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CONTINUOUS DEPENDENCE OF SOLUTIONS IN MAGNETO-ELASTICITY THEORY

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We prove continuous dependence on the intensity coefficient and continuous dependence on the external data in the theory of magneto-elasticity. We do not require the Lamé coefficients to be positive. We use logarithmic convexity arguments similar to those of Ames and Straughan (1992) in classical thermoelasticity.

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1. Introduction. In recent years, much attention has been directed to the knowledge of existence, uniqueness, and continuous dependence in several thermomechanical situations. We recall the book of Ames and Straughan [2] where the energy method is widely considered as a tool to obtain qualitative properties of solutions. We focus our interest on coupling elastic effects with magnetic effects. A derivation of the equations and recent papers on magneto-thermoelasticity and isothermal magneto-elasticity can be found in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

In this paper, we consider the dynamical theory of magneto-elasticity. The system of equations is

$$\rho\mathbf{u}_{,tt}-\mu\triangle\mathbf{u}-(\lambda+\mu)\nabla\operatorname{div}\mathbf{u}-\alpha[\nabla\times\mathbf{h}]\times\mathbf{H}=\rho\mathbf{m}, \tag{1.1}$$

$$\beta\mathbf{h}_{,t}+\nabla\times[\nabla\times\mathbf{h}]-\beta\nabla\times[\mathbf{v}\times\mathbf{H}]=\rho\mathbf{r}, \tag{1.2}$$

$$\operatorname{div}\mathbf{h}=0, \tag{1.3}$$

where $\mathbf{u}$ denotes the displacement, $\mathbf{v}=\mathbf{u}_t$ is the velocity, and $\mathbf{h}$ the magnetic field. A (known) constant magnetic field is denoted by $\mathbf{H}=(H,0,0)$, ρ, α, and β are positive constants, and $\mathbf{m}$ and $\mathbf{r}$ are the supply terms.

Here and from now on, we use summation and differentiation conventions: subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate; summation over repeated subscripts is implied.

The logarithmic convexity method is a very useful source of information about the qualitative behavior of the solutions of several kind of equations and systems (see, e.g., [2]). In particular, the method has been used to analyze the behavior of the solutions in classical thermoelasticity. Ames and Straughan [1] applied a logarithmic convexity technique to achieve continuous dependence on the supply terms and structural stability on the coupling term for the classical linear theory of thermoelasticity. They did not require the elasticity tensor to be sign-definite. All they needed was that the elasticity coefficients were symmetric.

The aim of this paper is to obtain a continuous dependence result on the intensity of the vector field **H** and the supply terms. Our main tool is also the logarithmic convexity method.

In this paper, we restrict our attention to homogeneous and isotropic materials. It is worth recalling that the extension to inhomogeneous and anisotropic materials would be straightforward.

Let B be a bounded domain in the three-dimensional Euclidean space whose boundary ∂B is smooth enough to allow the application of the divergence theorem. We assume that the set of equations (1.1), (1.2), and (1.3) holds in $B\times(0,t_1)$ for a time value $t_1<\infty$, and we assume the boundary conditions

$$\mathbf{u}=0,\quad \mathbf{h}\cdot\mathbf{n}=0,\quad [\nabla\times\mathbf{h}]\times\mathbf{n}=0,\quad \text{on } \partial B\times(0,\infty), \tag{1.4}$$

for all $t>0$. Here and from now on, we denote by $\mathbf{n}$ the normal vector to the boundary directed to the exterior. We impose the initial conditions

$$\mathbf{u}(\mathbf{x},0)=\mathbf{f}(\mathbf{x}),\quad \mathbf{v}(\mathbf{x},0)=\mathbf{g}(\mathbf{x}),\quad \mathbf{h}(\mathbf{x},0)=\mathbf{h}_0(\mathbf{x}),\quad \text{in } B. \tag{1.5}$$

For later use, we recall that the following inequality

$$\int_B (h_ih_i+h_{i,j}h_{i,j})\,dV \le C\int_B (h_{i,j}-h_{j,i})(h_{i,j}-h_{j,i})\,dV \tag{1.6}$$

holds with any vector field (h_i) that satisfies (1.3) and the second and third equalities of (1.4). Here, C is a constant that depends on the domain B.

Here are the contents of the paper. In Section 2, we prove some lemmas and we state some other preliminaries. In Section 3, we prove the continuous dependence result.

