) : z \in D\}$ has the finite intersection property. More precisely, for each $N \in \mathcal{F}(D)$, $\Gamma(N) \cap (\bigcap_{z \in N} F(z) \neq \varnothing)$. Further, if

(3) *$\bigcap_{z \in M} \operatorname{cl}F(z)$ is compact for some $M \in \mathcal{F}(D)$, then $\bigcap_{z \in D} \operatorname{cl}F(z) \neq \varnothing$.*

As a consequence of the above theorem, we obtain the following result which is a refinement of [3, Theorem 1.1] and [7, Theorem 3.3].

Theorem 2.2. *Let $(X,D;\Gamma)$ be a G-convex space such that, for each $A,B \in \mathcal{F}(D)$ with $A \subseteq B$, $\Gamma(A) \subseteq \Gamma(B)$. Suppose that $F : D \to 2^X \setminus \{\varnothing\}$ and $G : D \to 2^X \setminus \{\varnothing\}$ are two multivalued maps such that*

(1) *$F(x) \subseteq G(x)$ for all $x \in D$,*

(2) *F is a KKM map,*

(3) *for some* $M \in \mathcal{F}(D)$, $\bigcap_{x\in M} \operatorname{cl} F(x)$ *is compact,*
(4) *for each* $A \in \mathcal{F}(D)$ *with* $M \subseteq A$, $G : A \to 2^{\Gamma(A)}$ *is transfer closed-valued,*
(5) *for each* $A \in \mathcal{F}(D)$ *with* $M \subseteq A$,

$$\operatorname{cl}\left(\bigcap_{x\in A} G(x)\right) = \bigcap_{x\in A} G(x). \tag{2.1}$$

Then $\bigcap_{x\in D} G(x) \neq \emptyset$.

Proof. Let $A \in \mathcal{F}(D)$ with $M \subseteq A$. Consider a multivalued map $F_A : A \to 2^{\Gamma(A)} \setminus \{\emptyset\}$ defined by $F_A(x) := \operatorname{cl}_{\Gamma(A)}(F(x) \cap \Gamma(A))$ for all $x \in A$. Then $F_A(x)$ is closed in $\Gamma(A)$. Also F_A is a KKM map. In fact, if $B \in \mathcal{F}(A)$, then $\Gamma(B) \subseteq \Gamma(A)$ and $\Gamma(B) \subseteq \bigcup_{x\in B} F(x)$, thus $\Gamma(B) \subseteq (\bigcup_{x\in B} F(x)) \cap \Gamma(A) \subseteq \bigcup_{x\in B} F_A(x)$. So, by Theorem 2.1, we have

$$\bigcap_{x\in A} F_A(x) \neq \emptyset. \tag{2.2}$$

Let $\{A_i : i \in I\}$ be the family of all finite subsets of D containing the set M, partially ordered by $\subseteq$. Now, for each $i \in I$, let $X_i = \Gamma(A_i)$. By (2.2),

$$\bigcap_{x\in A_i} \operatorname{cl}_{X_i}\left(F(x) \cap X_i\right) \neq \emptyset, \quad \text{for each } i \in I. \tag{2.3}$$

Take any $x_i \in \bigcap_{x\in A_i} \operatorname{cl}_{X_i}(F(x) \cap X_i)$. For each $i \in I$, let $Y_i = \{x_j : j \geq i,\ j \in I\}$. Clearly, we have that $\{Y_i : i \in I\}$ has finite intersection property, and $Y_i \subseteq \bigcap_{x\in M} \operatorname{cl} F(x)$, for all $i \in I$. Hence, by condition (3), $\operatorname{cl} Y_i$ is compact. Therefore $\bigcap_{i\in I} \operatorname{cl} Y_i \neq \emptyset$. Choose any $\bar{x} \in \bigcap_{i\in I} \operatorname{cl} Y_i$. Also, for any $i, j \in I$ with $j \geq i$, we have

$$\begin{aligned} x_j \in \bigcap_{x\in A_j} \operatorname{cl}_{X_j}\left(F(x) \cap X_j\right) &\subseteq \bigcap_{x\in A_j} \operatorname{cl}_{X_j}\left(G(x) \cap X_j\right) \\ &= \bigcap_{x\in A_j} \left(G(x) \cap X_j\right) \subseteq \bigcap_{x\in A_i} \left(G(x) \cap X_j\right) \\ &\subseteq \bigcap_{x\in A_i} G(x). \end{aligned} \tag{2.4}$$

