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ON UNIFORMLY CLOSE-TO-CONVEX FUNCTIONS AND UNIFORMLY QUASICONVEX FUNCTIONS

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Two new subclasses of uniformly convex and uniformly close-to-convex functions are introduced. We obtain inclusion relationships and coefficient bounds for these classes.

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1. The class UCC(α)**.** Denote by S the family consisting of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

that are analytic and univalent in $\Delta = \{z : |z| < 1\}$ and by C, S^*, and K the subfamilies of functions that are, respectively, convex, starlike, and close to convex in Δ. Noor and Thomas [7] introduced the class of functions known as quasiconvex functions. A normalized function of the form (1.1) is said to be quasiconvex in Δ if there exists a convex function g with $g(0) = 0$, $g'(0) = 1$ such that for $z \in \Delta$,

$$\operatorname{Re} \frac{(zf'(z))'}{g'(z)} > 0. \tag{1.2}$$

Let Q denote the class of quasiconvex functions defined in Δ. It was shown that $Q \prec K$, where $\prec$ denotes subordination, so that every quasiconvex function is close to convex. Goodman [2, 3] introduced the classes UCV and UST of uniformly convex and uniformly starlike functions. In [10], Rønning defined the class UCV(α), $-1 \le \alpha < 1$, consisting of functions of the form (1.1) satisfying

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) - \alpha \ge \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in \Delta. \tag{1.3}$$

Geometrically, UCV(α) is the family of functions f for which $1 + zf''(z)/f'(z)$ takes values that lie inside the parabola $\Omega = \{\omega : \operatorname{Re}(\omega - \alpha) > |\omega - 1|\}$, which is symmetric about the real axis and whose vertex is $w = (1+\alpha)/2$.

Since the function

$$q_\alpha(z) = 1 + \frac{2(1-\alpha)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 \tag{1.4}$$

maps Δ onto this parabolic region, $f \in \mathrm{UCV}(\alpha)$ if and only if

$$1 + \frac{zf''(z)}{f'(z)} \prec q_\alpha(z). \tag{1.5}$$

Rønning [10] also defined the family $S_p(\alpha)$ consisting of functions $zf'(z)$ when f is in $\mathrm{UCV}(\alpha)$. In particular, f is in $S_p(\alpha)$ if and only if $zf'(z)/f(z) \prec q_\alpha(z)$.

Note for $g(z) = zf'(z)/f(z)$ that $g(z) + zg'(z)/g(z) = 1 + zf''(z)/f'(z)$, and hence a result of Miller and Mocanu [6] shows that $\mathrm{UCV}(\alpha) \subset S_p(\alpha)$.

Kumar and Ramesha [4] investigated the class UCC of uniformly close-to-convex functions consisting of normalized functions of the form (1.1) satisfying $f'(z)/g'(z) \prec q_0(z)$, where $g(z) \in C$ and $q_0(z)$ is given by (1.4) for $\alpha = 0$.

More generally, we give the following definition.

Definition 1.1. A function f is said to be uniformly close to convex of order α, $-1 \le \alpha < 1$, denoted by $\mathrm{UCC}(\alpha)$, if $f'(z)/g'(z) \prec q_\alpha(z)$, where $q_\alpha(z)$ is as defined by (1.4) and $g(z)$ is convex.

Since $\mathrm{Re}\, q_\alpha(z) > 0$, we see that $\mathrm{UCC}(\alpha)$ is a subclass of K. To see that $\mathrm{UCC}(\alpha)$ also contains the family $S_p(\alpha)$, we note for $f \in S_p(\alpha) \subset S^*$ that $f(z) = zg'(z)$ for some $g \in C$. Hence, $zf'(z)/f(z) = f'(z)/g'(z) \prec q_\alpha(z)$.

We have thus proved the following inclusion chain.

Theorem 1.2. *For* $-1 \le \alpha < 1$, $\mathrm{UCV}(\alpha) \prec S_p(\alpha) \prec \mathrm{UCC}(\alpha) \prec K$.

We next give a sufficient condition for a function to be in $\mathrm{UCC}(\alpha)$.