2. Preliminaries. We denote by $(u_i^{(1)},h_i^{(1)})$ the solution corresponding to the external data $(m_i^{(1)},r_i^{(1)})$ and intensity $H^{(1)}$. Let $(u_i^{(2)},h_i^{(2)})$ be the solution corresponding to the external data $(m_i^{(2)},r_i^{(2)})$ and intensity $H^{(2)}$. We introduce the notation

$$\begin{gathered} w_i=u_i^{(2)}-u_i^{(1)},\qquad l_i=h_i^{(2)}-h_i^{(1)},\qquad K=H^{(2)}-H^{(1)},\qquad \mathbf{K}=(K,0,0),\\ F_i=\rho\left(m_i^{(2)}-m_i^{(1)}\right),\qquad R_i=\rho\left(r_i^{(2)}-r_i^{(1)}\right). \end{gathered} \tag{2.1}$$

It follows that (w_i, l_i) satisfies the system

$$\rho \mathbf{w}_{,tt} - \mu \triangle \mathbf{w} - (\lambda+\mu)\nabla \operatorname{div}\mathbf{w} - \alpha[\nabla\times\mathbf{l}]\times\mathbf{H}^{(1)} - \alpha[\nabla\times\mathbf{h}^{(2)}]\times\mathbf{K} = \mathbf{F}, \tag{2.2}$$

$$\beta\mathbf{l}_{,t} + \nabla\times[\nabla\times\mathbf{l}] - \beta\nabla\times[\dot{\mathbf{w}}\times\mathbf{H}^{(1)}] - \beta\nabla\times[\mathbf{v}^{(2)}\times\mathbf{K}] = \mathbf{R}, \tag{2.3}$$

$$\operatorname{div}\mathbf{l} = 0, \tag{2.4}$$

the boundary conditions

$$\mathbf{w} = 0, \quad \mathbf{l}\cdot\mathbf{n} = 0, \quad [\nabla\times\mathbf{l}]\times\mathbf{n} = 0, \quad \text{on } \partial B\times(0,\infty), \tag{2.5}$$

and the initial conditions

$$\mathbf{w}(\mathbf{x},0) = \dot{\mathbf{w}}(0,\mathbf{x}) = \mathbf{l}(\mathbf{x},0) = 0, \quad \text{in } B. \tag{2.6}$$

Lemma 2.1. *Let*

$$V(t) = \int_0^t \int_B [\rho\dot{\mathbf{w}}\dot{\mathbf{w}} + \mu\nabla\mathbf{w}\cdot\nabla\mathbf{w} + (\lambda+\mu)(\operatorname{div}\mathbf{w})^2 + \alpha\mathbf{l}\cdot\mathbf{l}]\,dV\,ds. \tag{2.7}$$

Then,

$$\begin{aligned} V(t) = 2\int_0^t \int_B (t-s)\Big[& F_i\dot{w}_i + S_i l_i + \alpha K\Big(\dot{w}_2\Big(h_{2,1}^{(2)} - h_{1,2}^{(2)}\Big) + \dot{w}_3\Big(h_{3,1}^{(2)} - h_{1,3}^{(2)}\Big)\Big) \\ & + K\alpha\Big(l_1\Big(-\dot{u}_{2,2}^{(2)} - \dot{u}_{3,3}^{(2)}\Big) + l_2\dot{u}_{2,1}^{(2)} + l_3\dot{u}_{3,1}^{(2)}\Big) \\ & - \frac{\alpha}{\beta}(l_{i,j} - l_{j,i})(l_{i,j} - l_{j,i})\Big]\,dV\,ds, \end{aligned} \tag{2.8}$$

where

$$S_i(t) = \frac{\alpha}{\beta} R_i(t). \tag{2.9}$$

Proof. Differentiate twice and use the evolution equations and the boundary conditions to obtain

$$\begin{aligned} \frac{d^2V}{dt^2} &= 2\int_B [\mu w_{i,j}\dot{w}_{i,j} + (\lambda+\mu)w_{i,i}\dot{w}_{j,j} + \rho\dot{w}_i\ddot{w}_i + \alpha l_i\dot{l}_i]\,dV \\ &= 2\int_B [\mu w_{i,j}\dot{w}_{i,j} + (\lambda+\mu)w_{i,i}\dot{w}_{j,j}]\,dV \\ &\quad - 2\int_B [\mu w_{i,j}\dot{w}_{i,j} + (\lambda+\mu)w_{i,i}\dot{w}_{j,j} - \alpha H^1(\dot{w}_2(l_{2,1} - l_{1,2}) + \dot{w}_3(l_{3,1} - l_{1,3}))]\,dV \\ &\quad + 2\int_B \Big[F_i\dot{w}_i + \alpha K\Big(\dot{w}_2\Big(h_{2,1}^{(2)} - h_{1,2}^{(2)}\Big) + \dot{w}_3\Big(h_{3,1}^{(2)} - h_{1,3}^{(2)}\Big)\Big)\Big]\,dV \\ &\quad - 2\int_B [\alpha H^1(\dot{w}_2(l_{2,1} - l_{1,2}) + \dot{w}_3(l_{3,1} - l_{1,3}))]\,dV \end{aligned}$$