Therefore, $Y_i \subseteq \bigcap_{x\in A_i} G(x)$. Now, for any $x \in D$, there exists $i_0 \in I$ such that $x \in A_{i_0}$. It follows that

$$\bar{x} \in \operatorname{cl} Y_{i_0} \subseteq \operatorname{cl}\left(\bigcap_{z\in A_{i_0}} G(z)\right) = \bigcap_{z\in A_{i_0}} G(z) \subseteq G(x). \tag{2.5}$$

Then $\bar{x} \in G(x)$ for all $x \in X$, and the proof is completed. □

By Theorem 2.1 and the fact that $\bigcap_{x\in D} G(x) = \bigcap_{x\in D} \operatorname{cl} G(x)$, when G is transfer closed-valued, we can obtain the following result.

Theorem 2.3. *Let $(X,D;\Gamma)$ be a G-convex space. Suppose that $F : D \to 2^X \setminus \{\varnothing\}$ and $G : D \to 2^X \setminus \{\varnothing\}$ are two multivalued maps such that*

(1) $F(x) \subseteq G(x)$ *for all* $x \in D$,
(2) $\mathrm{cl}\, F$ *is a KKM map,*
(3) *for some* $M \in \mathcal{F}(D)$, $\bigcap_{x\in M} \mathrm{cl}\, F(x)$ *is compact,*
(4) G *is transfer closed-valued.*

Then $\bigcap_{x\in D} G(x) \neq \varnothing$.

The following examples show that Theorems 2.2 and 2.3 are different.

Example 2.4. Assume that $X = \mathbb{R}$ and $D = \mathbb{N}$. If we define $\Gamma(A) = \mathrm{co}(A+1)$ for every $A \in \mathcal{F}(D)$, then $(X,D;\Gamma)$ is a G-convex space and $\Gamma(A) \neq G\text{-}\mathrm{co}\, A$. Suppose that $F : D \to 2^X$ is defined as

$$F(x) = \begin{cases} \{1,2\} \cup ((-\infty,0) \cap \mathbb{Q}) & \text{if } x = 1, \\ (1,+\infty) & \text{if } x = 2, \\ \mathbb{R} & \text{if } x \neq 1,2. \end{cases} \tag{2.6}$$

By taking $M = \{1,2\}$ and $F = G$, all the conditions of Theorem 2.2 are satisfied and $\bigcap_{x\in D} F(x) = \{2\}$, but $\bigcap_{x\in D} \mathrm{cl}\, F(x) = \{1,2\}$. Therefore, F is not transfer closed-valued and so we cannot apply Theorem 2.3.

The following example is a modified form of [14, Example 1].

Example 2.5. If $X = [0,1]$, $D = \mathbb{Q} \cap X$, and $\Gamma(A) = [\min A, 1]$, for every $A \in \mathcal{F}(D)$, then $(X,D;\Gamma)$ is a G-convex space. Suppose that $F : D \to 2^X$ is defined by $F(x) = [x,1] \cap \mathbb{Q}$. If $F = G$, then all the conditions of Theorem 2.3 are satisfied. But F is not KKM map and moreover for $A = \{0, 0.5\}$, conditions (4) and (5) are not satisfied.

By a method similar to that of the proof of Theorem 2.2, we can obtain the following result which is an improvement of [2, Lemma 2] and [6, Lemma 3.1] on G-convex spaces.