Theorem 1.3. *If* $\sum_{n=2}^\infty n|a_n| \le (1-\alpha)/2$, *then* $f(z) = z + \sum_{n=2}^\infty a_n z^n$ *is in* $\mathrm{UCC}(\alpha)$, $-1 \le \alpha < 1$.

Proof. Setting $g(z) = z$, we have $f'(z)/g'(z) = f'(z) = 1 + \sum_{n=2}^\infty n a_n z^{n-1}$, so that for $z \in \Delta$,

$$\left|\frac{f'(z)}{g'(z)} - 1\right| < \sum_{n=2}^\infty n|a_n| \le 1 - \sum_{n=2}^\infty n|a_n| - \alpha \le \mathrm{Re}\, f'(z) - \alpha. \tag{1.6}$$

Thus $f'(z)/g'(z)$ lies in the parabolic region $\Omega = \{\omega : |\omega - 1| < \mathrm{Re}(\omega - \alpha)\}$. That is, $f'(z)/g'(z) \prec q_\alpha(z)$, where $q_\alpha(z)$ is as defined by (1.4). □

2. A convolution relation. We now prove a convolution result for the family $\mathrm{UCC}(\alpha)$. But first we need the following lemma.

Lemma 2.1 (see [8]). *Let $\phi(z) \in C$, $\psi \in S^*$. If $F(z)$ is analytic and $\mathrm{Re}\{F(z)\} > \alpha$, $-1 \le \alpha < 1$, then*

$$\mathrm{Re}\left\{\frac{\phi * F\psi}{\phi * \psi}\right\} > \alpha, \quad z \in \Delta. \tag{2.1}$$

The above result was proved in [11] for the case $\alpha = 0$.

Theorem 2.2. *If $f \in \mathrm{UCC}(\alpha)$, then to each $g \in S^*$, an $h \in S^*$ may be associated for which $\mathrm{Re}(f * g)/h > (1+\alpha)/2$, $z \in \Delta$.*

Proof. If $f \in \mathrm{UCC}(\alpha)$, then $f'(z)/g_1'(z) \prec q_\alpha(z)$, where $g_1(z) \in C$ and $q_\alpha(z)$ is defined by (1.4). Hence, $\mathrm{Re}(f'(z)/g_1'(z)) > (1+\alpha)/2$. Therefore, we can find an $\psi \in S^*$ for which

$$\mathrm{Re}\,\frac{zf'(z)}{\psi(z)} > \frac{1+\alpha}{2}. \tag{2.2}$$

Set $F(z) = zf'(z)/\psi(z)$. Then, for $g \in S^*$, there corresponds a $\phi \in C$ such that $z\phi' = g$. Also $f * g = zf' * \phi = \phi * F\psi$ and $h = \phi * \psi \in S^*$. By Lemma 2.1,

$$\mathrm{Re}\,\frac{\Phi * F\Psi}{\Phi * \Psi} = \mathrm{Re}\,\frac{f * g}{h} > \frac{1+\alpha}{2}, \tag{2.3}$$

and this proves the result. □

3. Coefficient estimates. We need the following result by Rogosinski [9] to obtain coefficient bounds for the class $\mathrm{UCC}(\alpha)$.

Lemma 3.1. *Let $h(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ be subordinate to $H(z) = 1 + \sum_{k=1}^{\infty} C_k z^k$. If $H(z)$ is univalent in Δ and $H(\Delta)$ is convex, then $|c_n| \le |C_1|$.*

Theorem 3.2. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathrm{UCC}(\alpha)$, then*

$$|a_n| \le (n-1)c + 1, \quad n \ge 2, \tag{3.1}$$

where $c = 4(1-\alpha)/\pi^2$.

Proof. Set

$$\Phi(z) = \frac{f'(z)}{g'(z)} = 1 + \sum_{k=1}^{\infty} c_k z^k \tag{3.2}$$

so that $\Phi(z) \prec q_\alpha(z)$, where $q_\alpha(z)$ is defined in (1.4).

Since $q_\alpha(z)$ is univalent and maps Δ onto a convex region, we may apply Lemma 3.1.