$$
\begin{aligned}
&+2\int_B \Big[S_i l_i + K\alpha\Big(l_1\Big(-\dot u^{(2)}_{2,2}-\dot u^{(2)}_{3,3}\Big)+l_2\dot u^{(2)}_{2,1}+l_3\dot u^{(2)}_{3,1}\Big)\\
&\qquad -\frac{\alpha}{\beta}(l_{i,j}-l_{j,i})(l_{i,j}-l_{j,i})\Big]dV\\
&=2\int_B \Big[F_i\dot w_i + S_i l_i + \alpha K\Big(\dot w_2\Big(h^{(2)}_{2,1}-h^{(2)}_{1,2}\Big)+\dot w_3\Big(h^{(2)}_{3,1}-h^{(2)}_{1,3}\Big)\Big)\\
&\qquad + K\alpha\Big(l_1\Big(-\dot u^{(2)}_{2,2}-\dot u^{(2)}_{3,3}\Big)+l_2\dot u^{(2)}_{2,1}+l_3\dot u^{(2)}_{3,1}\Big)-\frac{\alpha}{\beta}(l_{i,j}-l_{j,i})(l_{i,j}-l_{j,i})\Big]dV.
\end{aligned}
\tag{2.10}
$$

The lemma follows after two quadratures and from the equality

$$
\int_0^t\Big(\int_0^s f(\tau)d\tau\Big)ds=\int_0^t(t-s)f(s)ds, \tag{2.11}
$$

which is satisfied by every function $f(s)$. □

It will be useful to introduce the notation

$$
P_i(t)=\int_0^t l_i(s)ds. \tag{2.12}
$$

To obtain our results, we define the function

$$
H(t)=\int_0^t\int_B\Big(\rho w_i w_i+\frac{\alpha}{\beta}(t-s)(P_{i,j}-P_{j,i})(P_{i,j}-P_{j,i})\Big)dV\,ds. \tag{2.13}
$$

It is clear that

$$
\frac{dH}{dt}=2\int_0^t\int_B\Big(\rho w_i\dot w_i+\frac{\alpha}{2\beta}(P_{i,j}-P_{j,i})(P_{i,j}-P_{j,i})\Big)dV\,ds. \tag{2.14}
$$

The second derivative of the function H is given in the next lemma.

Lemma 2.2. *The second derivative of the function H is*

$$
\begin{aligned}
\frac{d^2H}{dt^2}&=4\int_0^t\int_B\Big[\rho\dot w_i\dot w_i+\frac{\alpha}{\beta}(t-s)(l_{i,j}-l_{j,i})(l_{i,j}-l_{j,i})\Big]dV\,ds\\
&\quad-4\int_0^t\int_B(t-s)\Big[(F_i\dot w_i+S_i l_i)+\alpha K\Big(\dot w_2\Big(h^{(2)}_{2,1}-h^{(2)}_{1,2}\Big)+\dot w_3\Big(h^{(2)}_{3,1}-h^{(2)}_{1,3}\Big)\Big)\\
&\qquad\qquad + K\alpha\Big(l_1\Big(-\dot u^{(2)}_{2,2}-\dot u^{(2)}_{3,3}\Big)+l_2\dot u^{(2)}_{2,1}+l_3\dot u^{(2)}_{3,1}\Big)\Big]dV\,ds\\
&\quad+2\int_B\Big[(F_i w_i+Q_i l_i)+\alpha K\Big(w_2\Big(h^{(2)}_{2,1}-h^{(2)}_{1,2}\Big)+w_3\Big(h^{(2)}_{3,1}-h^{(2)}_{1,3}\Big)\Big)\Big]dV\\
&\quad+2\int_B\Big[K\alpha\Big(l_1\Big(-u^{(2)}_{2,2}-u^{(2)}_{3,3}\Big)+l_2u^{(2)}_{2,1}+l_3u^{(2)}_{3,1}\Big)\\
&\qquad\qquad -K\alpha(l_1(-f_{3,3}-f_{2,2})+l_2f_{2,1}+l_3f_{3,1})\Big]dV,
\end{aligned}
\tag{2.15}
$$

where

$$Q_i(t) = \int_0^t S_i(s)\,ds. \tag{2.16}$$

Proof. A direct differentiation gives

$$\frac{d^2H}{dt^2} = 2\int_0^t\int_B \rho\dot{w}_i\dot{w}_i\,dV\,ds + 2\int_0^t\int_B\Big[\rho w_i\ddot{w}_i + \frac{\alpha}{\beta}(P_{i,j}-P_{j,i})(l_{i,j}-l_{j,i})\Big]dV\,ds. \tag{2.17}$$