Theorem 2.6. *Let $(X;\Gamma)$ be a G-convex space and let $G\text{-}\mathrm{co}\, A$ be closed for each $A \in \mathcal{F}(X)$. Suppose that $F : X \to 2^X \setminus \{\varnothing\}$ and $G : X \to 2^X \setminus \{\varnothing\}$ are two multivalued maps such that*

(1) $F(x) \subseteq G(x)$ *for all* $x \in X$,
(2) F *is a* KKM *map,*
(3) *for some* $M \in \mathcal{F}(X)$, $\bigcap_{x\in M} \mathrm{cl}\, F(x)$ *is compact,*
(4) *for each* $A \in \mathcal{F}(X)$ *with* $M \subseteq A$, G *is transfer closed-valued on* $G\text{-}\mathrm{co}\, A$,
(5) *for each* $A \in \mathcal{F}(X)$ *with* $M \subseteq A$,

$$\mathrm{cl}\left(\bigcap_{x\in G\text{-}\mathrm{co}\, A} G(x)\right) \cap G\text{-}\mathrm{co}\, A = \left(\bigcap_{x\in G\text{-}\mathrm{co}\, A} G(x)\right) \cap G\text{-}\mathrm{co}\, A. \tag{2.7}$$

Then $\bigcap_{x\in X} G(x) \neq \varnothing$.

Remark 2.7. (a) If, in Theorem 2.3, X is Hausdorff and $X = D$, then condition (3) can be replaced by the following condition:

(3′) there exists a compact subset K of X such that, for each $N \in \mathcal{F}(X)$, there exists a nonempty compact G-convex subset L_N of X such that $\bigcap_{x\in L_N} \mathrm{cl}\, F(x) \subseteq K$.

(b) If, in Theorem 2.6, for each $A \in \mathcal{F}(X)$, G-$\mathrm{co}\, A$ is compact, then, instead of conditions (3) and (4) we can assume that

(3′) there exists $M \in \mathcal{F}(X)$ such that $\mathrm{cl}(\bigcap_{x\in M} F(x))$ is compact,

(4′) for each $A \in \mathcal{F}(X)$ with $M \subseteq A$, F is transfer closed-valued on G-$\mathrm{co}\, A$.

Then the conclusion of Theorem 2.6 holds. In this case, we obtain a refinement of Lemma 2.3 of Ding and Tarafdar [4]. Also condition (3) of Theorem 2.6 can be replaced by the following condition:

(3′′) there exists $M \in \mathcal{F}(X)$ such that $\mathrm{cl}(\bigcap_{x\in M} G(x))$ is compact.

(c) Example 2.4 shows that, in general, $\Gamma(A) \neq G$-$\mathrm{co}\, A$. Therefore, Theorem 2.6 has its own applications.

Now, by Theorem 2.2, we obtain the following result, which gives an answer to problem (1.1).

Theorem 2.8. *Let $(X,D;\Gamma)$ be a G-convex space such that for each $A,B \in \mathcal{F}(D)$ with $A \subseteq B$, $\Gamma(A) \subseteq \Gamma(B)$. Suppose that f and g are two real bifunctions defined on $X \times D$ such that*

(1) *for each $(x,y) \in X \times D$, if $\alpha \le f(x,y) \le \beta$, then $\alpha \le g(x,y) \le \beta$;*

(2) *for each $A \in \mathcal{F}(D)$ and $B \subseteq A$ with $\emptyset \neq B \neq A$, either*

   (i) $\alpha \le \inf_{x\in\Gamma(A)} \max_{y\in B} f(x,y)$ *or*

   (ii) $\sup_{x\in\Gamma(A)} \min_{y\in A\setminus B} f(x,y) \le \beta$.

   For $B = A$, condition (i) holds, and for $B = \emptyset$, condition (ii) is satisfied;

(3) *there exist a compact subset K of X and $M \in \mathcal{F}(D)$ such that, for every $x \in X\setminus K$, there are a point $y \in M$ and a neighborhood $U(x)$ of x such that for any $z \in U(x)$, $f(z,y) < \alpha$ or $f(z,y) > \beta$;*

(4) *for each $A \in \mathcal{F}(D)$ with $M \subseteq A$, $g : \Gamma(A) \times A \to \mathbb{R}$ is α-transfer u.s.c. and β-transfer l.s.c. on the first variable on $\Gamma(A)$;*

(5) *for each $A \in \mathcal{F}(D)$ with $M \subseteq A$, $x \in X$ and for each net (x_λ) in X converging to x, if $\alpha \le g(x_\lambda, y) \le \beta$ for all $y \in A$, then $\alpha \le g(x,y) \le \beta$.*

Then there exists $\bar{x} \in X$ such that $\alpha \le g(\bar{x}, y) \le \beta$ for all $y \in D$.