Now

$$q_\alpha(z) = 1 + \frac{8(1-\alpha)}{\pi^2} z + \cdots, \quad \text{so that } |c_n| \le \frac{8(1-\alpha)}{\pi^2}. \tag{3.3}$$

With $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, we compare the coefficients of z^n for the expansion of $\phi(z)$ to obtain

$$(n+1)|a_{n+1}| = c_n + \sum_{k=1}^{n-1} (k+1) b_{k+1} c_{n-k} + (n+1) b_{n+1}. \tag{3.4}$$

Since $g(z)$ is convex, it is well known that $|b_n| \le 1$, $n = 1, 2, \ldots$. From (3.4), we get

$$(n+1)|a_{n+1}| \le cn(n+1) + (n+1), \tag{3.5}$$

and the proof is complete. □

4. The class UQC(α).

We now introduce a natural analogue to the class UCV(α) in terms of Alexander's result on convex functions [1, page 43].

Definition 4.1. A normalized function of the form (1.1) is said to be uniformly quasiconvex of order α, $-1 \le \alpha < 1$, in Δ, denoted by UQC(α), if there exists a convex function $g(z)$ with $g(0) = 0$, $g'(0) = 1$, such that

$$\frac{(zf'(z))'}{g'(z)} \prec q_\alpha(z), \tag{4.1}$$

where $q_\alpha(z)$ is as defined by (1.4).

Remark 4.2. (1) By setting $f(z) = g(z)$, we see that UCV(α) $\subset$ UQC(α).
(2) We see that $f \in$ UQC(α) if and only if $zf' \in$ UCC(α).

In view of the above remark, we obtain from Theorem 1.3 a sufficient coefficient bound for inclusion in the family UQC(α).

Theorem 4.3. *If $\sum_{n=2}^{\infty} n^2 |a_n| \le (1-\alpha)/2$, then $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in$ UQC(α).*

We next prove a theorem which shows that every function in UQC(α) is close to convex and hence univalent. We need a result due to Miller and Mocanu [5].

Lemma 4.4. *Let $M(z)$ and $N(z)$ be regular in Δ with $M(z) = N(z) = 0$ and let α be real. If $N(z)$ maps Δ onto a possibly many-sheeted region which is starlike with respect to the origin, then for $z \in \Delta$,*

$$\operatorname{Re} \frac{M'(z)}{N'(z)} > \alpha \Longrightarrow \operatorname{Re} \frac{M(z)}{N(z)} > \alpha. \tag{4.2}$$

Theorem 4.5. *If* $F(z) \in \mathrm{UQC}(\alpha)$*, then* $F(z) \in K$ *and hence it is univalent in* Δ*.*

Proof. Since

$$\frac{(zf'(z))'}{g'(z)} \prec q_\alpha(z) \Longrightarrow \mathrm{Re}\left\{\frac{(zf'(z))'}{g'(z)}\right\} > \frac{1+\alpha}{2}, \tag{4.3}$$

an application of Lemma 4.4, with $M(z) = zf'(z)$, $N(z) = g(z)$, proves the result. □

Theorem 4.6. *If* $f(z) \in \mathrm{UQC}(\alpha)$*, then* $H(z) = \int_0^z (tf'(t))'\,dt$ *is in* $\mathrm{UCC}(\alpha)$*.*

Proof. If $f(z) \in \mathrm{UQC}(\alpha)$, then there exists a function $g(z) \in C$ such that $(zf'(z))'/g'(z) \prec q_\alpha(z)$, where $q_\alpha(z)$ is as given by (1.4). The result now follows on observing that $H'(z) = (zf'(z))'$. □

We close with coefficient estimates for the class $\mathrm{UQC}(\alpha)$.

Theorem 4.7. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathrm{UQC}(\alpha)$*, then*

$$|a_n| \le \frac{(n-1)c+1}{n}, \quad n \ge 2, \tag{4.4}$$

where $c = 4(1-\alpha)/\pi^2$*.*

Proof. Proceeding on the same lines as in the proof of Theorem 3.2, we obtain the result. □

Remark 4.8. When $\alpha = 0$, $\mathrm{UQC}(0) = Q$ [6] and we see that the bounds are lower than the corresponding bounds for Q in [6].

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