Now, we make some calculations to determine the evolution of the second integral. If we multiply (2.2) by w_i, and integrate over B, we obtain

$$\begin{aligned}\int_B \rho w_i\ddot{w}_i\,dV &= -\int_B\big[\mu w_{i,j}w_{i,j}+(\lambda+\mu)w_{i,i}w_{j,j} - \alpha H^1(w_2(l_{2,1}-l_{1,2})+w_3(l_{3,1}-l_{1,3}))\big]dV \\ &\quad + \int_B\Big[F_iw_i + \alpha K\Big(w_2\Big(h^{(2)}_{2,1}-h^{(2)}_{1,2}\Big)+w_3\Big(h^{(2)}_{3,1}-h^{(2)}_{1,3}\Big)\Big)\Big]dV.\end{aligned} \tag{2.18}$$

If we integrate (2.3) with respect to the time parameter, multiply it by l_i, and integrate over B, we obtain

$$\begin{aligned}\frac{\alpha}{\beta}\int_B l_il_i\,dV &= -\int_B\Big[\alpha H^1(w_2(l_{2,1}-l_{1,2})+w_3(l_{3,1}-l_{1,3})) - \frac{\alpha}{\beta}(P_{i,j}-P_{j,i})(l_{i,j}-l_{j,i})\Big]dV \\ &\quad + \int_B\Big[Q_il_i + K\alpha\Big(l_1\Big(-u^{(2)}_{2,2}-u^{(2)}_{3,3}\Big)+l_2u^{(2)}_{2,1}+l_3u^{(2)}_{3,1}\Big) \\ &\qquad - K\alpha(l_1(-f_{3,3}-f_{2,2})+l_2f_{2,1}+l_3f_{3,1})\Big]dV.\end{aligned} \tag{2.19}$$

It follows that

$$\begin{aligned}\int_B\Big(\rho w_i\ddot{w}_i + \frac{\alpha}{\beta}(P_{i,j}-P_{j,i})(l_{i,j}-l_{j,i})\Big)dV &= -\int_B\Big[\mu w_{i,j}w_{i,j}+(\lambda+\mu)w_{i,i}w_{j,j}+\frac{\alpha}{\beta}l_il_i\Big]dV \\ &\quad + \int_B\Big[(F_iw_i+Q_il_i)+\alpha K\Big(w_2\Big(h^{(2)}_{2,1}-h^{(2)}_{1,2}\Big)+w_3\Big(h^{(2)}_{3,1}-h^{(2)}_{1,3}\Big)\Big)\Big]dV \\ &\quad + \int_B\Big[K\alpha\Big(l_1\Big(-u^{(2)}_{2,2}-u^{(2)}_{3,3}\Big)+l_2u^{(2)}_{2,1}+l_3u^{(2)}_{3,1}\Big) \\ &\qquad - K\alpha(l_1(-f_{3,3}-f_{2,2})+l_2f_{2,1}+l_3f_{3,1})\Big]dV.\end{aligned} \tag{2.20}$$

Then, we obtain

$$
\begin{aligned}
\frac{d^2H}{dt^2} = 4\int_0^t\int_B \rho\dot{w}_i\dot{w}_i\,dV\,ds \\
&-2\int_0^t\int_B\Big[\mu w_{i,j}w_{i,j}+(\lambda+\mu)w_{i,i}w_{j,j}+\frac{\alpha}{\beta}l_il_i+\rho\dot{w}_i\dot{w}_i\Big]dV\,ds \\
&+2\int_B\Big[(F_iw_i+Q_il_i)+\alpha K\Big(w_2\Big(h^{(2)}_{2,1}-h^{(2)}_{1,2}\Big)+w_3\Big(h^{(2)}_{3,1}-h^{(2)}_{1,3}\Big)\Big)\Big]dV \\
&+2\int_B\Big[K\alpha\Big(l_1\Big(-u^{(2)}_{2,2}-u^{(2)}_{3,3}\Big)+l_2u^{(2)}_{2,1}+l_3u^{(2)}_{3,1}\Big) \\
&\qquad -K\alpha(l_1(-f_{3,3}-f_{2,2})+l_2f_{2,1}+l_3f_{3,1})\Big]dV.
\end{aligned}
\tag{2.21}
$$

Lemma 2.2 is a consequence of Lemma 2.1 and equality (2.21). □

We now state a lemma concerning the behaviour of the magnetic field, which will also be used in the next section.

Lemma 2.3. *There exist three positive constants A, B^*, and C^* such that*

$$
\int_0^t\int_B l_il_i\,dV\,ds \le \int_0^{t_1}\int_B[AS_iS_i+B^*\rho\dot{w}_i\dot{w}_i+C^*K^2]\,dV\,ds, \tag{2.22}
$$

for $t\le t_1$.