Proof. Assume that $F, G : D \to 2^X$ are defined by

$$F(y) = \{x \in X : \alpha \le f(x,y) \le \beta\},$$
$$G(y) = \{x \in X : \alpha \le g(x,y) \le \beta\}. \tag{2.8}$$

By condition (1), $F(y) \subseteq G(y)$ for all $y \in D$. Condition (2) implies that F is a KKM map, because if there exists $A \in \mathcal{F}(D)$ such that $\Gamma(A) \not\subseteq \bigcup_{y\in A} F(y)$, then there is a point $\hat{x} \in \Gamma(A)$ such that $f(\hat{x}, y) < \alpha$ or $f(\hat{x}, y) > \beta$, for all $y \in A$. Let $B = \{y \in A : f(\hat{x}, y) < \alpha\}$, then $B = A$ or $\emptyset$, or $\emptyset \neq B \neq A$. In the case when

$B = A$ or $B = \emptyset$, we have $\max_{y\in A} f(\hat{x},y) < \alpha$ or $\min_{y\in A} f(\hat{x},y) > \beta$. If $\emptyset \neq B \neq A$, then $\max_{y\in B} f(\hat{x},y) < \alpha$ and $\min_{y\in A\setminus B} f(\hat{x},y) > \beta$ which contradicts condition (2). Also, by condition (3) we have $\bigcap_{y\in M} \operatorname{cl} F(y) \subseteq K$. Now, we show that condition (4) implies that $G : A \to 2^{\Gamma(A)}$ is transfer closed-valued for each $A \in \mathcal{F}(D)$ with $M \subseteq A$. Let (x,y) be a point in $\Gamma(A)\times A$ and $x \notin \Gamma(A)\cap G(y)$. Then $g(x,y) < \alpha$ or $g(x,y) > \beta$. If $g(x,y) < \alpha$, then there exist $y' \in A$ and a neighborhood $U(x)$ of x in $\Gamma(A)$ such that $g(z,y') < \alpha$ for all $z \in U(x)$. Thus, $x \notin \operatorname{cl}_{\Gamma(A)}(\Gamma(A)\cap G(y'))$. Similarly, we can prove the case when $g(x,y) > \beta$. Moreover if $x \in \operatorname{cl}(\bigcap_{y\in A} G(y))$, then there exists a net (x_λ) in $\bigcap_{y\in A} G(y)$ such that $x_\lambda \to x$. Therefore, $\alpha \le g(x_\lambda,y) \le \beta$ for all $y \in A$, and by condition (5), we have $\alpha \le g(x,y) \le \beta$. Hence $x \in \bigcap_{y\in A} G(y)$ and so, by Theorem 2.2, we have $\bigcap_{y\in D} G(y) \neq \emptyset$. □

Remark 2.9. (a) If in Theorem 2.8 instead of condition (4) we assume the following condition:

(4′) g is α-transfer u.s.c. and β-transfer l.s.c. on the first variable on X,

then, by Theorem 2.3 and without condition (5), we can obtain another answer for problem (1.1). In the above case, if $X = D$ and X is Hausdorff, then by Remark 2.7(a), condition (3) can be replaced by the following condition:

(3′) there exists a compact subset K of X such that, for every $N \in \mathcal{F}(X)$ there is a nonempty compact G-convex subset L_N of X such that for every $x \in X\setminus K$, there are a point $y \in L_N$ and a neighborhood $U(x)$ of x such that for any $z \in U(x)$ we have $f(z,y) < \alpha$ or $f(z,y) > \beta$.