Proof. In view of (2.3), we have

$$
\begin{aligned}
\int_0^t\int_B l_il_i\,dV\,ds &= \frac{1}{\beta}\int_0^t\int_B \beta l_il_i\,dV\,ds \\
&= -\frac{1}{\beta}\int_0^t\int_B\frac{\partial}{\partial s}[(t-s)\beta l_il_i]\,dV\,ds \\
&\quad +\frac{2}{\beta}\int_0^t\int_B(t-s)\beta l_i\frac{\partial l_i}{\partial s}\,dV\,ds \\
&= \frac{2}{\beta}\int_0^t\int_B(t-s)\Big[R_il_i-(l_{i,j}-l_{j,i})(l_{i,j}-l_{j,i}) \\
&\qquad -\beta H^{(1)}(\dot{w}_2(l_{2,1}-l_{1,2})+\dot{w}_3(l_{3,1}-l_{1,3})) \\
&\qquad -\beta K\Big(\dot{u}^{(2)}_2(l_{2,1}-l_{1,2})+\dot{u}^{(2)}_3(l_{3,1}-l_{1,3})\Big)\Big]dV\,ds.
\end{aligned}
\tag{2.23}
$$

The use of the arithmetic-geometric mean inequality leads to the following estimates:

$$\begin{aligned}
&\int_0^t\int_B (t-s)R_i l_i\,dV\,ds \le \frac{\epsilon_1}{2}\int_0^t\int_B t_1 R_i R_i\,dV\,ds + \frac{1}{2\epsilon_1}\int_0^t\int_B t_1 l_i l_i\,dV\,ds,\\
&\int_0^t\int_B (t-s)\big(\dot w_2(l_{2,1}-l_{1,2}) + \dot w_3(l_{3,1}-l_{1,3})\big)\,dV\,ds\\
&\qquad \le \frac{\epsilon_2}{2}\int_0^t\int_B t_1(l_{i,j}-l_{j,i})(l_{i,j}-l_{j,i})\,dV\,ds + \frac{1}{2\epsilon_2}\int_0^t\int_B t_1 \dot w_i \dot w_i\,dV\,ds,\\
&\int_0^t\int_B (t-s)K\Big(\dot u_2^{(2)}(l_{2,1}-l_{1,2}) + \dot u_3^{(2)}(l_{3,1}-l_{1,3})\Big)\,dV\,ds\\
&\qquad \le \frac{\epsilon_3}{2}\int_0^t\int_B t_1 K^2\,dV\,ds + \frac{1}{2\epsilon_3}\int_0^t\int_B t_1 \dot u_i^{(2)}(l_{i,j}-l_{j,i})(l_{i,j}-l_{j,i})\,dV\,ds,
\end{aligned} \tag{2.24}$$

where ϵ_1, ϵ_2, and ϵ_3 are arbitrary positive constants.

If we assume that $\dot u_i^{(2)}$ is uniformly bounded on the interval $[0,t_1]$, we can make a suitable choice of the parameters ϵ_i $(i=1,2,3)$ to obtain the estimate (2.22), where A, B^*, and C^* can be easily computed. □

3. Continuous dependence. In this section, we obtain continuous dependence and structural stability results. We assume that the functions

$$\sup_B \left|h_{i,j}^{(2)}\right|^2, \qquad \sup_B \left|\dot u_{i,j}^{(2)}\right|^2, \qquad \sup_B \left|u_{i,j}^{(2)} - f_{i,j}\right|^2, \tag{3.1}$$

are uniformly bounded by a constant M.

Here, we introduce a family of functions

$$H_\omega(t) = H(t) + \omega, \tag{3.2}$$

where ω is an arbitrary positive constant.

Lemma 3.1. *Let*

$$\omega = \int_0^{t_1}\int_B \left(F_iF_i + 2K^2 + S_iS_i + Q_iQ_i\right)dV\,ds. \tag{3.3}$$

Then, there exists a positive constant ξ such that

$$H_\omega \frac{d^2H_\omega}{dt^2} - \left(\frac{dH_\omega}{dt}\right)^2 \ge -\xi H_\omega^2, \tag{3.4}$$

for $t \le t_1$.