(b) If in Theorem 2.8 $X = D$ and $G\text{-}\operatorname{co} A$ is compact for any $A \in \mathcal{F}(X)$, then we can conclude Theorem 2.8 by replacing conditions (3), (4), and (5) by the following conditions:

(3′) there exist a compact subset K of X and $M \in \mathcal{F}(X)$ such that, for every $x \in X\setminus K$, there is a point $y \in M$ such that $f(x,y) < \alpha$ or $f(x,y) > \beta$;

(4′) for each $A \in \mathcal{F}(X)$ with $M \subseteq A$, $f : G\text{-}\operatorname{co} A \times G\text{-}\operatorname{co} A \to \mathbb{R}$ is α-transfer u.s.c. and β-transfer l.s.c. on the first variable on $G\text{-}\operatorname{co} A$;

(5′) for each $A \in \mathcal{F}(X)$ with $M \subseteq A$, $x,y \in G\text{-}\operatorname{co} A$, and for each net (x_λ) in X converging to x, if $\alpha \le g(x_\lambda,z) \le \beta$ for all $z \in \Gamma(\{x,y\})$, then $\alpha \le g(x,y) \le \beta$.

(c) In part (a), if X is a nonempty convex subset of a Hausdorff topological vector space, then we can obtain a refinement of [1, Theorem 2.3] and [8, Theorem 3.1].

Theorem 2.10. *Let $(X;\Gamma)$ be a Hausdorff G-convex space, for any finite subset A of X, and let $G\text{-}\operatorname{co} A$ be compact. Suppose that f, g_1, and g_2 are real bifunctions on $X\times X$ satisfying the following conditions:*

(1) *$g_1(x,x) \ge \alpha$ and $g_2(x,x) \le \beta$, for all $x \in X$;*

(2) *for every $x \in X$ and for every $A \in \mathcal{F}(X)$ if $A \subseteq \{y \in X : f(x,y) < \alpha$ or $f(x,y) > \beta\}$, $\Gamma(A) \subseteq \{y \in X : g_1(x,y) < \alpha$ or $g_2(x,y) > \beta\}$;*

(3) *there exist compact subset K of X and $M \in \mathcal{F}(X)$ such that the set $\{y \in M : f(x,y) < \alpha$ or $f(x,y) > \beta\}$ is nonempty for each $x \in X\setminus K$;*

(4) *for each $A \in \mathcal{F}(X)$ with $M \subseteq A$, $f : G\text{-co}\,A \times G\text{-co}\,A \to \mathbb{R}$ is α-transfer u.s.c. and β-transfer l.s.c. on the first variable on $G\text{-co}\,A$;*

(5) *for each $A \in \mathcal{F}(X)$ with $M \subseteq A$, $x, y \in G\text{-co}\,A$, and for each net (x_λ) in X converging to x, if $\alpha \le f(x_\lambda, z) \le \beta$ for all $z \in \Gamma(\{x,y\})$, then $\alpha \le f(x,y) \le \beta$.*

Then there exists $\bar{x} \in X$ such that $\alpha \le f(\bar{x}, y) \le \beta$ for each $y \in X$.

Proof. Let $F : X \to 2^X$ be defined by

$$F(y) = \{x \in X : \alpha \le f(x,y) \le \beta\}. \tag{2.9}$$

First, we show that F is a KKM map. Assume that there exists $A \in \mathcal{F}(X)$ such that $\Gamma(A) \not\subseteq \bigcup_{y \in A} F(y)$. Therefore, $\Gamma(A)$ contains a point x_0 which is not in $\bigcup_{y \in A} F(y)$. Hence, by condition (2), we have $g_1(x_0, x_0) < \alpha$ or $g_2(x_0, x_0) > \beta$. This contradicts condition (1). Condition (3) implies that $\bigcap_{y \in M} F(y) \subseteq K$. As in the proof of Theorem 2.8, condition (4) implies condition (4′) of Remark 2.7, and condition (5) implies condition (5) of Theorem 2.6. Therefore, by Theorem 2.6 and part (b) of Remark 2.7, we have $\bigcap_{y \in X} F(y) \neq \emptyset$. □

Remark 2.11. If, in Theorem 2.10, instead of conditions (3) and (4), we have the following conditions:

(3′) there exists a compact subset K of X such that for every $N \in \mathcal{F}(X)$ there is a nonempty compact G-convex subset L_N of X such that for every $x \in X \setminus K$ there are a point $y \in L_N$ and a neighborhood $U(x)$ of x such that for any $z \in U(x)$, we have $f(z,y) < \alpha$ or $f(z,y) > \beta$;

(4′) f is α-transfer u.s.c. and β-transfer l.s.c. on the first variable on X.

Then, by Remark 2.7(a) and without condition (5) we can obtain a refinement of [1, Theorem 2.2]. Also if g_1 and g_2 are identical and equal to f, then we obtain an improvement of [8, Theorem 3.1].

3. Some applications.

In this section, we give some applications of Theorem 2.8 and Remark 2.9.

Theorem 3.1. *Let $(X, D; \Gamma)$ be a G-convex space such that for each $A, B \in \mathcal{F}(D)$ with $A \subseteq B$, $\Gamma(A) \subseteq \Gamma(B)$. Suppose that f_1 and g_1 are two real bifunctions defined on $D \times X$ such that*

(1) *for each $(y,x) \in D \times X$, if $f_1(y,x) \le c$, then $g_1(y,x) \le c$,*

(2) *for each $A \in \mathcal{F}(D)$, $\sup_{x \in \Gamma(A)} \min_{y \in A} f_1(y,x) \le c$,*

(3) *there exist a compact subset K of X and $M \in \mathcal{F}(D)$ such that, for every $x \in X \setminus K$, there exist a point $y \in M$ and a neighborhood $U(x)$ of x such that for any $z \in U(x)$, $f_1(y,z) > c$,*

(4) *for each $A \in \mathcal{F}(D)$ with $M \subseteq A$, $g_1 : A \times \Gamma(A) \to \mathbb{R}$ is c-transfer l.s.c. on the second variable on $\Gamma(A)$,*

(5) *for each $A \in \mathcal{F}(D)$ with $M \subseteq A$ and each net (x_λ) in X converging to x, if $g_1(y, x_\lambda) \le c$ for all $y \in A$, then $g_1(y,x) \le c$.*

Then there exists $\bar{x} \in X$ such that $g_1(y, \bar{x}) \le c$ for all $y \in D$.

Proof. Define $f, g : X \times D \to \mathbb{R}$ by $f(x,y) = e^{f_1(y,x)}$ and $g(x,y) = e^{g_1(y,x)}$. If $\alpha = 0$ and $\beta = e^c$, then it is easy to see that all of the conditions of Theorem 2.8 are satisfied. Therefore, there is a point $\bar{x} \in X$ such that $0 \le g(\bar{x}, y) \le e^c$ for all $y \in D$, that is, $g_1(y,\bar{x}) \le c$ for all $y \in D$. □

Corollary 3.2. *Let $(X,D;\Gamma)$ be a G-convex space such that for each $A,B \in \mathcal{F}(D)$ with $A \subseteq B$, $\Gamma(A) \subseteq \Gamma(B)$. Suppose that φ and ψ are two real bifunctions defined on $X \times D$ such that*

(1) *for each $(x,y) \in X \times D$, if $\varphi(x,y) \ge 0$, then $\psi(x,y) \ge 0$,*
(2) *for each $A \in \mathcal{F}(D)$, $\inf_{x\in\Gamma(A)} \max_{y\in A} \varphi(x,y) \ge 0$,*
(3) *there exist a compact subset K of X and $M \in \mathcal{F}(D)$ such that for every $x \in X \setminus K$ there exist a point $y \in M$ and a neighborhood $U(x)$ of x such that for any $z \in U(x)$, $\varphi(z,y) < 0$,*
(4) *for each $A \in \mathcal{F}(D)$ with $M \subseteq A$, $\psi : \Gamma(A) \times A \to \mathbb{R}$ is 0-transfer u.s.c. on the first variable on $\Gamma(A)$,*
(5) *for each $A \in \mathcal{F}(D)$ with $M \subseteq A$ and each net (x_λ) in X converging to x, if $\psi(x_\lambda, y) \ge 0$ for all $y \in A$, then $\psi(x,y) \ge 0$.*

Then there exists $\bar{x} \in X$ such that $\psi(\bar{x}, y) \ge 0$ for all $y \in D$.