Proof. From the definition of H_ω and Lemma 2.2, it follows that

$$\begin{aligned}
&H_\omega \frac{d^2 H_\omega}{dt^2} - \left(\frac{dH_\omega}{dt}\right)^2 \\
&= 4N^2 + 4\omega \int_0^t \int_B \left(\rho \dot{w}_i \dot{w}_i + (t-s)\frac{\alpha}{\beta}(l_{i,j} - l_{j,i})(l_{i,j} - l_{j,i})\right) dV\, ds \\
&\quad - H_\omega \Bigg(4 \int_0^t \int_B (t-s)\Big[(F_i \dot{w}_i + S_i l_i) + \alpha K\left(\dot{w}_2\left(h_{2,1}^{(2)} - h_{1,2}^{(2)}\right) + \dot{w}_3\left(h_{3,1}^{(2)} - h_{1,3}^{(2)}\right)\right) \\
&\qquad\qquad + K\alpha\left(l_1\left(-\dot{u}_{2,2}^{(2)} - \dot{u}_{3,3}^{(2)}\right) + l_2 \dot{u}_{2,1}^{(2)} + l_3 \dot{u}_{3,1}^{(2)}\right)\Big] dV\, ds \\
&\qquad + 2 \int_0^t \int_B \Big[(F_i w_i + Q_i l_i) + \alpha K\left(w_2\left(h_{2,1}^{(2)} - h_{1,2}^{(2)}\right) + w_3\left(h_{3,1}^{(2)} - h_{1,3}^{(2)}\right)\right)\Big] dV \\
&\qquad + 2 \int_B \Big[K\alpha\left(l_1\left(-u_{2,2}^{(2)} - u_{3,3}^{(2)}\right) + l_2 u_{2,1}^{(2)} + l_3 u_{3,1}^{(2)}\right) \\
&\qquad\qquad - K\alpha(l_1(-f_{3,3} - f_{2,2}) + l_2 f_{2,1} + l_3 f_{3,1})\Big] dV\, ds\Bigg),
\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
N^2 &= I_1 I_2 - I_3^2, \\
I_1 &= \int_0^t \int_B \left(\rho w_i w_i + (t-s)\frac{\alpha}{\beta}(P_{i,j} - P_{j,i})(P_{i,j} - P_{j,i})\right) dV\, ds, \\
I_2 &= \int_0^t \int_B \left[\rho \dot{w}_i \dot{w}_i + (t-s)\frac{\alpha}{\beta}(l_{i,j} - l_{j,i})(l_{i,j} - l_{j,i})\right] dV\, ds, \\
I_3 &= \int_0^t \int_B \left(\rho w_i \dot{w}_i + (t-s)\frac{\alpha}{\beta}(P_{i,j} - P_{j,i})(l_{i,j} - l_{j,i})\right) dV\, ds.
\end{aligned} \tag{3.6}$$

In view of the Schwarz inequality, we have $N^2 \geq 0$.

Now, we estimate some integrals which are on the right-hand side of equality (3.5). After some uses of the Hölder inequality and inequality (1.6), we can obtain the existence of constants a_i such that

$$\begin{aligned}
&\int_0^t \int_B (t-s) F_i \dot{w}_i \, dV\, ds \leq a_1 \left(\int_0^t \int_B \rho \dot{w}_i \dot{w}_i \, dV\, ds\right)^{1/2} \left(\int_0^t \int_B F_i F_i \, dV\, ds\right)^{1/2}, \\
&\int_0^t \int_B (t-s) S_i l_i \, dV\, ds \leq a_2 \left(\int_0^t \int_B (t-s)(l_{i,j} - l_{j,i})(l_{i,j} - l_{j,i}) dV\, ds\right)^{1/2} \\
&\qquad\qquad \times \left(\int_0^t \int_B S_i S_i \, dV\, ds\right)^{1/2}, \\
&\int_0^t \int_B (t-s)\alpha K\left(\dot{w}_2\left(h_{2,1}^{(2)} - h_{1,2}^{(2)}\right) + \dot{w}_3\left(h_{3,1}^{(2)} - h_{1,3}^{(2)}\right)\right) dV\, ds \\
&\qquad \leq a_3 \left(\int_0^t \int_B \rho \dot{w}_i \dot{w}_i \, dV\, ds\right)^{1/2} \left(\int_0^t \int_B K^2 \, dV\, ds\right)^{1/2},
\end{aligned}$$

$$\begin{aligned}&\int_0^t\int_B (t-s)K\alpha\Big(l_1\Big(-\dot{u}_{2,2}^{(2)}-\dot{u}_{3,3}^{(2)}\Big)+l_2\dot{u}_{2,1}^{(2)}+l_3\dot{u}_{3,1}^{(2)}\Big)dV\,ds\\&\quad\le a_4\Big(\int_0^t\int_B (t-s)\,(l_{i,j}-l_{j,i})\,(l_{i,j}-l_{j,i})\,dV\,ds\Big)^{1/2}\\&\qquad\times\Big(\int_0^t\int_B K^2\,dV\,ds\Big)^{1/2}.\end{aligned}\tag{3.7}$$

From (3.7), it follows that

$$\begin{aligned}&\int_0^t\int_B (t-s)\Big[(F_i\dot{w}_i+S_il_i)+\alpha K\Big(\dot{w}_2\Big(h_{2,1}^{(2)}-h_{1,2}^{(2)}\Big)+\dot{w}_3\Big(h_{3,1}^{(2)}-h_{1,3}^{(2)}\Big)\Big)\\&\qquad\qquad+K\alpha\Big(l_1\Big(-\dot{u}_{2,2}^{(2)}-\dot{u}_{3,3}^{(2)}\Big)+l_2\dot{u}_{2,1}^{(2)}+l_3\dot{u}_{3,1}^{(2)}\Big)\Big]dV\,ds\\&\quad\le D\Big(\int_0^t\int_B \big(\rho\dot{w}_i\dot{w}_i\,dV\,ds+(t-s)\,(l_{i,j}-l_{j,i})\,(l_{i,j}-l_{j,i})\big)dV\,ds\Big)^{1/2}\\&\qquad\times\Big(\int_0^t\int_B (F_iF_i+S_iS_i+K^2)dV\,ds\Big)^{1/2},\end{aligned}\tag{3.8}$$

where D is an easily computable constant that depends on the constitutive coefficients, the initial conditions, the time t_1, and the domain.

The arithmetic-geometric mean inequality implies that

$$\begin{aligned}&4H_\omega\int_0^t\int_B (t-s)\Big[(F_i\dot{w}_i+S_il_i)+\alpha K\Big(\dot{w}_2\Big(h_{2,1}^{(2)}-h_{1,2}^{(2)}\Big)+\dot{w}_3\Big(h_{3,1}^{(2)}-h_{1,3}^{(2)}\Big)\Big)\\&\qquad\qquad+K\alpha\Big(l_1\Big(-\dot{u}_{2,2}^{(2)}-\dot{u}_{3,3}^{(2)}\Big)+l_2\dot{u}_{2,1}^{(2)}+l_3\dot{u}_{3,1}^{(2)}\Big)\Big]dV\,ds\\&\quad\le D^2H_\omega^2+4\Big(\int_0^t\int_B \big(\rho\dot{w}_i\dot{w}_i\,dV\,ds+(t-s)\,(l_{i,j}-l_{j,i})\,(l_{i,j}-l_{j,i})\big)dV\,ds\Big)\\&\qquad\times\Big(\int_0^t\int_B (F_iF_i+S_iS_i+K^2)dV\,ds\Big).\end{aligned}\tag{3.9}$$

Similarly, we can obtain several constants b_i such that

$$\begin{aligned}&\int_0^t\int_B F_iw_i\,dV\,ds\le b_1\Big(\int_0^t\int_B \rho w_iw_i\,dV\,ds\Big)^{1/2}\Big(\int_0^t\int_B F_iF_i\,dV\,ds\Big)^{1/2},\\&\int_0^t\int_B Q_il_i\,dV\,ds\le\Big(\int_0^t\int_B l_il_i\,dV\,ds\Big)^{1/2}\Big(\int_0^t\int_B Q_iQ_i\,dV\,ds\Big)^{1/2},\end{aligned}\tag{3.10}$$

$$\begin{aligned}&\int_0^t\int_B \alpha K\Big(w_2\Big(h_{2,1}^{(2)}-h_{1,2}^{(2)}\Big)+w_3\Big(h_{3,1}^{(2)}-h_{1,3}^{(2)}\Big)\Big)dV\,ds\\&\qquad\le b_3\Big(\int_0^t\int_B \rho w_iw_i\,dV\,ds\Big)^{1/2}\Big(\int_0^t\int_B K^2\,dV\,ds\Big)^{1/2},\end{aligned}\tag{3.11}$$

$$\begin{aligned}&\int_0^t\int_B \Big[K\alpha\Big(l_1\Big(\Big(-u_{2,2}^{(2)}-u_{3,3}^{(2)}\Big)-(-f_{3,3}-f_{2,2})\Big)\\&\qquad+l_2\Big(u_{2,1}^{(2)}-f_{2,1}\Big)+l_3\Big(u_{3,1}^{(2)}-f_{3,1}\Big)\Big)\Big]dV\,ds\\&\qquad\le b_4\Big(\int_0^t\int_B l_il_i\,dV\,ds\Big)^{1/2}\Big(\int_0^t\int_B K^2\,dV\,ds\Big)^{1/2}.\end{aligned}\tag{3.12}$$

Thus,

$$
\begin{aligned}
&2H_\omega \int_0^t \int_B \Big(F_i w_i + \alpha K\Big(w_2\Big(h_{2,1}^{(2)} - h_{1,2}^{(2)}\Big) + w_3\Big(h_{3,1}^{(2)} - h_{1,3}^{(2)}\Big)\Big)\Big) dV\,ds \\
&\quad \le EH_\omega^2 + H_\omega\Big(\int_0^t \int_B (F_iF_i + K^2)\,dV\,ds\Big).
\end{aligned}
\tag{3.13}
$$

In (3.13), E is a constant that can be computed in terms of the constitutive coefficients, the initial conditions, the time t_1, and the domain.