Proof. It is enough in Theorem 3.1 to set $c = 0$, $f_1(y,x) = -\varphi(x,y)$, and $g_1(y,x) = -\psi(x,y)$. □

If (X,Γ) is a G-convex space, then $g : X \to \mathbb{R}$ is G-quasiconvex if $\{x \in X : g(x) < \lambda\}$ is G-convex for each $\lambda \in \mathbb{R}$.

Remark 3.3. If in Corollary 3.2 $X = D$, for each $x \in X$, $y \mapsto \varphi(x,y)$ is G-quasiconvex, and $\varphi(x,x) \ge 0$, then condition (2) of Corollary 3.2 is satisfied. So Corollary 3.2 improves [9, Corollary 2].

If $X = D$, X is Hausdorff space and $G\text{-co}\,A$ is compact for any $A \in \mathcal{F}(X)$, then instead of conditions (3), (4), and (5) of Theorem 3.1 we can suppose that

(3′) there exist a compact subset K of X and $M \in \mathcal{F}(X)$ such that, for every $x \in X \setminus K$, there exists a point $y \in M$ such that $f_1(y,x) > c$;
(4′) for each $A \in \mathcal{F}(X)$ with $M \subseteq A$, f_1 is c-transfer l.s.c. on the second variable on $G\text{-co}\,A$,
(5′) for each $A \in \mathcal{F}(X)$ with $M \subseteq A$, $x,y \in G\text{-co}\,A$, and each net (x_λ) in X converging to x, if $g_1(z,x_\lambda) \le c$ for all $z \in \Gamma(\{x,y\})$, then $g_1(y,x) \le c$.

In the above case we obtain a refinement of [2, Theorem 2], [6, Theorem 3.2], and [15, Theorems 2.2 and 2.3].

The following corollary improves [9, Corollary 3].

Corollary 3.4. *Let $(X;\Gamma)$ be a Hausdorff G-convex space and let $G\text{-co}\,A$ be compact for all $A \in \mathcal{F}(X)$. Suppose that Y is a topological space, $T : X \to 2^Y$ is a multivalued mapping having a continuous selection f, and $\phi : X \times Y \times X \to \mathbb{R}$ is a function such that*

(1) $\phi(x,y,z)$ *is* G*-quasiconvex in* z*,*
(2) $\phi(x,f(x),z) \geq 0$ *for all* $x \in X$*,*
(3) *there exist a compact subset* K *of* X *and* $M \in \mathcal{F}(X)$ *such that, for every* $x \in X \setminus K$ *and* $y \in Y$ *there exists a point* $z \in M$ *such that* $\phi(x,y,z) < 0$*,*
(4) *for each* $A \in \mathcal{F}(X)$ *with* $M \subseteq A$*,* $\phi(x,y,z)$ *is* 0*-transfer u.s.c. in* (x,y) *on* $G\text{-co}\,A$*,*
(5) *for each* $A \in \mathcal{F}(X)$ *with* $M \subseteq A$*,* $x,z \in G\text{-co}\,A$*, and for each net* (x_λ) *in* X *converging to* x*, if* $\phi(x_\lambda, f(x_\lambda), z') \geq 0$ *for all* $z' \in \Gamma(\{x,z\})$*, then* $\phi(x,f(x),z) \geq 0$*.*

Then there exist an $\bar{x} \in X$ *and* $\bar{y} \in T(\bar{x})$ *such that* $\phi(\bar{x},\bar{y},z) \geq 0$ *for all* $z \in X$*.*

Proof. Let $\varphi(z,x) = \psi(z,x) = -\phi(x,f(x),z)$ for $(x,z) \in X \times X$. Then ψ satisfies all of the requirements of Remark 3.3. Therefore, by Theorem 3.1, we have the conclusion. □

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