In view of the estimates (2.22), (3.10), and (3.13), we can see that

$$
\begin{aligned}
&2H_\omega \int_0^t \int_B Q_i l_i\,dV\,ds \\
&\quad \le 2H_\omega\Big(\int_0^t \int_B (AS_iS_i + C^*K^2)\,dV\,ds\Big)^{1/2}\Big(\int_0^t \int_B Q_iQ_i\,dV\,ds\Big)^{1/2} \\
&\qquad + 2H_\omega\Big(B^* \int_0^t \int_B \rho \dot{w}_i \dot{w}_i\,dV\,ds\Big)^{1/2}\Big(\int_0^t \int_B Q_iQ_i\,dV\,ds\Big)^{1/2}, \\
&2H_\omega \int_0^t \int_B \Big[K\alpha\Big(l_1\Big(\Big(-u_{2,2}^{(2)} - u_{3,3}^{(2)}\Big) + (f_{3,3} + f_{2,2})\Big) \\
&\qquad\qquad + l_2\Big(u_{2,1}^{(2)} - f_{2,1}\Big) + l_3\Big(u_{3,1}^{(2)}\Big) - f_{3,1}\Big)\Big]\,dV\,ds \\
&\quad \le N^*H_\omega\Big(\int_0^t \int_B (AS_iS_i + C^*K^2)\,dV\,ds\Big)^{1/2}\Big(\int_0^t \int_B K^2\,dV\,ds\Big)^{1/2} \\
&\qquad + N^*H_\omega\Big(B^* \int_0^t \int_B \rho \dot{w}_i \dot{w}_i\,dV\,ds\Big)^{1/2}\Big(\int_0^t \int_B K^2\,dV\,ds\Big)^{1/2}.
\end{aligned}
\tag{3.14}
$$

Again, N^* is an easily computable positive constant. If we use the arithmetic-geometric mean inequality, we obtain

$$
\begin{aligned}
&2H_\omega \int_0^t \int_B \Big(Q_i l_i + \Big[K\alpha\Big(l_1\Big(\Big(-u_{2,2}^{(2)} - u_{3,3}^{(2)}\Big) + (f_{3,3} + f_{2,2})\Big) \\
&\qquad\qquad\qquad + l_2\Big(u_{2,1}^{(2)} - f_{2,1}\Big) + l_3\Big(u_{3,1}^{(2)} - f_{3,1}\Big)\Big)\Big]\Big)\,dV\,ds \\
&\quad \le 4F^2H_\omega^2 + \Big(\int_0^t \int_B ((C^* + 1)K^2 + AS_iS_i + Q_iQ_i)\,dV\,ds\Big) \\
&\qquad + \frac{B^*F^2}{2}H_\omega^2 + 4\Big(\int_0^t \int_B \rho \dot{w}_i \dot{w}_i\,dV\,ds\Big)\Big(\int_0^t \int_B (K^2 + Q_iQ_i)\,dV\,ds\Big),
\end{aligned}
\tag{3.15}
$$

where F can be computed in terms of the constitutive coefficients, the initial conditions, the time t_1, and the domain.

From (3.5), (3.9), (3.13), and (3.15), we conclude that we can explicitly determine a constant ξ satisfying (3.4). □

Theorem 3.2. *Let (w_i, l_i) be a solution of the problem determined by system (2.2), (2.3) with initial conditions (2.4), and boundary conditions (2.5). Then, there* *exists a positive constant M^* such that*

$$H_\omega(t) \le M^* \left(\int_0^{t_1} \int_B (F_i F_i + 2K^2 + S_i S_i + Q_i Q_i) dV\, ds \right)^{1-t/t_1}, \tag{3.16}$$

for all $t \le t_1$, where ω is given in (3.3).

Proof. If we define the function

$$P(t) = \ln\left[H_\omega(t) \exp\left(\frac{1}{2} \xi t^2 \right) \right], \tag{3.17}$$

then

$$\frac{d^2 P}{dt^2} = H_\omega^{-2} \left(H_\omega \frac{d^2 H_\omega}{dt^2} - \left(\frac{dH_\omega}{dt} \right)^2 + \xi H_\omega^2 \right). \tag{3.18}$$

Thus, according to (3.4),

$$\frac{d^2 P}{dt^2} \ge 0. \tag{3.19}$$

Jensen's inequality gives

$$H_\omega(t) \le [H_\omega(0)]^{1-t/t_1} [H_\omega(t_1)]^{t/t_1} \exp\left[\frac{1}{2} \xi t (t_1 - t) \right], \tag{3.20}$$

for $t \in [0, t_1]$. The theorem is proved taking

$$M^* = \max(1, H_\omega(t_1)) \exp\left[\frac{1}{8} \xi t_1^2 \right]. \tag{3.21}$$

□

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